Applications of General Relativity
Solutions 3

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Exercise 1 [Trajectories in the Reissner-Nordström space-time] (7 points)

a) Although the test particle is not charged, because the metric depends on Q through r_Q, the particle's equations of motion depend on Q as well. This is because the Einstein's field equations depend on the mass-energy, and there is a non-zero energy contribution generated by the electric field due to the charge through the EM stress energy tensor:

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F^{\rho}_{\ \nu} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{1}$$

with

$$F^{0i} = -F^{i0} = -E^i, \quad \text{and} \quad F^{ij} = -F^{ji} = \frac{1}{\sqrt{-g}}\epsilon^{ijk}B_k. \tag{2}$$

For a static field determined by the charge Q placed at the center of the coordinate system, $B^k = 0$ and $E^i = Q\,x^i/r^3$.

b) First of all, we should note that the term $g_{rr} = \left(1 - r_S/r + r_Q^2/r^2\right)^{-1}$ is singular at two radial positions $r_\pm$ given by the condition $r^2 + r_S r + r_Q^2 = 0$:

$$r_\pm = \frac{r_S}{2}\left(1 \pm \sqrt{1 - \frac{4r_Q^2}{r_S^2}}\right), \tag{3}$$

with the ratio:

$$\frac{4r_Q^2}{r_S^2} = \frac{Q^2}{GM^2} \ll 1, \tag{4}$$

for typical physical application where $Q \ll \sqrt{G}M$. We can describe the motion of a particle through the lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, where $\dot{} = d/d\lambda$. Since the metric is spherically symmetric, we can study orbit on a specific plane $\theta = \pi/2$ without loss of generality, and our lagrangian reads:

$$\mathcal{L} = \Delta(r)\,c^2\,\dot{t}^2 - \frac{\dot{r}^2}{\Delta(r)} - r^2\dot{\phi}^2, \tag{5}$$

where $\Delta(r) = 1 - r_S/r + r_Q^2/r^2$. We can immediately identify 3 constants of motion:

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0 \quad \Rightarrow \quad r^2\dot{\phi} = L, \tag{6}$$

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = 0 \quad \Rightarrow \quad \Delta(r)\dot{t} = \frac{\sqrt{E}}{c}, \tag{7}$$

$$\mathcal{L} = \Delta(r)\, c^2\, \dot{t}^2 - \frac{\dot{r}^2}{\Delta(r)} - r^2 \dot{\phi}^2 = \left\{ \begin{array}{ll} c^2 & \text{massive particle} \\ 0 & \text{massless particle} \end{array} \right\} = \epsilon. \tag{8}$$

Inserting equations (6) and (7) into (8) we get:

$$\dot{r}^2 + V(r) = \dot{r}^2 + \Delta(r) \left(\epsilon + \frac{L^2}{r^2} \right) = E. \tag{9}$$

This equation represents the conservation of the specific total energy E and then we can study the orbits simply plotting the effective potential $V(r)$, parametrized by the specific angular momentum L. Let first focus on the case of a massive particle (i.e. $\epsilon = c^2$) approaching the black hole from far away. As in the Schwarzschild solution, there is a critical angular momentum L_{crit} above which bounded orbits are possible. Figure 1[1] shows (for a fixed $L > L_{crit}$) the potential $V(r)$ expressed as:

$$\widetilde{V}(x) \equiv \frac{V(r/r_S)}{c^2} = \left(1 - \frac{1}{x} + \frac{r_Q^2}{r_S^2 x^2} \right) \left(1 + \frac{L^2}{r_S^2 c^2 x^2} \right). \tag{10}$$

In newtonian terms, $V(r)$ represents the sum of the potential and the rest energy mc^2 of the test particle normalized by the mass m, which explains the $\lim_{x \to +\infty} \widetilde{V}(x) = 1$. When $E/c^2 < 1$,

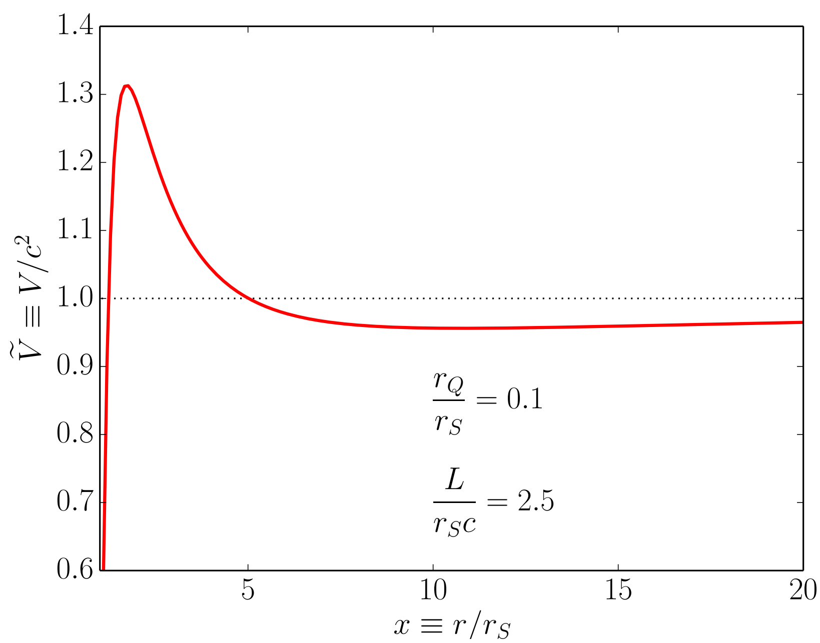

Figure 1: Effective potential $V(r)$ for a massive test particle (i.e. $\epsilon = c^2$).

the test particle is on a bounded orbit. In particular, if $E/c^2 = \widetilde{V}_{\min}$ (i.e. the energy is equal to the minimum of the potential), the particle is on a stable circular orbit. When $1 \le E/c^2 \le \widetilde{V}_{\max}$ the particle can only reach a minimal distance from the central mass, before

[1] Note that $r_{\pm}$ are both below the lower limit of the plot.

being scattered back to infinity. When $E/c^2 = \widetilde{V}_{\rm max}$, the particle is on an *unstable* circular orbit, whose radius is called *innermost stable circular orbit* radius. When $E/c^2 > \widetilde{V}_{\rm max}$, the particle can directly fall onto the central mass.

For a massless particle, $\epsilon = 0$ and the effective potential becomes:

$$\widetilde{V}(x) \equiv \frac{V(r/r_S)}{c^2} = \left(1 - \frac{1}{x} + \frac{r_Q^2}{r_S^2 x^2}\right) \frac{L^2}{r_S^2 c^2 x^2}. \tag{11}$$

The rest mass energy now is clearly 0. Figure 2[2] shows the effective potential for a massless particle. In this case, the test particle can not move on a bounded, stable orbit. When

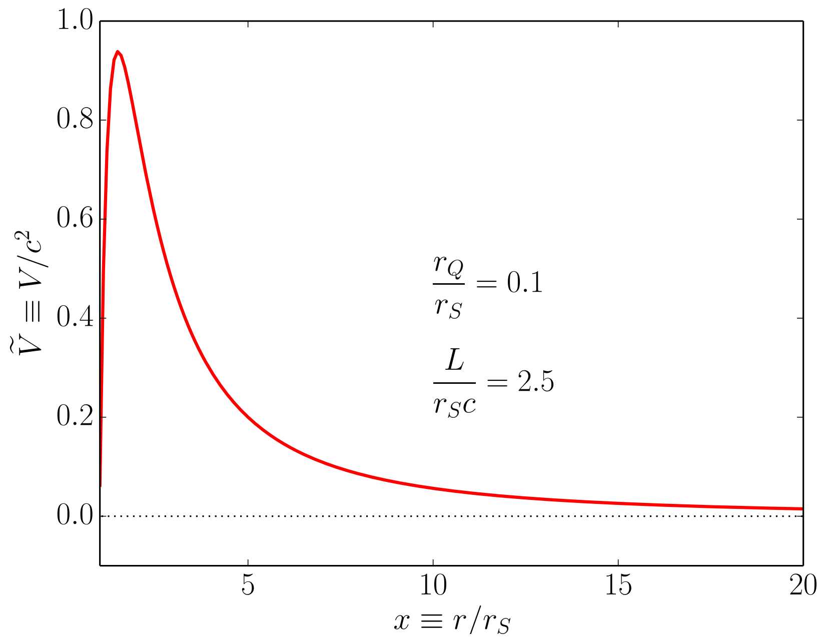

Figure 2: Effective potential $V(r)$ for a massless test particle (i.e. $\epsilon = 0$).

$0 < E/c^2 < \widetilde{V}_{\rm max}$ the test particle can either reach a minimum distance from the central mass or fall onto that. When $E/c^2 = \widetilde{V}_{\rm max}$, the particle in on an *unstable* circular orbit. When $E/c^2 > \widetilde{V}_{\rm max}$, the particle can directly fall onto the central mass or get farer from that.

c) We consider the geodetic precession of a gyroscope with a spin S^μ moving along a geodesic u^μ. S^μ and u_μ satisfy the equations:

$$S^\mu u_\mu = 0, \quad (u^\nu \nabla_\nu) S^\mu = 0, \quad (u^\nu \nabla_\nu) u^\mu = 0. \tag{12}$$

The solution is similar to that discussed during the lecture, in particular we have:

$$\begin{aligned} \omega &= \omega_0 \sqrt{\Delta\left(1 - \frac{r}{2\Delta}\frac{d\Delta}{dr}\right)} = \omega_0 \sqrt{1 - \frac{3r_S}{2r} + \frac{2r_Q^2}{r^2}} = \\ &= \omega_0 \sqrt{1 - \frac{3GM}{rc^2} + \frac{2GQ^2}{r^2c^4}}. \end{aligned} \tag{13}$$

[2] Again, $r_\pm$ are both below the lower limit of the plot.

Exercise 2 [The linearized EFE] (5 points)

a) The linearized Einstein Field Equations read:

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{c}\eta_{\mu\nu}\right). \tag{14}$$

For $T_{\mu\nu} = \mathrm{diag}\{\rho c^2, 0, 0, 0\}$, we have $T = \rho c^2$ and then:

$$\Box h_{\mu\nu} = -\frac{8\pi G}{c^2}\rho\,\delta_{\mu\nu}. \tag{15}$$

Since the source is static, ρ does not depend on t, then $h_{\mu\nu}$ does not depend on t and $\Box$ reduces to $-\triangle$. We finally have:

$$\triangle h_{\mu\nu} = \frac{8\pi G}{c^2}\rho\,\delta_{\mu\nu}, \quad \Rightarrow \quad \triangle\left(\frac{c^2 h_{\mu\nu}}{2}\right) = 4\pi G\rho\,\delta_{\mu\nu}, \tag{16}$$

which is the Poisson equation with:

$$h_{\mu\nu} = \frac{2\phi}{c^2}\delta_{\mu\nu}, \tag{17}$$

where ϕ is the newtonian potential associated to ρ. Finally, we obtain the metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (1 + 2\phi/c^2)c^2dt^2 - (1 - 2\phi/c^2)\left(dx^2 + dy^2 + dz^2\right). \tag{18}$$

b) As we did in exercise 1, we can define the lagrangian:

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \left(1 + \frac{2\phi}{c^2}\right)c^2\dot{t}^2 - \left(1 - \frac{2\phi}{c^2}\right)\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) = c^2. \tag{19}$$

Again, the symmetry allows us to chose $\theta = \pi/2$. We have the constants of motion:

$$\left(1 - \frac{2\phi}{c^2}\right)r^2\dot{\phi} = L, \tag{20}$$

$$\left(1 + \frac{2\phi}{c^2}\right)\dot{t} = \frac{\sqrt{E}}{c}, \tag{21}$$

$$\left(1 + \frac{2\phi}{c^2}\right)c^2\dot{t}^2 - \left(1 - \frac{2\phi}{c^2}\right)\left(\dot{r}^2 + r^2\dot{\phi}^2\right) = c^2. \tag{22}$$

We then get:

$$E - \left(1 + \frac{2\phi}{c^2}\right)\left(1 - \frac{2\phi}{c^2}\right)\dot{r}^2 - \frac{1 + \dfrac{2\phi}{c^2}}{1 - \dfrac{2\phi}{c^2}}\frac{L^2}{r^2} = c^2\left(1 + \frac{2\phi}{c^2}\right). \tag{23}$$

The term $(1 + 2\phi/c^2)(1 - 2\phi/c^2) = 1 - 4\phi^2/c^4 \simeq 1$, because $4\phi^2/c^2$ is a second order term in ϕ/c^2, whereas:

$$\frac{1 + \frac{2\phi}{c^2}}{1 - \dfrac{2\phi}{c^2}} \simeq \left(1 + \frac{2\phi}{c^2}\right)\left(1 + \frac{2\phi}{c^2}\right) \simeq 1 + \frac{4\phi}{c^2}, \tag{24}$$

as before. We finally get:

$$E = \dot{r}^2 + c^2 + 2\phi + \frac{L^2}{r^2} + \frac{4\phi L^2}{r^2c^2} \equiv \dot{r}^2 + V_{\mathrm{eff}}. \tag{25}$$