On the evolution of the parton distribution functions to N^{3}LO accuracy in QCD

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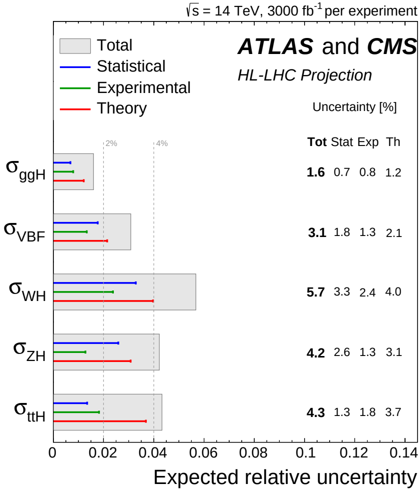
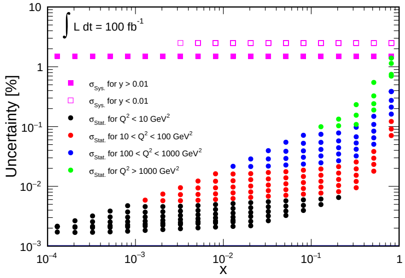
Outline

Outline

The precision frontier



(CERN Yellow Reports 2019)

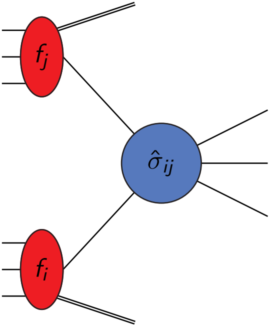


(EIC Yellow Report 2021) Experimental errors O(1%)

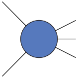
- Higgs production at the HL-LHC
- Inclusive DIS at the EIC

Theoretical predictions: ingredients

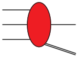
$$\sigma_{PP} = \sum_{i,j} \int dx_1 dx_2 \; f_i(x_1, \mu) f_j(x_j, \mu) \; \hat{\sigma}_{ij}(s\, x_1\, x_2, \mu) \equiv \sum_{i,j} f_i \otimes f_j \otimes \hat{\sigma}_{ij}$$



- Partonic cross section $\hat{\sigma}_{ij}(Q, \mu)$



- Non-perturbative PDFs $f_i(x_i, \mu)$



- Factorisation scale $\mu \gg \Lambda_{\mathrm{QCD}}$

QCD at N^{3}LO

	Q [GeV]	$\delta\sigma^{\mathrm{N^3LO}}$	δ(scale)	δ(PDF-TH)
NCDY	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm 2.5\%$
CCDY(W^+)	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm 2.1\%$
CCDY(W^-)	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm 2.13\%$

J. Baglio, C. Duhr, B. Mistlberger, R. Szafron 2022

- N^{3}LO corrections $O(\%)$, scale uncertainties sub-percent.
- δ(PDF-TH): Additional error due missing N^{3}LO PDFs.

$$\delta(\text{PDF-TH}) = \frac{1}{2}\left|\frac{\sigma^{\text{NNLO}}(\text{NNLO PDF}) - \sigma^{\text{NNLO}}(\text{NLO PDF})}{\sigma^{\text{NNLO}}(\text{NNLO PDF})}\right|$$

Scale evolution of the PDFs

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} q_{\rm ns} \\ q_{\rm s} \\ g \end{pmatrix} = \begin{pmatrix} P_{\rm ns} & 0 & 0 \\ 0 & P_{\rm qq} & P_{\rm qg} \\ 0 & P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm ns} \\ q_{\rm s} \\ g \end{pmatrix}$$

- Flavour non-singlet quark combinations $q_{\rm ns}$

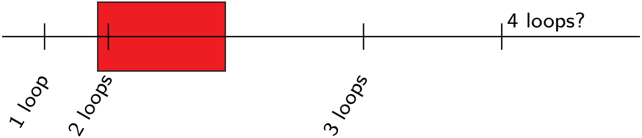
$$q^{\pm}_{\rm ns}(x,\mu^2) = (q_{\rm i} \pm \overline{q}_{\rm i}) - (q_{\rm k} \pm \overline{q}_{\rm k}), \quad q^{\rm v}_{\rm ns}(x,\mu^2) = \sum_i (q_i - \overline{q}_i),$$

- Flavour singlet PDFs

$$q_{\rm s}(x,\mu^2) = \sum_{\rm i=u,d,s} (f_{\rm i} + f_{\bar{\rm i}}), \quad g(x,\mu^2)$$

- Splitting functions $P_{\rm ij} = \underbrace{a\, P^{(0)}_{\rm ij}}_{\rm LO} + \underbrace{a^2\, P^{(1)}_{\rm ij}}_{\rm NLO} + \underbrace{a^3\, P^{(2)}_{\rm ij}}_{\rm NNLO} + \underbrace{a^4\, P^{(3)}_{\rm ij}}_{\rm N^3LO}, \; a = \frac{\alpha_s}{4\pi}$

State of the art - incomplete review



- One loop: 1974 - OPE for DIS Gross,Wilczek; Politzer,Georgi
- Two loops: still OPE
  - 1977 - Floratos,Ross,Sachrajda; Gonzalez-Arroyo,Lopez,Ynduràin
  - 1980 - Curci,Furmanski,Petronzio
  - 1991 - Hamberg,van Neerven
- Three loops: new approaches
  - 2004 - N^3LO DIS Moch,Vermaseren,Vogt
  - 2015 - N^3LO $PP \to H$, DY. . . Anastasiou,Duhr,Dulat,Herzog,Mistlberger;. . .
  - 2019 - Beam functions Luo,Yang,Zhu,Zhu; Ebert,Mistlberger,Vita; Baranowski,Behring,Melnikov,Tancredi,Wever
  - 2021 - OPE again: Blümlein,Marquard,Schneider,Schönwald; Gehrmann,von Manteuffel,Yang

Towards N^{3}LO splitting functions

- **Exact results**
  - $P^{(3)}_{\rm ns}$ in the **planar limit** (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017)
  - $P^{(3)}_{\rm ij}$ in the large-n_f limit (Gracey 1994,1998; Davies,Vogt,Ruijl,Ueda,Vermaseren 2016)
  - n_f^2 term in $P^{(3)}_{\rm qq}$ (Gehrmann,Sotnikov,von Manteuffel,Yang 2023)
  - n_f^2 term in $P^{(3)}_{\rm gq}$ (**GF**,Herzog,Moch,Vermaseren,Vogt 2023)
  - Flavour non-singlet: $n_f C_F^3$ term (Gehrmann,Sotnikov,von Manteuffel,Yang 2023)
- Fixed moments
  - $P^{(3)}_{\rm ns}(N \leq 16)$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017)
  - $P^{(3)}_{\rm qq}(N \leq 12)$ and $P^{(3)}_{\rm qg,gq,gg}(N \leq 10)$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2023)
- **High-precision approximations**
  - $q_{\rm ns}$ evolution to 1%-level (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017).
- **This talk:** evolution of the **singlet** to the same accuracy.
  Main tool: calculation of **moments** up to $N = 20$.

Outline

The OPE approach in DIS

Factorisation of DIS structure functions in moment space

(Christ,Hasslacher,Mueller 1972)

$$F_2(N, Q^2, P^2) = \int_0^1 x^{N-2} F_i(x) dx = \sum_{\mathrm{j=ns,q,g}} C_2^{(\mathrm{j})}\left(N, \frac{Q^2}{\mu^2}\right) \underbrace{\mathrm{A}_{\mathrm{j\,P}}^{\mathsf{ren}}\left(\mathrm{N}, \frac{\mathrm{P}^2}{\mu^2}\right)}_{\langle P|\mathcal{O}_\mathrm{j}(N)|P\rangle_{\mathsf{ren}}}$$

- $C_2^{(\mathrm{j})}(N)$ short-distance Wilson coefficient involving **hard** scale Q.
- $\mathrm{A}_{\mathrm{jP}}^{\mathsf{ren}}(\mathrm{N})$ **moments** of the PDFs (Collins,Soper 1981).

$$\mathcal{O}_{\mathrm{ns}}(N) = \mathcal{S}_T \left\{\bar{\psi}_{i_1} (\lambda^\rho) \gamma_{\mu_1} D_{\mu_2}^{i_1 i_2} \ldots D_{\mu_N}^{i_{N-1} i_N} \psi_{i_N}\right\},$$
$$\mathcal{O}_q(N) = \mathcal{S}_T \left\{\bar{\psi}_{i_1} \gamma_{\mu_1} D_{\mu_2}^{i_1 i_2} \ldots D_{\mu_N}^{i_{N-1} i_N} \psi_{i_N}\right\},$$
$$\mathcal{O}_g(N) = \mathcal{S}_T \left\{F_{\rho\mu_1}^{a_1} D_{\mu_2}^{a_1 a_2} \ldots D_{\mu_{N-1}}^{a_{N-2} a_{N-1}} F^{a_N;\rho}{}_{\mu_N}\right\},$$

- **Scale dependence** μ **cancels** between $C^{(j)}(N)$ and $\mathrm{A}_{\mathrm{jP}}^{\mathsf{ren}}$.

Moments of the splitting functions

Scale dependence of the PDFs from UV renormalisation

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}_{\mathrm{i}}^{\mathrm{ren}}(N) = \gamma_{\mathrm{ij}}(N)\, \mathcal{O}_{\mathrm{j}}^{\mathrm{ren}}(N), \quad \gamma_{\mathrm{ij}}(N) = \sum_{k>0} a^k \gamma_{\mathrm{ij}}^{(k-1)}$$

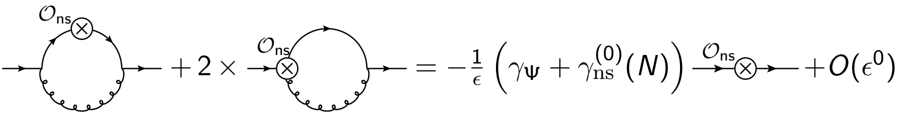
- Direct relation with the moments of the splitting functions

$$\gamma_{\mathrm{ij}}^{(k)}(N) = -\int_0^1 dx\, x^{N-1}\, P_{\mathrm{ij}}^{(k)}(x)$$

- UV behaviour of the Operator Matrix Elements $\mathrm{A}_{\mathrm{j\,P}}$
  - Renormalisation of **partonic** OMEs $\mathrm{A}_{\mathrm{j\,k}}$, $\mathrm{k} = \mathrm{g}, \mathrm{q}$.

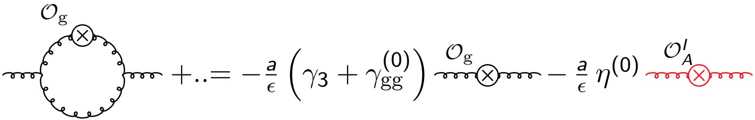
Off-shell renormalisation: non-singlet operators

The OME $\mathrm{A}^{\mathsf{ren}}_{\mathsf{ns\,q}}$ determines $\gamma_{\rm ns}$



- The calculation of the OMEs $\mathrm{A_{ij}}$ is automated to **4 loops** $\rightarrow$ `Forcer` (Ruijl,Ueda,Vermaseren 2017)
- $\gamma^{(3)}_{\mathsf{ns}}$ computed up to $N = 16$ (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017) and extended to $N = 22$ (Moch,Ruijl,Ueda,Vermaseren,Vogt to appear)

Troubles in the singlet sector



- Presence of unphysical *alien* operators (Gross,Wilczek 1974)

$$\mathcal{O}_A^I = (\partial^2 A^a - \partial(\partial . A^a))(\partial^{N-2} A^a) + \dots$$

- $\eta = \frac{a}{\epsilon}\eta^{(0)} + a^2\eta^{(1)} + \dots$ is the mixing between $\mathcal{O}_g$ and $\mathcal{O}_A^I$
- $\eta^{(0)} = -\frac{C_A}{N(N-1)}$ (Dixon,Taylor 1974; Hamberg,van Neerven 1991)

  $\eta^{(1)}$ (Hamberg,van Neerven 1991)

  $\eta^{(2)}$ (Gehrmann,von Manteuffel,Yang 2023)

Venturing out to two loops

$$\mathcal{O}_{\text{EOM}} = \underbrace{\eta\,(D.F)^a\,\left(\partial^{N-2}A^a\right)}_{\mathcal{O}_A^I} + \underbrace{g\,f^{aa_1a_2}\,(D.F)^a \sum_{\substack{i,j\geq 0\\ i+j=N-3}} \kappa_{i\,j}^{(1)}\,\left(\partial^i A^{a_1}\right)\left(\partial^j A^{a_2}\right)}_{\mathcal{O}_A^{II}} + ..,$$

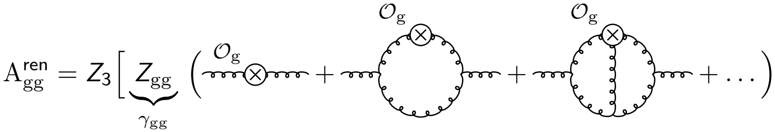
$$\mathcal{O}_c = \underbrace{-\eta\,\left(\partial\bar{c}^a\right)\left(\partial^{N-1}c^a\right)}_{\mathcal{O}_c^I} - \underbrace{g\,f^{aa_1a_2}\,\left(\partial\bar{c}^a\right) \sum_{\substack{i,j\geq 0\\ i+j=N-3}} \eta_{i\,j}^{(1)}\,\left(\partial^i A^{a_1}\right)\left(\partial^j c^{1+a_2}\right)}_{\mathcal{O}_c^{II}} + ..,$$

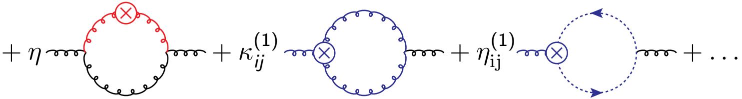
$\kappa_{ij}^{(1)}$, $\eta_{ij}^{(1)}$ extracted from 3-point correlators

$$+\,..\,\Big|_{\frac{1}{\epsilon}} = -\sum_{i,j}\eta_{ij}^{(1)}$$

Aliens in action

The OME $\mathrm{A}^{\mathrm{ren}}_{\mathrm{gg}}$ includes γ_{gg} provided we include all the terms



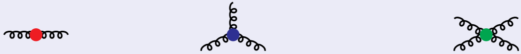


Discrepancies $\gamma^{(1)}_{\mathrm{gg}}$ due to missing red and blue terms (Hamberg,van Neerven 1991).

Systematic study to four loops

$\mathcal{O}_{\text{EOM}} \to$ generated by a field redefinition

$$\mathcal{O}_{\text{EOM}} \sim (D.F)^a \Big[\eta\, \partial^{N-2} A^a \quad +\kappa^{(1)}_{ij}\, \partial^i A^{a_1} \partial^j A^{a_2} \quad +\kappa^{(1)}_{ijk}\, \partial^i A^{a_1} \partial^j A^{a_2} \partial^k A^{a_3} + \dots$$



$\mathcal{O}_c \to$ generated by a BRST-type operator

$$\mathcal{O}_c \sim (\partial \bar{c}^a)\, \Big[\eta\, \partial^{N-1} c^a \quad +\eta^{(1)}_{ij}\, \partial^i A^{a_1} \partial^{1+j} c^{a_2} \quad +\eta^{(1)}_{ijk}\, \partial^i A^{a_1} \partial^j A^{a_2} \partial^{1+k} c^{a_3} + \dots$$

BRST and anti-BRST symmetries give relations between κ, η

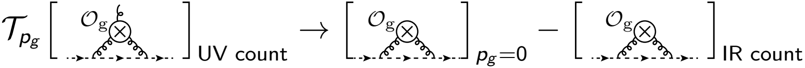
Bases of alien operators

$$\eta_{ij}^{(1)} = 2\kappa_{ij}^{(1)} + \eta(N) \begin{pmatrix} i+j+1 \\ i \end{pmatrix} = -\sum_{s=0}^{i} (-1)^{s+j} \begin{pmatrix} s+j \\ j \end{pmatrix} \eta_{i-s,j+s}^{(1)} \qquad (1)$$

- **Fix N**: linear system of equations. Say $N = 6$

$$5\eta(6) + 4\kappa_{1\,2}^{(1)} - 6\kappa_{0\,3}^{(1)} = 0.$$

- For $N \leq 20$ we computed **basis** of $\kappa_{ijk\ldots}$ using Taylor expansion $\mathcal{T}$ and IR R-operation (Chetyrkin,Tkachov 1982;Chetyrkin,Smirnov 1984;Herzog,Ruijl 2017)



Agreement with (Gehrmann,von Manteuffel,Yang 2023).

Results - fixed moments

"Top row" $\gamma_{\rm qq}^{(3)}$ and $\gamma_{\rm qg}^{(3)}$

- $\gamma_{\rm qq}^{(3)}(N \leq 20)$ 2302.07593 - (**GF**,Herzog,Moch,Vogt)
- $\gamma_{\rm qg}^{(3)}(N \leq 20)$ 2307.04158 - (**GF**,Herzog,Moch,Vogt)

"Bottom row" $\gamma_{\rm gq}^{(3)}$ and $\gamma_{\rm gg}^{(3)}$

- $\gamma_{\rm gq}^{(3)}(N \leq 20)$ 2404.09701 - (**GF**,Herzog,Moch,Pelloni,Vogt)
- **NEW** $\boldsymbol{\gamma_{\rm gg}^{(3)}(N \leq 20)}$ to appear - (**GF**,Herzog,Moch,Pelloni,Vogt)

$$
\begin{aligned}
\gamma_{\rm gg}^{(3)}(N=2) &= 654.463 n_f - 245.611 n_f^2 + 0.924991 n_f^3, \\
&\dots \\
\gamma_{\rm gg}^{(3)}(N=20) &= 90499.3 - 26132.3 n_f + 1178.50 n_f^2 + 25.6433 n_f^3.
\end{aligned}
$$

Checks

The moments are compared with the results in the literature for

- Large-n_f (Gracey 1994,1998; Davies,Ruijl,Ueda,Vermaseren,Vogt 2016)
- Quartic Casimir contributions (Moch,Ruijl,Ueda,Vermaseren,Vogt 2018)
- ζ_4 terms from the no-π^2 theorem (Jamin,Miravitllas 2018; Baikov,Chetyrkin 2018; Kotikov,Teber 2019)
- $\gamma^{(3)}_{\rm qq}(N \leq 12)$ and $\gamma^{(3)}_{\rm qg,gq,gg}(N \leq 10)$ from DIS structure functions (Moch,Ruijl,Ueda,Vermaseren,Vogt 2021-2023)
- $\gamma^{(3)}_{\rm qq}\Big|_{n_f^2}$ (Gehrmann,Sotnikov,von Manteuffel,Yang 2023)

Analytic reconstruction

Use moments to constrain an ansatz for $\gamma_{ij}^{(3)}(N)$

- Denominator structure $D_a = 1/(N+a)$
- Harmonic sums $S_{\pm m_1,\dots m_k} = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\dots,m_k}$

Large ansatz: at weight $w = 7 \to 2 \cdot 3^{w-1} = 1458$ S-sums.

- Restrict to solutions with **integer** coefficients (Lenstra^2,Lovász 1982)!
- Some further constraints e.g. **reciprocity** $N \to -1-N$ (Gribov,Lipatov 1972;Blümlein,Ravindran,van Neerven 2001;Chen,Yang,Zhu,Zhu 2020) in the **non-singlet** (Moch,Ruijl,Ueda,Vermaseren,Vogt 2017).

Complete reconstruction

- Coefficients of ζ_k multiply S of lower weight e.g. **new:** $\gamma^{(3)}_{\mathrm{ij}}\big|_{\zeta_5}$

$$\begin{aligned}\gamma_{gg}(N)\big|_{\zeta_5 C_A^4} = \; & C_A^4\Big(-12016/27 - 1120/3\nu - 640/3\nu^2 \\ & + 3008/9\eta + 640\eta^2 - 8/27N(N+1) + 1760/3S_1 \\ & + 640/3S_1\left\{2\nu - 2\eta - S_1\right\}\Big)\end{aligned}$$

$\eta = D_0 - D_1$, $\nu = D_{-1} - D_2$.

- Colour factors with simpler structure $\rightarrow \gamma^{(3)}_{\mathrm{gq}}(N)\Big|_{n_f^2}$
  - Only S-sum up to weight 4
  - Simpler diagrams computed with `Forcer` up to $N = 60$
  - Complete reconstruction of $\gamma^{(3)}_{\mathrm{gq}}(N)\Big|_{n_f^2}$ and translation to x-space (**GF**,Herzog,Moch,Vermaseren,Vogt 2023)

Outline

Approximate approach

Parameterise $P_{\rm ij}^{(3)}(x)$ with an ansatz that

- **reproduces** the computed moments
- **matches** the known endpoint behaviour at $x \to 1$ and $x \to 0$

Several collaborations are developing approximate N^{3}LO PDFs

(Cridge,Harland-Lang,McGowan,Thorne 2022;NNPDF 2024)

Recent benchmarking of different approximate evolutions

(Cooper-Sarkar,Cridge,Giuli,Harland-Lang,Hekhorn,Huston,Magni,Moch,Thorne 2024)

Example: quark pure singlet $P_{\mathrm{ps}} = P_{\mathrm{qq}} - P^{+}_{\mathrm{ns}}$

Large-x behaviour

$$P^{(3)}_{\mathrm{ps}}(x \to 1) = \sum_{j\geq 1} \sum_{k \leq 4} C_{j,k}\,(1-x)^j \log^k (1-x),$$

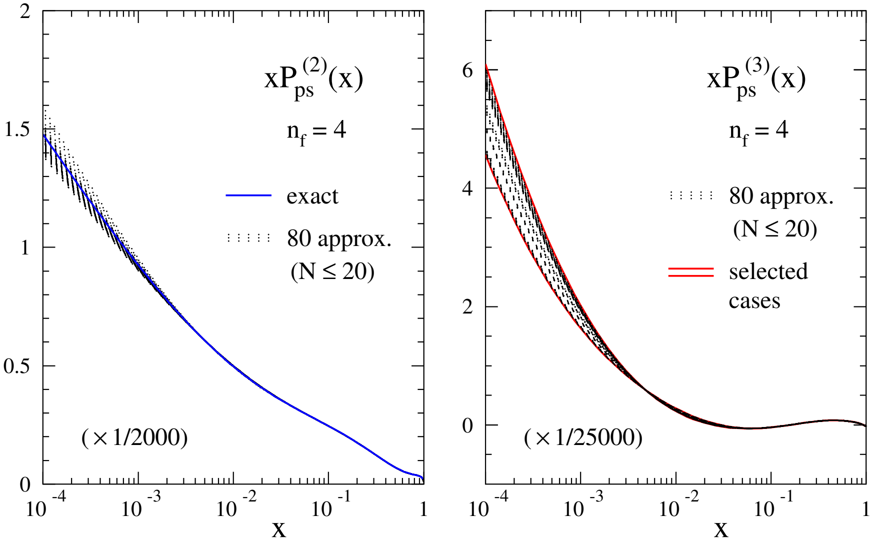
where $C_{1,4}$ and $C_{1,3}$ are known (Soar,Moch,Vermaseren,Vogt 2009)

Small-x behaviour

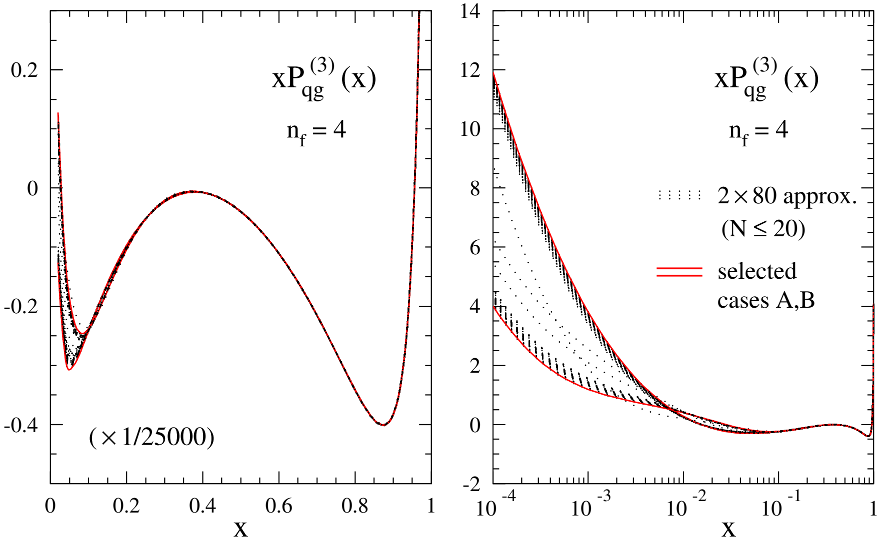
$$P^{(3)}_{\mathrm{ps}}(x \to 0) = \sum_{l=1}^{3} E_l \frac{\log^{3-l} x}{x} + \sum_{l=0}^{6} F_l \log^{6-l} x$$

- Leading BFKL logarithm E_1 known (Catani,Hautman 1994)
- Non-BFKL logarithm known for $l < 3$ (Davies,Kom,Moch,Vogt 2022)

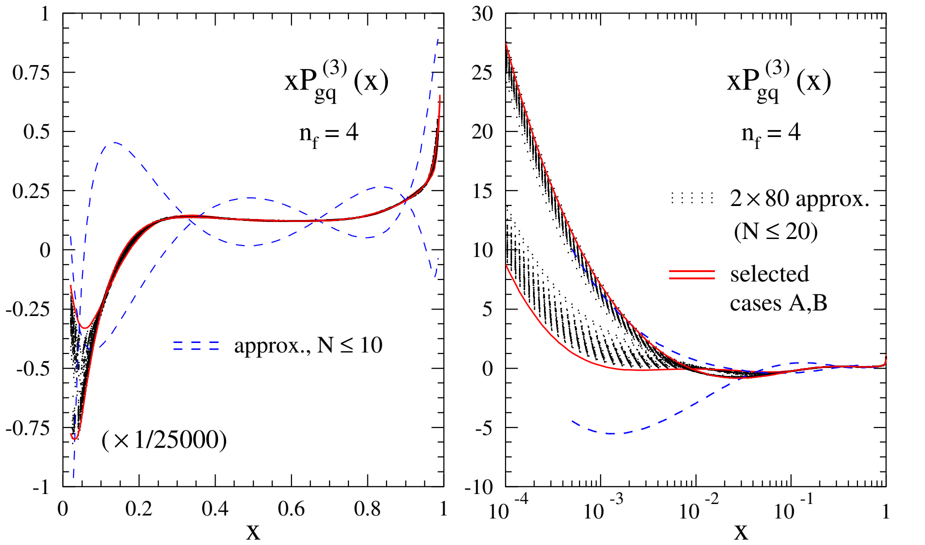
Approximate $P_{\rm ps}^{(3)}(x)$



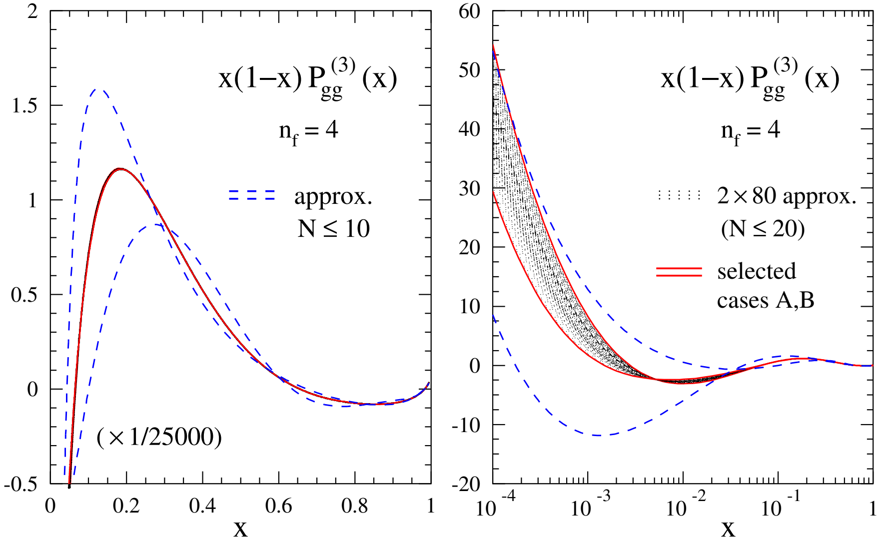
Approximate $P_{qg}^{(3)}(x)$



Approximate $P_{gq}^{(3)}(x)$



NEW: Approximate $P_{gg}^{(3)}(x)$



Evolution of the parton densities

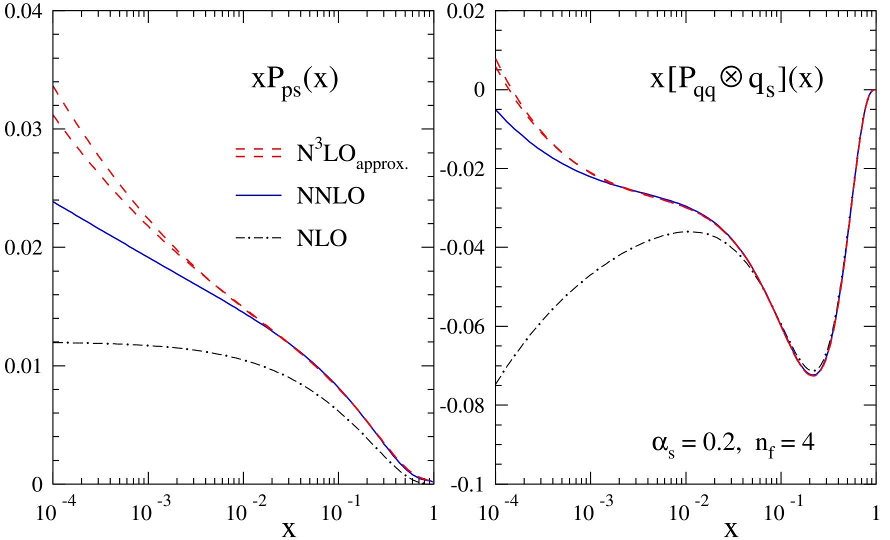
The splitting functions enter the PDF evolution via **convolution**

$$\dot{f}_{\rm i} \equiv \mu^2 \frac{dq_{\rm i}}{d\mu^2} = \sum_j \int_x^1 \frac{dz}{z} \underbrace{P_{\rm ij}(z)}_{\text{approx.}} f_{\rm j}\left(\frac{x}{z}\right), \quad {\rm j} = {\rm q}, {\rm g}$$

Note the interplay between $P(z \sim x \to 0)$ and $f\left(\frac{x}{z} \to 1\right)$.
$P(z \sim x \to 0)$ has the largest uncertainty
$f\left(\frac{x}{z} \to 1\right)$ is **suppressed**. Model PDFs (Moch,Vermaseren,Vogt 2004)

$$\begin{aligned} x\, g(x) &= 1.6\, x^{-0.3}\, (\mathbf{1}-\mathbf{x})^{\mathbf{4.5}}\, (1 - 0.6x^{0.3}), \\ x\, q_{\rm s}(x) &= 0.6\, x^{-0.3}\, (\mathbf{1}-\mathbf{x})^{\mathbf{3.5}}\, (1 + 5.0x^{0.8}) \end{aligned}$$

Example: $P_{\mathrm{qq}} \otimes q_{\mathrm{s}}$



$\mathrm{N}^3\mathrm{LO}$ evolution of the PDFs

Use approximate splitting functions and model PDFs to construct

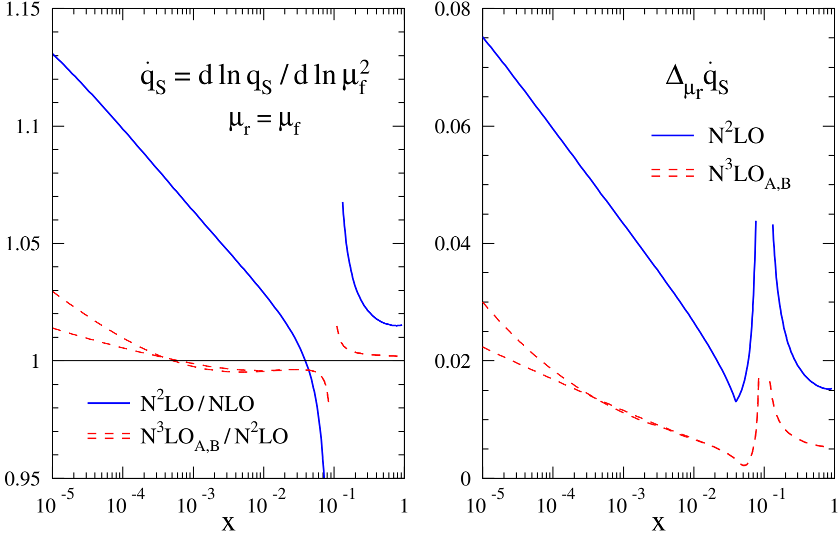
$$\dot{q}_\mathrm{s} = P_\mathrm{qq} \otimes q_\mathrm{s} + P_\mathrm{qg} \otimes g$$
$$\dot{g} = P_\mathrm{gq} \otimes q_\mathrm{s} + P_\mathrm{gg} \otimes g$$

Focus on

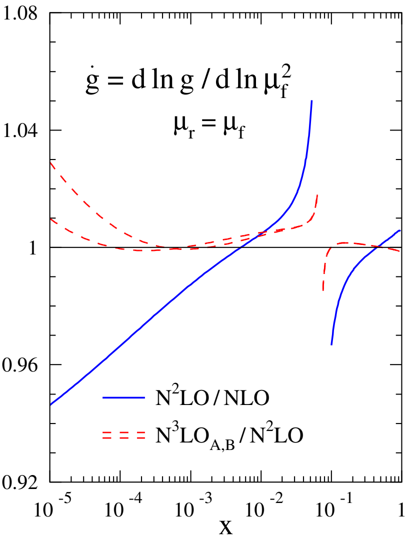
- **Impact** of the $\mathrm{N}^3\mathrm{LO}$ correction $\frac{\dot{q}_\mathrm{s}^{\mathrm{N}^3\mathrm{LO}}}{\dot{q}_\mathrm{s}^\mathrm{NNLO}}, \frac{\dot{g}_\mathrm{s}^{\mathrm{N}^3\mathrm{LO}}}{\dot{g}_\mathrm{s}^\mathrm{NNLO}}$
  - Spurious pole at $x \sim 0.1$ where $\dot{q}_s$, $\dot{g}$ vanish.
- **Stability** under variations of the renormalisation scale

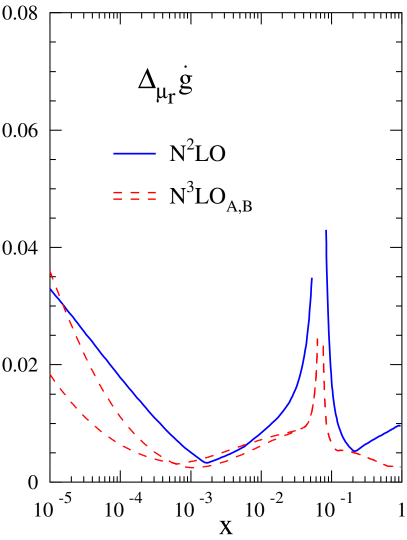
$$\Delta_{\mu_r} \dot{q}_\mathrm{s} = \frac{1}{2} \frac{\max[\dot{q}_\mathrm{s}(\mu_r^2 = \lambda \mu_f^2)] - \min[\dot{q}_\mathrm{s}(\mu_r^2 = \lambda \mu_f^2)]}{\mathrm{average}[\dot{q}_\mathrm{s}(\mu_r^2 = \lambda \mu_f^2)]}, \qquad \lambda = \frac{1}{4} \ldots 4$$

Scale evolution of the quark singlet



NEW: Scale evolution of the gluon





Outline

Conclusion

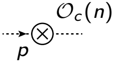
- We computed the moments of **all** the four-loop splitting functions up to $N = 20$.
- We constructed a highly precise parameterisation of the **scale evolution** of the parton densities.
  - N^3LO corrections below $(2 \pm 1)\%$ up to $x = 10^{-5}$.
  - Renormalisation scale variations $\Delta\dot{q} \sim \Delta\dot{g} \approx 2\%$ for $x \gtrsim 10^{-5}$.
- Can we compute the four-loop splitting functions exactly?
  - Try to get new infomation of the small-x logarithms

How to go to arbitrary N?

We relied on **fixed N** assumption for the

1. calculation of the OMEs $\mathrm{A_{ij}}$
2. computing a basis of alien couplings $\kappa_{ij}^{(1)}$, ...

1: *Generating function* method (Ablinger,Blümlein,Hasselhuhn,Klein,Schneider,Wißbrock 2012;Ablinger,Behring,Blümlein,De Freitas,von Manteuffel,Schneider 2014;Gehrmann,von Manteuffel,Yang 2023)

$\sum_n \tau^n$  $= \sum_n \tau^n (\Delta.p)^n = \frac{1}{1-\tau\,\Delta.p} \sim$ 

This approach converts the operator vertices (physical and alien) into *propagators* (Gehrmann,Sotnikov,von Manteuffel,Yang 2023)[2].

Alien couplings for arbitrary N

S. Van Thurenhout noticed **all-N** structure of the *alien mixing constants* at leading order

$$\kappa_{ij}^{(1)}, \eta_{ij}^{(1)} \sim \eta(N) \left[a_1 (-1)^j + a_2 \begin{pmatrix} i+j+1 \\ i \end{pmatrix} + a_3 \begin{pmatrix} i+j+1 \\ j \end{pmatrix} \right]$$

$\rightarrow$ Generalised anstaz for the mixing of the **higher point** aliens.

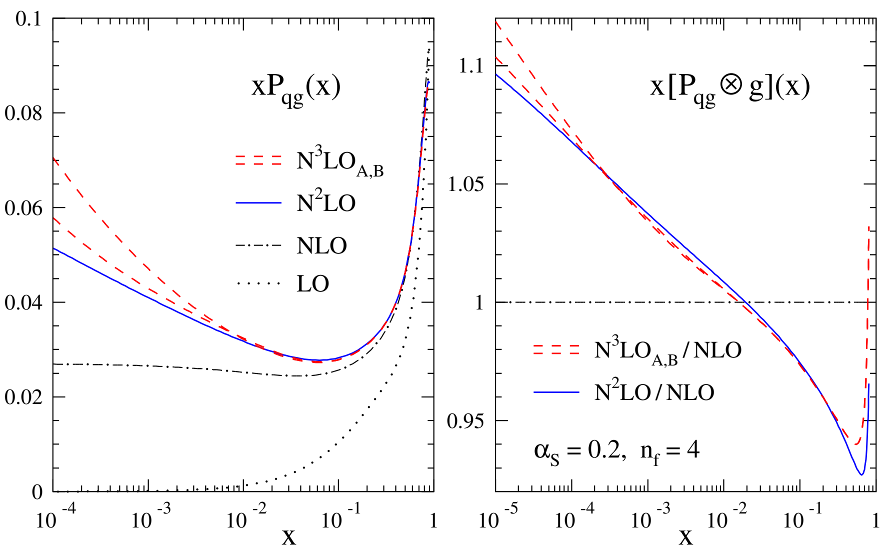
(anti-)BRST constraints fix $\kappa_{ijk}^{(i=1,2)}$ and $\kappa_{ijkl}^{(1)}$ up to **boundary values**.

- **NEW:**

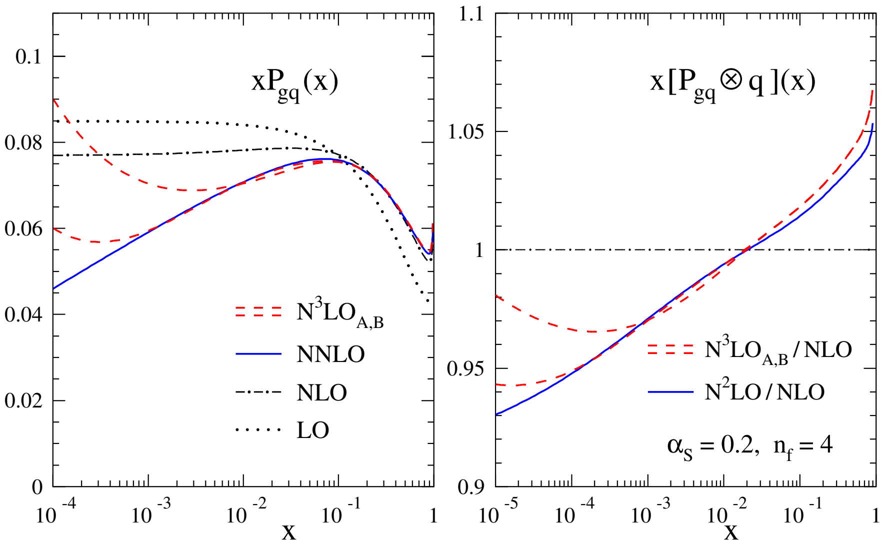
Solutions for all the required leading-order mixing constants from symmetry constraints (GF,Herzog,Moch,**Van Thurenhout** 2024).

Thank you!

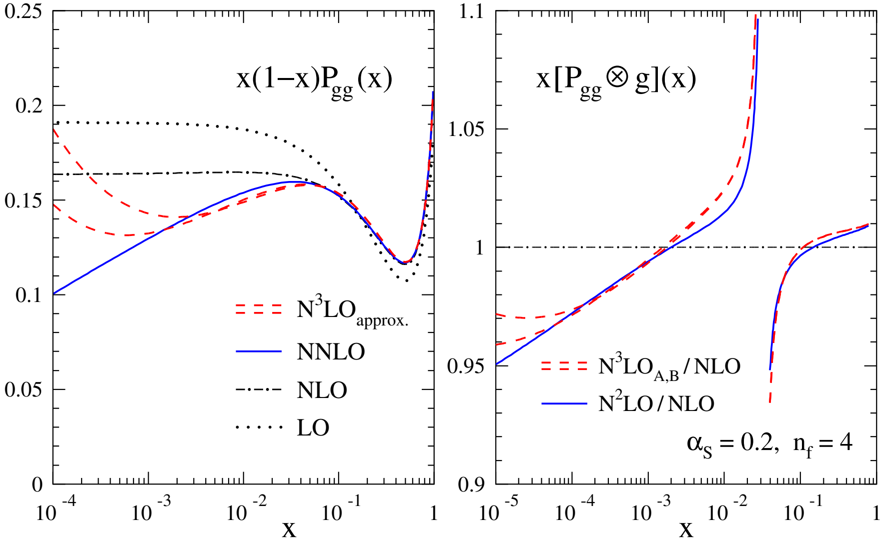
Quark evolution: gluon-to-quark contribution



Gluon evolution: quark-to-gluon contribution



Gluon evolution: gluon-to-gluon contribution



Gluonic aliens

$$\mathcal{L} = -\frac{1}{4} F^{a\,\mu\nu} F^{a}_{\mu\nu} + c_{\mathrm{g}} \mathcal{O}_{\mathrm{g}} \qquad \mathcal{L} \to \mathcal{L} + \underbrace{(D.F)^a \mathcal{G}^a}_{\mathcal{O}_{\mathrm{EOM}}}$$

$$\hookrightarrow A^a_\mu \to A^a_\mu + \Delta_\mu \, \mathcal{G}^a(A, \partial A, \dots) \nearrow$$

To linear order in $\Delta_{\mu_1} \dots \Delta_{\mu_N}$

Gluonic aliens

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$$A^a_\mu \to A^a_\mu + \Delta_\mu \, \mathcal{G}^a(A, \partial A, \dots)$$

To linear order in $\Delta_{\mu_1} \dots \Delta_{\mu_N}$

$$\mathcal{O}_{\mathrm{EOM}} = (D.F)^a \Big[\underbrace{\eta\, \partial^{N-2} A^a}_{\mathcal{O}^{I}_A} + g f^{a a_1 a_2} \sum_{i_1+i_2=N-3} \underbrace{\kappa_{i_1 i_2} (\partial^{i_1} A^{a_1})(\partial^{i_2} A^{a_2})}_{\mathcal{O}^{II}_A}$$

$$+ g^2 \sum_{\substack{i_1+i_2+i_3\\N-4}} \underbrace{\Big(\kappa^{(1)}_{i_1 i_2 i_3} f^{a a_1 z} f^{a_2 a_3 z} + \kappa^{(2)}_{i_1 i_2 i_3} d_4^{a a_1 a_2 a_3} + \kappa^{(3)}_{i_1 i_2 i_3} d_{\widehat{4ff}}^{a a_1 a_2 a_3} \Big) (\partial^{i_1} A^{a_1})..(\partial^{i_3} A^{a_3})}_{\mathcal{O}^{III}_A}$$

$$+ g^3 \sum_{\substack{i_1+..+i_4\\N-5}} \underbrace{\Big(\kappa^{(1)}_{i_1 \dots i_4} (f\,f\,f)^{a a_1 a_2 a_3 a_4} + \kappa^{(2)}_{i_1 \dots i_4} d_{4f}^{a a_1 a_2 a_3 a_4} \Big) (\partial^{i_1} A^{a_1})..(\partial^{i_4} A^{a_4})}_{\mathcal{O}^{IV}_A} + O(g^4) \Big]$$

Generalised gauge invariance

New gauge transformation $A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \tilde{\delta}_\omega A^a_\mu$

$$\delta_\omega A^a_\mu = D^{ab}_\mu \omega^b,$$
$$\tilde{\delta}_\omega A^a_\mu = -\Delta_\mu \left(\delta_\omega \mathcal{G}^a - g\, f^{abc} \mathcal{G}^b \omega^c\right)$$

Generalised BRST transformation $A^a_\mu \to A^a_\mu + \lambda s'(A^a_\mu)$, $s' = s + \tilde{s}$

$$s(A^a_\mu) = D^{ab}_\mu c^b,$$
$$\tilde{s}(A^a_\mu) = -\Delta_\mu \left(s(\mathcal{G}^a) - g\, f^{abc} \mathcal{G}^b c^c\right),$$

are **nihilpotent**.

$$(s')^2 = 0.$$

Ghost aliens

$$\widetilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + c_{\mathrm{g}}\, \mathcal{O}_{\mathrm{g}}^{(N)} + \mathcal{O}_{\mathrm{EOM}}^{(N)}}_{\mathcal{L}_{\mathrm{GGI}}} + \mathbf{s'} \left[\bar{c}^a \left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a \right) \right]$$

- $s\left[\bar{c}^a \left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a\right)\right]$ generates gauge fixing and ghost terms.
- $\tilde{s}\left[\bar{c}^a \left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a\right)\right]$ generates **ghost** aliens

$$\mathcal{O}_G^{(N)} = \bar{c}^a \partial \left(s\left(\mathcal{G}^a\right) - g\, f^{abc}\, \mathcal{G}^b\, c^c \right)$$

- The **couplings** in $\mathcal{O}_G^{(N)}$ are related to those in $\mathcal{O}_{\mathrm{EOM}}^{(N)}$ by BRST.
- Further anti-BRST relations impose more constraints

Spin N	2	4	6	8	10	12	14	16
# aliens	1	2	5	12	25	50	87	140