MMP I
Exercise Sheet 5

HS 21
Prof. Ph. Jetzer

L. Buonocore, M. Loechner, X. Liu, M. Ebersold
`https://www.physik.uzh.ch/en/teaching/PHY312`

Issued: 21.10.2021
Due: 28.10.2021

Exercise 1 [Heat Conduction in a Ball (6 points)]

Let $K = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ be a unit ball. We are looking for the solution $u(t, \mathbf{x})$ of the heat conduction equation for $t > 0$ and x $\in K$. The differential equation is given by

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = \Delta u(t, \mathbf{x}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u(t, \mathbf{x}).$$

with the following boundary and initial conditions:

1. $u(t, \mathbf{x}) = 0$ if $\mathbf{x} \in \partial K$ (surface of the ball)
2. The distribution of the temperature at $t = 0$ is given by the function $T(r) \in C^2(\mathbb{R}_+)$ with $r = |\mathbf{x}|$ and $T(1) = 0$.

Why is $u(t, \mathbf{x}) = u(t, r)$ a clever ansatz?

a) Show that $\Delta u(t, r) = \frac{\partial^2}{\partial r^2} u(t, r) + \frac{2}{r} \frac{\partial}{\partial r} u(t, r)$.
Hint: Compute the general expression of the Laplacian in spherical coordinates and simplify it under the assumption that the function does not depend on the angles.

b) Let's use the ansatz $u(t, r) = f(t) g(r)$ with $g(r) = \frac{h(r)}{r}$. Solve the differential equations separately for g and f.
Hint: Since $u(t, r)$ is a physical distribution of heat, it must be regular at $r = 0 \quad \forall\, t$.

c) Now take into account the boundary condition $u(t, 1) = 0$.

d) Write down the general solution by superposing the already constructed solutions f and g. Why are you allowed to do that?

e) Express the solution, in terms of the initial temperature distribution $T(r)$.
Hint: Fourier series

– please turn over –

Exercise 2 [Growth of Bacteria (5 points)]

Let the rate of growth dN/dt of a colony of bacteria be proportional to

a) the square root of the number present at any time

b) the number present at any time

c) the number present but the population is being reduced at a constant rate by the removal of bacteria for experimental purposes

d) the number present but due to competition for the limited resources the growth rate also decreases with the size of the population
Hint: $\frac{dN}{dt} = \alpha \left(1 - \frac{N}{\beta}\right) N$, where α is the growth rate and β is the equilibrium value

For each one of these different cases, write and solve the differential equation for the number N as a function of time when there are N_0 bacteria at $t = 0$. What are the solutions for $N_0 = 0$? Do they make sense?