5.5 The inertia ellipsoid

The inertia tensor allows us to express two independent quantities $\vec{L}$ and T, which are both conserved in absence of external forces. The dynamics of a free rotating rigid body is therefore characterized by two quadratic conserved quantities:

$$2T = I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 = \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3} \qquad (5.21)$$

$$|\vec{L}|^2 = (I_1 \omega_1)^2 + (I_2 \omega_2)^2 + (I_3 \omega_3)^2 = L_1^2 + L_2^2 + L_3^2 \qquad (5.22)$$

The trajectories of the $\omega_i(t)$ are curves lying on the interception of the geometric figures defined by (5.21) and (5.22). By construction, these are defined in the body frame, choosing the inertia axes as reference axes (by convention: $I_1 > I_2 > I_3$).

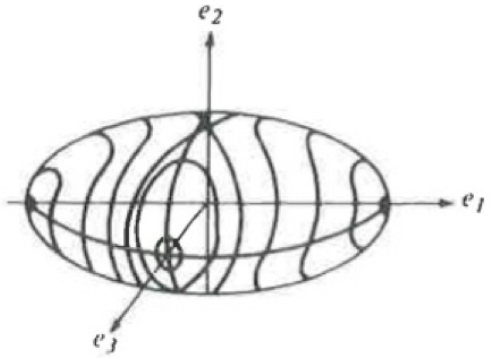
* The 3D figure described by (5.21) is denoted inertia ellipsoid
* Using the L_i as variables (rather than the ω_i), the function (5.22) defines a sphere.

Using these two facts it follows that

I. If $|\vec{L}|^2 < 2T I_3$ or $|\vec{L}|^2 > 2T I_1$ there is no interception (no physical motion possible for such values of $\vec{L}$ and T).

II. For $2T I_3 < |\vec{L}|^2 < 2T I_2$, the intercepion are two closed curves centered on the extrema of the inertia ellissoid along the minor ($\vec{e}_3$) axis.

III. For $2T I_2 < |\vec{L}|^2 < 2T I_1$, the intercepion are two closed curves centered on the extrema of the inertia ellissoid along the major ($\vec{e}_1$) axis.



IV. For $(\vec{L})^2 = 2T I_2$ the interception are two curves crossing at the extremal points of $\vec{e}_2$.

All these curves are allowed trajectories of $\vec{\omega}(t)$, and are in one-two-one correspondance with the stable (II & III) and unstable (IV) configurations discussed before.

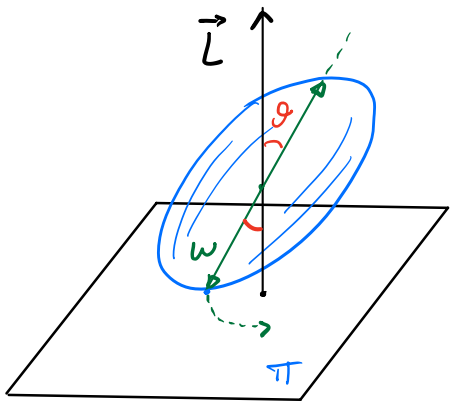
So far we discussed the motion in the body frame, let's now try to understand what happens in an inertial frame. In that frame the inertia ellipsoid moves. By construction: the motion of the ellipsoid coincides with the motion of the body.

Note that $\vec{L} = \vec{\nabla}_\omega T$

where $\nabla_\omega = \left(\frac{\partial}{\partial \omega_1}, \frac{\partial}{\partial \omega_3}, \frac{\partial}{\partial \omega_3}\right)$ $\Rightarrow$ $\vec{L}$ is always perpendicular to the ellipsoid

$\Downarrow$

This observation allows us to define the invariable plane, defined as the plane Π, tangent to the ellipsoid, perpendicular to $\vec{L}$



The point where the ellipsoid is tangent to Π is what defines $\vec{\omega}$.

* The conservation of $\vec{L}$ implies that the invariant plane is a fixed plane in the inertial frame
* Since $T \propto \vec{L} \cdot \vec{\omega}$, the angle ϑ between $\vec{\omega}$ and $\vec{L}$ is constant

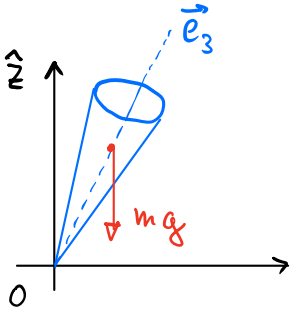
$\Downarrow$

The ellipsoid "rolls without slipping" on the invariant plane (i.e. $\vec{\omega}$ precesses around $\vec{L}$, as we have seen in the case of the symmetric top).

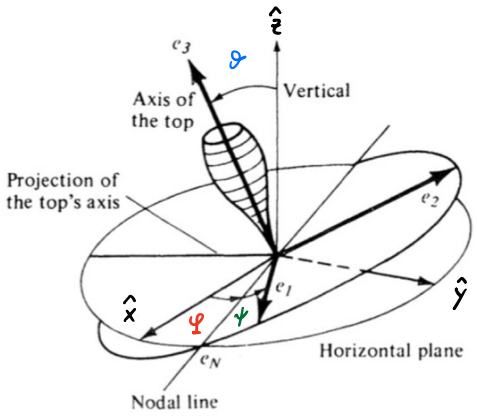
5.6 The heavy top (Lagrange's top)

Having understood the motion of rigid bodies in absence of external forces, we are ready to face the more complicated (and often unsolvable problem) of rigid-body motion in presence of forces and constraints.

A particularly interesting case is that of a symmetric top, fixed on a point O, subject to the gravitational force $\Rightarrow$



The inertia ellipsoid of this body is centered in O, it has two equal axes ($I_1 = I_2$), and the c.o.m. lies on the third axis ($\vec{e}_3$)



We are interested in describing the motion in an inertial frame $\{\hat{x}, \hat{y}, \hat{z}\} \equiv \{\vec{n}_i\}$.

As usual, the axes of the body frame are denoted by $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

The relation between the 2 frames is described by Euler's angles $\{\vartheta, \varphi, \psi\}$ (see figure).

The SO(3) rotation matrix R, such that $\vec{e}_a = R_{ab}\,\vec{n}_b$, is obtained via

1. Rotation of φ along $\hat{z}$ $\quad (\hat{x} \to \vec{e}_n)$ $\qquad \varphi \in [0, 2\pi]$
2. " " ϑ " $\vec{e}_n$ $\quad (\hat{z} \to \vec{e}_3)$ $\qquad \vartheta \in [0, \pi]$
3. " " ψ along $\vec{e}_3$ $\quad (\vec{e}_n \to \vec{e}_1)$ $\qquad \psi \in [0, 2\pi]$

$\Downarrow$

$$R = R_{\vec{e}_3}(\psi)\, R_{\vec{e}_n}(\vartheta)\, R_{\hat{z}}(\varphi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Any SO(3) matrix can unambiguously decomposed in this way.

In principle we can use the relation

$$\omega_a = \frac{1}{2}\,\epsilon_{abc}\,\left[\dot{R}(\vartheta,\psi,\varphi)\,R^{T}(\vartheta,\psi,\varphi)\right]_{bc}$$

to express ω_a in terms of Euler's angles. However, there is a faster way: by construction

$$\vec{\omega} = \dot{\varphi}\,\hat{z} + \dot{\vartheta}\,\vec{e}_n + \dot{\psi}\,\vec{e}_3 \qquad (5.23)$$

Since

$$\begin{cases} \vec{e}_n = \cos\psi\,\vec{e}_1 - \sin\psi\,\vec{e}_2 \\ \vec{z} = \cos\vartheta\,\vec{e}_3 + \sin\vartheta\,(\sin\psi\,\vec{e}_1 + \cos\psi\,\vec{e}_2) \end{cases}$$

we deduce that

$$\begin{cases} \omega_1 = \dot{\varphi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi \\ \omega_2 = \dot{\varphi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi \\ \omega_3 = \dot{\psi} + \dot{\varphi}\cos\vartheta \end{cases} \qquad (5.24)$$

These relations allow us to compute the kinetic energy of the body and finally the Lagrangian, in terms of Euler's angles:

$$T = \frac{1}{2}\sum_i I_i\,\omega_i^2 = \frac{1}{2}I_1\left[\dot{\varphi}^2\sin^2\vartheta + \dot{\vartheta}^2\right] + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\varphi}\cos\vartheta\right)^2 \qquad (5.25)$$

$$U = mg\ell\cos\vartheta \qquad L = T - U$$

where we have used the relation $I_1 = I_2$. The latter implies that φ and ψ are cyclic variables $\Rightarrow$ 2 conserved quantities, which are nothing but the projection of the angular momentum on $\vec{e}_3$ and $\vec{z}$:

$$\begin{cases} p_\psi = \dfrac{\partial L}{\partial\dot{\psi}} = I_3\left(\dot{\psi} + \dot{\varphi}\cos\vartheta\right) = I_3\,\omega_3 = L_3 \\ p_\varphi = \dfrac{\partial L}{\partial\dot{\varphi}} = I_1\left(\dot{\varphi}\sin^2\vartheta\right) + I_3\left(\dot{\psi} + \dot{\varphi}\cos\vartheta\right)\cos\vartheta = L_z \end{cases} \qquad (5.26)$$

The situation is apparently similar to that of the free top (where both L_3 and L_z are conserved). However, in this case this happens with a non-trivial $\vartheta(t)$.

In order to solve the e.o.m. let's first note that, defining

$$a = \frac{L_3}{I_1} \qquad \text{and} \qquad b = \frac{L_z}{I_1}$$

then

$$\dot{\varphi} = \frac{b - a\cos\vartheta}{\sin^2\vartheta} \qquad \text{and} \qquad \dot{\psi} = \frac{I_1 a}{I_3} - \frac{(b - a\cos\vartheta)\cos\vartheta}{\sin^2\vartheta}$$

Namely we found two eqs. of the form $\dot{\varphi} = f_\varphi(\vartheta)$ and $\dot{\psi} = f_\psi(\vartheta)$ that, once $\vartheta(t)$ is known, can be integrated to determine $\varphi(t)$ & $\psi(t)$.

The e.o.m. for $\vartheta(t)$ is deduced using the third conserved quantity of the motion, namely the energy, that we can re-write as

$$\begin{cases} E = T + U = E' + \frac{1}{2} I_3 \omega_3^2 \leftarrow \text{irrelevant being constant} \\ E' = \frac{1}{2} I_1 \dot{\vartheta}^2 + V_{eff}(\vartheta) \end{cases} \tag{5.27}$$

where

$$V_{eff}(\vartheta) = \frac{I_1 (b - a\cos\vartheta)^2}{2\sin^2\vartheta} + Mgl\cos\vartheta \tag{5.28}$$

A final useful change of variable is obtained defining

$$u = \cos(\vartheta) \quad \Rightarrow \quad -1 \le u \le 1$$

$$\alpha = \frac{2E'}{I_1} \qquad \text{and} \qquad \beta = \frac{2Mgl}{I_1}$$

Such that the three first order differential eq.s assume the form:

$$\begin{cases} \dot{u}^2 = (1 - u^2)(\alpha - \beta u) - (b - au)^2 \equiv f(u) \end{cases} \tag{5.29}$$

$$\begin{cases} \dot{\varphi} = \frac{b - au}{1 - u^2} \qquad \dot{\psi} = \frac{I_1 a}{I_3} - \frac{u(b - au)}{1 - u^2} \end{cases} \tag{5.30}$$

In principle, one could try to integrate explicitly (5.29). This leads to the (difficult) integral $\int [f(u)]^{-1/2} du$, known as elliptic integral. However, there is a simpler way to understand the motion analysing the properties of $f(u)$.