1)

THE LAPLACE OPERATOR AND THE SCHRÖDINGER EQUATION

- Laplace operator in Cartesian coordinates

Let us consider the wave equation with the ansatz $\psi(\underline{r},t) = e^{i\omega t}\,\psi(\underline{r})$

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \quad \Rightarrow \quad \nabla^2\psi + k^2\psi = 0 \qquad k^2 = \frac{\omega^2}{c^2} = \text{constant}$$

↳ Helmholtz equation

We make the ansatz $\psi(\underline{r}) = X(x)\,Y(y)\,Z(z)$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = X''YZ + XY''Z + XYZ'' + k^2XYZ = 0$$

$$\Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2$$

$$\frac{X''}{X} = \underbrace{-\frac{Y''}{Y} - \frac{Z''}{Z}}_{\text{independent on } X} - k^2 \equiv -\ell^2$$

← the sign here is just for convention!

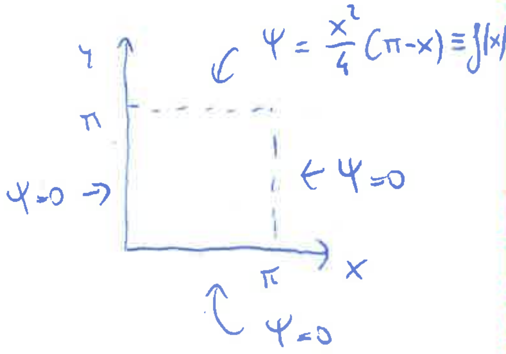
Analogously we can write

$$\frac{Y''}{Y} = -m^2 \qquad \frac{Z''}{Z} = -n^2 \qquad \ell^2 + m^2 + n^2 = k^2$$

$\Rightarrow$ Separation of variables: a partial differential equation becomes equivalent to three ordinary diff. eqs.

EXAMPLE

We search for a solution of $\nabla^2\psi = 0$ in 2 dimensions with Dirichlet boundary conditions



$$\psi(x,y) = X(x)\,Y(y) \quad \Rightarrow \quad X'' = -\ell^2 X$$
$$Y'' = \ell^2 Y$$

given the boundary conditions $\Rightarrow$ choose $\ell^2 > 0$!

2)

$X_\ell(x) = a_{x\ell} \sin \ell x + b_{x\ell} \cos \ell x \qquad Y_\ell(y) = a_{y\ell}\, e^{\ell y} + b_{y\ell}\, e^{-\ell y}$

$\Rightarrow$ the general solution is $\psi(x,y) = \sum_\ell \left(k_{\ell 1} \sin \ell x\, e^{\ell y} + k_{\ell 2} \cos \ell x\, e^{\ell y} + k_{\ell 3} \sin \ell x\, e^{-\ell y} + k_{\ell 4} \cos \ell x\, e^{-\ell y} \right)$

Note that at this step ℓ is arbitrary

$\Rightarrow$ the sum over ℓ should be an integral!

we now impose the boundary conditions:

- $\psi(0,y) = 0 \Rightarrow \sum_\ell \left(k_{\ell 2} e^{\ell y} + k_{\ell 4} e^{-\ell y} \right) = 0 \quad \forall y \Rightarrow k_{\ell 2} = k_{\ell 4} = 0$

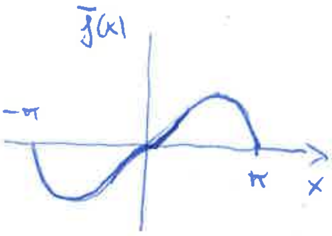
- $\psi(x,0) = 0 \Rightarrow \sum_\ell \left(k_{\ell 1} \sin \ell x + k_{\ell 3} \sin \ell x \right) = 0 \ \forall x \Rightarrow k_{\ell 1} = -k_{\ell 3}$

- $\psi(\pi, y) = 0 \Rightarrow \sum_\ell k_{\ell 1} \left(e^{\ell y} \sin \ell \pi - e^{-\ell y} \sin \ell \pi \right) = 0 \Rightarrow$ It must vanish $\forall y$ $\Rightarrow \ell \in \mathbb{Z}$

$\Rightarrow$ we can limit the summation to integer values! (limit ourselves to we can consider only $\ell > 0$)

- $\psi(x,\pi) = \sum_{\ell \in \mathbb{N}} 2 k_{\ell 1} \sin \ell x \sinh \ell \pi = f(x)$

To exploit this condition, let us extend $f(x) \to \bar{f}(x)$ such that $\bar{f}(x) = \begin{cases} \frac{x^2}{4}(\pi - x) & x \geq 0 \\ -\frac{x^2}{4}(\pi + x) & x < 0 \end{cases}$



$$\bar{f}(x) = \sum_{m=1}^{\infty} b_m \sin mx$$

only $\sin mx$ because the function is odd!

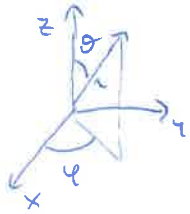
$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \bar{f}(x) \sin mx = -\frac{1 + 2(-1)^m}{m^3}$$

$$\Rightarrow \sum_{\ell=1}^{\infty} 2 k_{\ell 1} \sin \ell x \sinh \ell \pi = \sum_{\ell=1}^{\infty} b_\ell \sin \ell x$$

$$\Rightarrow 2 k_{\ell 1} = \frac{b_\ell}{\sinh \ell \pi}$$

$$\Rightarrow \psi(x,y) = \sum_{\ell=1}^{\infty} -\frac{1 + 2(-1)^\ell}{\ell^3 \sinh \ell \pi} \sin \ell x \sinh \ell y$$

LAPLACIAN IN SPHERICAL COORDINATES



$$x = r\sin\vartheta\cos\varphi \qquad y = r\sin\vartheta\sin\varphi \qquad z = r\cos\vartheta \quad \Rightarrow \quad \operatorname{tg}\varphi = \frac{y}{x}, \quad \sin^2\vartheta = \frac{x^2+y^2}{x^2+y^2+z^2}, \quad r^2 = x^2+y^2+z^2$$

We first look for an expression of the gradient

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} + \frac{\partial \vartheta}{\partial x}\frac{\partial}{\partial \vartheta}$$

$$= \frac{x}{r}\frac{\partial}{\partial r} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)\frac{\partial}{\partial \varphi} + \frac{1}{\sqrt{1-\frac{x^2+y^2}{r^2}}}\,\frac{xz^2}{\sqrt{x^2+y^2}\,r^3}\frac{\partial}{\partial \vartheta}$$

$$= \boxed{\sin\vartheta\cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r\sin\vartheta}\frac{\partial}{\partial \varphi} + \frac{\cos\vartheta\cos\varphi}{r}\frac{\partial}{\partial \vartheta}}$$

analogously

$$\boxed{\frac{\partial}{\partial y} = \sin\vartheta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial \varphi} + \frac{\cos\vartheta\sin\varphi}{r}\frac{\partial}{\partial \vartheta}}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \cancel{\frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi}} + \frac{\partial \vartheta}{\partial z}\frac{\partial}{\partial \vartheta}$$

$$= \cos\vartheta\frac{\partial}{\partial r} - \frac{1}{\sqrt{1-\frac{x^2+y^2}{r^2}}}\,\frac{\sqrt{x^2+y^2}\,z}{r^3}\frac{\partial}{\partial \vartheta} = \boxed{\cos\vartheta\frac{\partial}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial}{\partial \vartheta}}$$

After some algebra we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial f}{\partial \vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial \varphi^2}}$$

we also have $\displaystyle \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$

4)

Let us now study the Laplace equation in spherical coordinates

$$\nabla^2 \psi = 0 \qquad \psi = \psi(r, \theta, \varphi)$$

separation of variables $\quad \psi(r,\theta,\varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi)$

$\to$ the $\frac{1}{r}$ factor is just for convention;

$\hookrightarrow$ but it makes calculations simpler

$$\frac{\partial^2}{\partial r^2}\left(\frac{u}{r}\right) + \frac{2}{r}\frac{\partial}{\partial r}(u/r) = \frac{\partial}{\partial r}\left(\frac{u'}{r} - \frac{u}{r^2}\right) + \frac{2}{r}\left(\frac{u'}{r} - \frac{u}{r^2}\right) = \frac{u''}{r}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\psi(r,\theta,\varphi) = 0$$

implies

$$PQu'' + \frac{uQ}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\, P') + \frac{uPQ''}{r^2\sin^2\theta} = 0$$

$$\underbrace{r^2\sin^2\theta\,\frac{u''}{u} + \frac{\sin\theta}{P}\frac{d}{d\theta}(\sin\theta\, P')}_{\text{independent on } \varphi} + \frac{Q''}{Q} = 0$$

$\Rightarrow$ Q''/Q must be independent on φ $\quad \frac{Q''}{Q} \equiv -m^2$

$\Rightarrow Q(\varphi) = e^{\pm im\varphi}$ $\quad$ in order for this function to have the same value at 0 and 2π we must have $m \in \mathbb{Z}$!

$$r^2\frac{u''}{u} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}(\sin\theta\, P') - \frac{m^2}{\sin^2\theta}}_{\text{independent on } r} = 0$$

$$\Rightarrow \quad r^2\frac{u''}{u} = \text{const} \equiv \ell(\ell+1)$$

We have now recast the PDE in two ordinary diff. equations

$$\begin{cases} \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\sin\theta\, P') + \left(\ell(\ell+1) - \dfrac{m^2}{\sin^2\theta}\right)P = 0 \\[2ex] u'' - \dfrac{\ell(\ell+1)}{r^2}u = 0 \end{cases}$$

5) Let us focus on the first equation and set $x = \cos\vartheta \Rightarrow \sin^2\vartheta = 1 - x^2$

$$\frac{d}{d\vartheta} = \frac{dx}{d\vartheta}\frac{d}{dx} = \sin\vartheta \frac{d}{dx} \quad \Rightarrow \quad \frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta \frac{d}{d\vartheta}\right) = \frac{d}{dx}(1-x^2)\frac{d}{dx}$$

and the equation becomes

$$\boxed{\frac{d}{dx}(1-x^2)\frac{dP}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0} \quad *$$

In the case $m=0$ this is the equation of ORDINARY LEGENDRE POLYNOMIALS

Rodriguez formula

$$(1-x^2)P'' - 2xP' + \ell(\ell+1)P = 0 \qquad P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell (x^2-1)^\ell}{dx^\ell}$$

In the case $m \neq 0$ the solution of the diff. equation are the associated Legendre Polynomials

Let us set $P = (1-x^2)^{m/2} z(x)$

$$P' = (1-x^2)^{m/2} z' - mx(1-x^2)^{\frac{m}{2}-1} z$$

$$P'' = (1-x^2)^{\frac{m}{2}} z'' + \frac{m}{2}(-2x)(1-x^2)^{\frac{m}{2}-1} z' - m(1-x^2)^{\frac{m}{2}-1} z - mx(-2x)\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2} z$$
$$- mx(1-x^2)^{\frac{m}{2}-1} z'$$

$$= (1-x^2)^{\frac{m}{2}} z'' - 2mx(1-x^2)^{\frac{m}{2}-1} z' + m\left(x^2(m-1)-1\right)(1-x^2)^{\frac{m}{2}-2}$$

$$= (1-x^2)^{\frac{m}{2}} z'' - 2mx(1-x^2)^{\frac{m}{2}-1} z' + \left(\frac{d^2}{dx^2}(1-x^2)^{\frac{m}{2}}\right) z$$

Replacing into the original equation and dividing by $(1-x^2)^{m/2}$ we get

$$(1-x^2) z'' - 2x(m+1) z' + \left[\ell(\ell+1) - m(m+1)\right] z = 0 \qquad (1)$$

$m=0 \Rightarrow z(x) = P_\ell(x)$ is a solution

Taking the derivative we get

$$(1-x^2) z''' - 2x z'' - 2(m+1) z' - 2x(m+1) z'' + \left[\ell(\ell+1) - m(m+1)\right] z' = 0$$

setting $V = z'$

$$(1-x^2) V'' - 2(m+2) x V' + \left[\ell(\ell+1) - (m+1)(m+2)\right] V = 0 \qquad (2)$$

We get (2) from (1) with $m \to m+1$ and $z \to V$

* Note that up to this point $\lambda = \ell(\ell+1)$ is an arbitrary constant

However for arbitrary values of λ the solutions of the differential equation are DIVERGENT at $x \to \pm 1$

It is only for $\lambda = \ell(\ell+1)$, with ℓ integer that ONE OF THE TWO SOLUTIONS is finite at $x = \pm 1$

6)

$\Rightarrow$ $P_\ell(x)$ is a solution of (1) with $m=0$

$P_\ell'(x)$ is a solution of (1) with $m=1$

$\vdots$

$P_\ell^{(N)}(x)$ is a solution of (1) with $m=N$

$\Rightarrow$ the general solution of the original equation is $\boxed{P_\ell^{(m)}(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x)}$

Associated Legendre Polynomials

↳ vanishes for $m > \ell$!

In summary: combining the solutions of the equations for θ and φ we define:

$$Y_\ell^{(m)}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^{(m)}(\cos\theta)\, e^{im\varphi}\,(-1)^m \qquad m \geq 0$$

SPHERICAL HARMONICS

$$Y_\ell^{(-m)}(\theta,\varphi) = (-1)^m \left(Y_\ell^{m}\right)^*$$

These functions provide an orthonormal basis for the sphere with unit radius:

$$\int d\Omega_2 \left(Y_\ell^{(m)}\right)^* Y_{\ell'}^{(m')} = \delta_{\ell\ell'}\delta_{mm'}$$

basis for $L^2[S^2]$

$$Y_0^{(0)}(\theta,\varphi) = \sqrt{\frac{1}{4\pi}} \qquad Y_1^{(0)}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{(\pm1)}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

We now move to consider the radial equation

$$u'' - \frac{\ell(\ell+1)}{r^2}u = 0$$

this is an Euler equation $\Rightarrow$ ansatz $u = r^\alpha$

$$\alpha^2 - \alpha - \ell(\ell+1) = 0 \Rightarrow \begin{cases} \alpha = (\ell+1) \\ \alpha = -\ell \end{cases}$$

$\Rightarrow$ The general solution is $u(r) = k_1 r^{\ell+1} + k_2 r^{-\ell}$

↑ only this solution has a regular behaviour as $r \to 0$

6 BIS

In studying quantum mechanics you will learn that

- $r^{\ell+1}$ is relevant for bound states (regular as $r \to 0$)

- $r^{-\ell}$ is relevant for scattering problems (regular as $r \to \infty$)

If we limit ourselves to solutions that are regular as $r \to 0$

we can write the general solution of $\nabla^2 \psi = 0$ as

$$\boxed{\psi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell,m}\, r^{\ell}\, Y_{\ell}^{m}(\theta,\varphi)}$$

7) EXAMPLE: The electric potential in an empty sphere

$$\begin{cases} \nabla^2 \varphi = 0 & \text{for } r < R \\ \varphi(r,\vartheta,\varphi) = f(\vartheta,\varphi) & r = R \end{cases}$$

we look for solutions of the form $\varphi(r,\vartheta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell,m}\, r^\ell\, Y_\ell^m(\vartheta,\varphi)$

$$\varphi(R,\vartheta,\varphi) = f(\vartheta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell,m}\, R^\ell\, Y_\ell^m(\vartheta,\varphi)$$

$$\int_0^\pi \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi\, Y_\ell^{*m}(\vartheta,\varphi)\, f(\vartheta,\varphi) = c_{\ell,m}\, R^\ell$$

LAPLACE OPERATOR IN CYLINDRICAL COORDINATES

$\nabla^2\varphi$ in cylindrical coordinates becomes $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right)\varphi(r,\varphi,z) = 0$

$$\varphi = R(r)\, Q(\varphi)\, Z(z) \Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{Q''}{Q} + \underbrace{\frac{Z''}{Z}}_{k^2} = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{Q''}{Q} + r^2k^2 = 0 \qquad \Rightarrow Z(z) = \alpha_k e^{kz} + \beta_k e^{-kz}$$

$\hookrightarrow$ constant $= -\nu^2$ $\Rightarrow Q(\varphi) = \gamma_\nu \sin\nu\varphi + \delta_\nu \cos\nu\varphi$

ν integer to have correct behavior with φ

$$\Rightarrow r^2R'' + rR' + (r^2k^2 - \nu^2)R = 0 \qquad \text{Bessel equation!}$$

$R = J_{\pm\nu}(kr)$ $\Rightarrow$ but ν is integer, so we can write the general solution as

$$R(r) = j_\nu J_\nu(kr) + m_\nu N_\nu(kr)$$

general solution

$$\varphi(r,\varphi,z) = \left(j_\nu J_\nu(kr) + m_\nu N_\nu(kr)\right)\left(\gamma_\nu \sin\nu\varphi + \delta_\nu\cos\nu\varphi\right)\left(\alpha_k e^{kz} + \beta_k e^{-kz}\right)$$

$\rightarrow$ constants to be fixed by using boundary conditions

SCHRÖDINGER EQUATION

$$H = \frac{p^2}{2m} + V(x)$$ Classical Hamiltonian of a particle with mass m (energy)

$\left(\underline{p} = m\underline{v} \Rightarrow \frac{p^2}{2m} = \frac{1}{2} m \underline{v}^2 \right.$ kinetic term $\Big)$

In quantum mechanics physical observables become operators on the HS of physical states

$\Rightarrow \underline{p} \to \hat{\underline{p}} = -i\hbar \underline{\nabla}$ with this substitution the Hamiltonian becomes an operator

$$H \to \hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(\underline{x})$$

Laplace operator!

$\hat{H}\psi = E\psi$ classical problem in quantum mechanics: find the eigenvalues of the Hamiltonian and the corresponding eigenfunctions ψ

$\Rightarrow$ stationary states

Suppose that the potential is spherically symmetric $\Rightarrow V(\underline{x}) \to V(r)$

We can use what we have learned on the Laplace operator in spherical coordinates

The equation becomes

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]\psi + V(r)\psi = E\psi$$

Before proceeding further let us introduce the angular momentum operator:

$$\underline{L} = \underline{r} \wedge \underline{p} \to \underline{r} \wedge \hat{\underline{p}} = -i\hbar\, \underline{r} \wedge \underline{\nabla}$$

$L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$

If we compute $L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ we get

$$L_z = -i\hbar\frac{\partial}{\partial\varphi} \quad !$$

This means that if we go back to the solution of the Laplace equation in spherical coordinates

we get $L_z Q(\varphi) = -i\hbar\frac{\partial}{\partial\varphi} e^{im\varphi} = -i\hbar\, im\, e^{im\varphi} = \hbar m\, Q(\varphi)$

$\Rightarrow \hbar m$ are eigenvalues of L_z !

This means that the spherical harmonics Y_ℓ^m correspond to angular momentum $m\hbar$

9) Let us now consider $L^2 = L_x^2 + L_y^2 + L_z^2$

By using the expression of the gradient in spherical coordinates one can show that

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right]$$

but this is exactly the expression we find in the Laplacian!

If $P_\ell^m(\cos\theta)$ and $Q(\varphi) = e^{im\varphi}$ are the solutions of the angular part of the Laplace equation we have

$$\left[\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta} \right) \right] P = 0 \qquad \text{and} \qquad m^2 = -\frac{Q''}{Q}$$

$$\Rightarrow \quad L^2 PQ = \hbar^2 \ell(\ell+1) PQ$$

the coefficients $\ell(\ell+1)$ represent the eigenvalues of the L^2 operator!

So we have $L^2 \psi(r,\theta,\varphi) = \hbar^2 \ell(\ell+1) \psi(r,\theta,\varphi)$ and $L_z \psi(r,\theta,\varphi) = \hbar m \psi$

with $-\ell \leq m \leq \ell$

The Schrödinger equation becomes

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \frac{L^2}{\hbar^2} \right] \psi + V(r)\psi = E\psi$$

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \ell(\ell+1) + \underbrace{(V(r) - E)}_{\cdot\left(-\frac{2m}{\hbar^2}\right)} \right] \psi = 0$$

$$\psi = \frac{u(r)}{r} Y_\ell^{(m)}(\theta,\varphi)$$

$$\Rightarrow \quad u'' - \frac{1}{r^2} \ell(\ell+1) u + \frac{2m}{\hbar^2} (E - V(r)) u = 0$$

similar to what we did for $\nabla^2\psi = 0$ but now the radial equation is different

Hydrogen atom $V(r) = -Ze^2/r$

$\Rightarrow$ $E < 0$ bound states and the solutions of the radial equation can be expressed by using Laguerre Polynomials