

1.5 Constraints

From what we discussed so far, it might appear that solving mechanics problem is a straightforward exercise

- ⊙ determine the forces
- ⊙ fix the initial conditions
- ⊙ solve coupled diff. eq.^s of the type $\ddot{\vec{x}}_i = \frac{1}{m_i} \vec{F}_i(\vec{x}_k, \dot{\vec{x}}_k)$

Actually in most physical problems this is not at all an easy exercise for two closely connected reasons

- ⊙ difficulty in determining the structure of all relevant forces
- ⊙ implementation of physical constraints [→ examples]

These 2 points are closely connected since the forces which are more difficult to be evaluated are those associated to constraints.

Constraints can be classified into 2 main groups

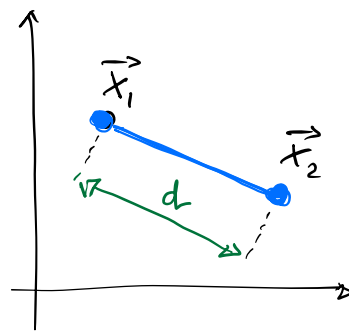
I. holonomic constraints

If they can be expressed as

$$f_a(\vec{x}_i, t) = 0 \quad a = 1 \dots K \quad (1.28)$$

E.g.: coordinates of rigid bodies satisfy constraints of this type

$$(\vec{x}_1 - \vec{x}_2)^2 - d^2 = 0$$



II. non-holonomic constraints

All those which cannot be put in the form (1.28)

E.g.: boundaries determined by walls

$$|\vec{x}_i|^2 \leq R^2 \quad (\text{particles inside a sphere of radius } R)$$

On general grounds, constraints imply two types of difficulties:

- Ⓐ The "original" coordinates, hence Newton's set of diff. eq.^s, are not any more independent ($\leftrightarrow$ redundant variables).
- Ⓑ The forces exert by the constraints are not given and are usually difficult to be computed.

For holonomic constraints, the problem (A) is solved introducing an appropriate set of generalized coordinates $\{q_i\}$

$$N \text{ particles} \rightarrow 3 \times N \text{ degrees of freedom } \{\vec{x}_i\} \quad i=1 \dots N$$

(in absence of constr.)

⊕

$$K \text{ holon. const.} \rightarrow K \text{ eq.}^s \text{ among the } \{\vec{x}_i\}$$

⇓

$$(3N-K) \text{ d.o.f. (with constr.)} \rightarrow \{q_i\} \quad i=1 \dots 3N-K$$

(if sufficiently smooth...) The transformations from the initial set of variables to the new one define a mapping between $\mathbb{R}^{3N-K}$ and a manifold $\mathcal{M} \in \mathbb{R}^{3N}$

$$\varphi: \begin{cases} \vec{x}_1 = \vec{x}_1(q_1, \dots, q_{3N-K}, t) \\ \vdots \\ \vec{x}_N = \vec{x}_N(q_1, \dots, q_{3N-K}, t) \end{cases} \quad (1.29)$$

$$\{q_i\} \in \mathcal{U} \subset \mathbb{R}^{3N-K}$$

↑
domain of the
new variables

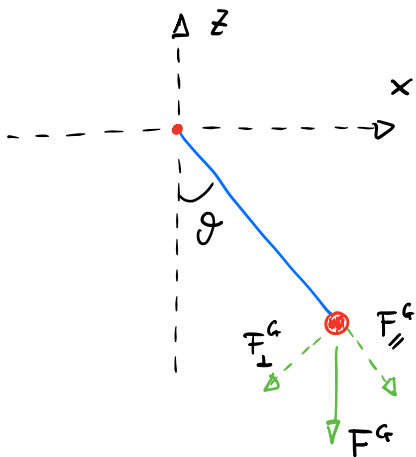
$$\varphi: \mathcal{U} \rightarrow \mathcal{M} \in \mathbb{R}^{3N}$$

Generalized coordinates turn out to be very useful in many physical problems, even in absence of constraints (change of coord. can make the problem simpler).

However, they usually do not have a Cartesian structure (they can also have different physical dimensions) $\rightarrow$ often non trivial to evaluate the forces in terms of the new variables (especially in the case of constraints $\leftrightarrow$ problem (B) not solved yet...).

1.6 D'Alembert's principle

To illustrate the above points about constraints & generalize coord. let's consider the familiar case of the pendulum:



$$\vec{F}^{\text{tot}} = \vec{F}^G + \vec{F}^{\text{constr.}}$$

$$\begin{cases} \ddot{x} = F_x^{\text{tot}}(x, z, \dot{x}, \dot{z}) \\ \ddot{z} = F_z^{\text{tot}}(x, z, \dot{x}, \dot{z}) \end{cases}$$

$$|\vec{F}_G| = mg$$

$$F_{\parallel}^G = -mg \cos \theta$$

$$F_{\perp}^G = -mg \sin \theta$$

$$F_{\parallel}^{\text{constr.}} = -F_{\parallel}^G$$

$$F_{\perp}^{\text{constr.}} = 0$$

(1.30)



$$\begin{cases} F_{\parallel}^{\text{tot}} = 0 \\ F_{\perp}^{\text{tot}} = -mg \sin \theta \end{cases}$$

Rather boring exercise writing down the e.o.m. in x, z coord....

But there is a much simpler way to solve the problem:
change coordinates $\oplus$ use conservation laws

$$\begin{cases} x = L \sin \vartheta \\ z = -L \cos \vartheta \end{cases} \rightarrow \begin{aligned} T &= \frac{1}{2} m (\dot{x}^2 + \dot{z}^2) = \frac{1}{2} m L^2 \dot{\vartheta}^2 \\ U &= mgz = -mgL \cos \vartheta \end{aligned} \quad (1.31)$$

$$E = T + U = \frac{1}{2} m L^2 \dot{\vartheta}^2 - mgL \cos \vartheta = \text{const.} \quad (1.32)$$

$$\frac{dE}{dt} = 0 = m L^2 \dot{\vartheta} \ddot{\vartheta} + mg \sin \vartheta \dot{\vartheta} = 0 \Rightarrow \boxed{\ddot{\vartheta} = -\frac{g}{L} \sin \vartheta} \quad (1.33)$$

The fact that the system is conservative $\oplus$ the fact that there is a single d.o.f. is what makes the problem particularly simple (in the right variables...).

In this example it is easy to check we are still dealing with Newton's law ($m \ddot{\mathbf{r}}_1 = \mathbf{F}_1$). But in more complicated systems it is much more difficult to identify the structure of the forces.

On the other hand, we can try to generalize the concept that constraints do not produce work [D'Alembert principle]



Let's consider $\vec{X}(t) = \vec{X}(q_j(t), t)$ $j=1 \dots d \leftarrow$ number of degrees of freedom

$$\boxed{\frac{\partial \vec{X}}{\partial q_j} \cdot \vec{F}^{\text{constr}} = 0} \quad (1.34)$$

↑
velocity on the
"constrained manifold"

$$\left[\begin{aligned} \text{E.g.:} \\ \vec{X} \equiv (L \sin \vartheta, -L \cos \vartheta) \quad \frac{\partial \vec{X}}{\partial \vartheta} \equiv (L \cos \vartheta, L \sin \vartheta) \end{aligned} \right]$$

Combining D'Alembert principle, (1.34), with Newton's law:

$$\vec{p} \cdot \frac{\partial \vec{x}}{\partial \dot{q}_j} = \vec{F}^{\text{tot}} \cdot \frac{\partial \vec{x}}{\partial \dot{q}_j} = \vec{F}^G \cdot \frac{\partial \vec{x}}{\partial \dot{q}_j} = -\vec{\nabla} U \cdot \frac{\partial \vec{x}}{\partial \dot{q}_j} \quad (1.35)$$

the constraints drops

The result is clearly valid for any conservative force. Let's now look in component form, setting $m=0$

$$m \ddot{x}_i \frac{\partial x_i}{\partial \dot{q}_j} = -\frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial \dot{q}_j} = -\frac{\partial U}{\partial \dot{q}_j} \quad (1.36)$$

Goal: manipulate (1.36) in order to obtain an equation that depends only on the new variables q_i & $\dot{q}_i = \frac{d}{dt} q_i$

Key points:

- ⊙ pay attention to partial derivatives (with respect to x_i or q_i) vs. complete differential with respect to time ($\dot{x}_i$ and $\dot{q}_i$)
- ⊙ note that $x_k(q_i(t), t)$ does not depend explicitly on $\dot{q}_i$ (same for U)

$$\rightarrow \dot{x}_i \equiv \frac{dx_i}{dt} = \frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial q_k} \dot{q}_k \quad (1.37)$$

$$\rightarrow \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left[\frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial q_k} \dot{q}_k \right] = \frac{\partial x_i}{\partial q_j} \quad (1.38)$$

Using (1.38) we can re-write the l.h.s. of (1.36) as

$$m \ddot{x}_i \frac{\partial x_i}{\partial \dot{q}_j} = \frac{d}{dt} \left[m \dot{x}_i \frac{\partial x_i}{\partial \dot{q}_j} \right] - m \dot{x}_i \frac{d}{dt} \left[\frac{\partial x_i}{\partial \dot{q}_j} \right] \quad (1.39)$$

= I_A - I_B

Let's now look at the Kinetic energy

(19)

$$T = \frac{1}{2} m \dot{x}_i \dot{x}_i = T(q_k, \dot{q}_k, t) \quad \frac{\partial T}{\partial \dot{q}_j} = m \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \quad (1.40)$$

$$\Rightarrow I_A = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \quad (1.41)$$

$$I_B = m \dot{x}_i \left[\frac{\partial^2 x_i}{\partial q_j \partial t} + \frac{\partial^2 x_i}{\partial q_i \partial q_k} \dot{q}_k \right] = m \dot{x}_i \frac{\partial}{\partial \dot{q}_j} [\dot{x}_i] = \frac{\partial}{\partial \dot{q}_j} T \quad (1.42)$$

Putting all ingredients together, (1.36) has been transformed in

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_j} \right] - \frac{\partial T}{\partial q_j} = - \frac{\partial U}{\partial q_j} \quad (1.43)$$

Let's finally define the function

$$L = T(q_i, \dot{q}_i, t) - U(q_i, t) \quad (1.44)$$

which is known as the Lagrangian of the system.

Eq. (1.43), obtained by combining D'Alembert principle & Newton law, then becomes

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0} \quad \text{Euler-Lagrange eq.} \quad (1.45)$$

Exercise: determine L and check (1.45) in the case of the pendulum.

As we will discuss in the next lectures, the concept of Lagrangian goes well beyond the problem of constraints and allows us to make a deeper (more general) formulation of the principles of mechanics.