MMP I
Solution Sheet 6

HS 21
Prof. Ph. Jetzer

L. Buonocore, M. Loechner, X. Liu, M. Ebersold
https://www.physik.uzh.ch/en/teaching/PHY312

Issued: 28.10.2021
Due: 04.11.2021

Exercise 1 [Ordinary Differential Equations (7 points [1, 1.5, 1.5, 1, 2])]

Find a general solution $y(x)$ to the following differential equations:

a)

$$y' = \frac{1}{xy}, \quad y(1) = -1$$

Separation:

$$\begin{aligned}
&ydy = \frac{1}{x}dx \Rightarrow \int_{y_0}^{y} \tilde{y} d\tilde{y} = \int_{x_0}^{x} \frac{1}{\tilde{x}} d\tilde{x} \\
&\Rightarrow \frac{1}{2}y^2 - \frac{1}{2}y_0^2 = \ln(x) - \ln(x_0) = \ln(\frac{x}{x_0}) \\
&\Rightarrow y^2 = 2\ln(\frac{x}{x_0}) + y_0^2 \\
&\Rightarrow y(x) = \pm\sqrt{2\ln(\frac{x}{x_0}) + y_0^2} \quad (\pm\text{: two branches!}) \\
&x_0 = 1, y_0 = -1 \Rightarrow \underline{\underline{y(x) = -\sqrt{2\ln(x) + 1}}} \\
&(y_0 = -1\text{: only negative branch allowed})
\end{aligned}$$

b)

$$y' = \frac{y}{x} + \frac{x}{y}$$

Substitution:

$$v := \frac{y}{x} \Rightarrow y' = f(v) = v + \frac{1}{v}$$

$$v' = \frac{y'x - x'y}{x^2} = \frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x}(y' - v) = \frac{f(v) - v}{x} = \frac{dv}{dx}$$

Separation:

$$
\begin{aligned}
\frac{dv}{f(v)-v} = \frac{dx}{x} &\Rightarrow \int_{v_0}^{v} \frac{d\tilde{v}}{f(\tilde{v})-\tilde{v}} = \int_{x_0}^{x} \frac{d\tilde{x}}{\tilde{x}} \\
&\Rightarrow \int_{v_0}^{v} \frac{d\tilde{v}}{f(\tilde{v})-\tilde{v}} = \int_{v_0}^{v} \tilde{v} d\tilde{v} = \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \ln(\frac{x}{x_0}) \\
&\Rightarrow v(x) = \pm\sqrt{2\ln(\frac{x}{x_0} + v_0^2)} \\
&\Rightarrow \underline{\underline{y(x) = \pm x\sqrt{2\ln(\frac{x}{x_0}) + \frac{y_0^2}{x_0^2}}}}
\end{aligned}
$$

c)

$$
\begin{aligned}
\frac{dy}{dx} = \frac{3x-y+4}{x+y} &\Rightarrow (3x-y+4)dx = (x+y)dy \\
&\Rightarrow (\underbrace{3x-y+4}_{P(x)})dx + (\underbrace{-x-y}_{Q(x)})dy = 0
\end{aligned}
$$

Is this partial differential equation exact?

$\rightarrow P_y = -1 = Q_x$ ✓ Yes

$\rightarrow$ Find partial $f(x,y)$ with $(P,Q) = \nabla f = (f_x, f_y)$

$$
\begin{aligned}
\Rightarrow f(x,y) &= \int P(x,y)dx + \phi(y) + C \\
&= \int (3x-y+4)dx + \phi(x) + C \\
&= \underline{\frac{3}{2}x^2 - yx + 4x + \phi(y) + C} \\
f(x,y) &= \int Q(x,y)dy + \psi(x) + C \\
&= \int (x+y)dx + \psi(x) + C \\
&= \underline{-xy - \frac{1}{2}y^2 + \psi(x) + C} \\
\Rightarrow \phi(y) &= -\frac{1}{2}y^2, \quad \psi(x) = \frac{3}{2}x^2 + 4x \\
\Rightarrow f(x,y) &= \frac{3}{2}x^2 - \frac{1}{2}y^2 + 4x - yx + C = \text{const (integral curve)} \\
&\text{(put the const into } C \rightarrow C'\text{)} \\
&\Leftrightarrow \frac{3}{2}x^2 - \frac{1}{2}y^2 + 4x - yx + C' = 0, \\
&y^2 + 2xy - 3x^2 - 8x - 2C' = 0 \\
\Rightarrow y(x) &= \frac{-2x \pm \sqrt{4x^2 + 12x^2 + 32x + 8C'}}{2}
\end{aligned}
$$

$$\underline{\underline{y(x) = -x \pm 2\sqrt{x^2 + 2x + \tilde{C}}}}$$

with two branches due to $\pm$

d)

$$\frac{dy}{dx} = y^2 \cos(x) \Rightarrow \int_{y_0}^{y} \frac{1}{(y')^2} dy' = \int_{x_0}^{x} cos(x') dx'$$

$$-\frac{1}{y} + \frac{1}{y_0} = \sin(x) - \sin(x_0) \Rightarrow \frac{1}{y} = \frac{1}{y_0} - \sin(x) + \sin(x_0)$$

$$\Rightarrow \underline{\underline{y(x) = -\frac{1}{\sin(x) + C}, \quad C = -\sin(x_0) - \frac{1}{y_0}}}$$

e)

$$\begin{aligned} y'(x) &= \cos(x + y) + \sin(x - y) \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + \sin(x)\cos(y) - \sin(y)\cos(x) \\ &= \cos(x)\left(\cos(y) - \sin(y)\right) + \sin(x)\left(\cos(y) - \sin(y)\right) \\ &= \underline{\left(\cos(x) + \sin(x)\right)\left(\cos(y) - \sin(y)\right)} \end{aligned}$$

$\Rightarrow$ Separation:

$$\frac{dy}{\cos(y) - \sin(y)} = (\cos(x) + \sin(x)) dx$$

$$\Rightarrow \underbrace{\int \frac{dy}{\cos(y) - \sin(y)}}_{(2)} = \underbrace{\int (\cos(x) + \sin(x)) dx}_{(1)} + C$$

with C to be determined later.

Integral (1):

$$\int (\cos(x) + \sin(x)) dx = \sin(x) - \cos(x)$$

Integral (2): Using the parametrization:

$$\sin(y) = \frac{2t}{1 + t^2}, \cos(y) = \frac{1 - t^2}{1 + t^2}, \tan(y) = \frac{2t}{1 - t^2}$$

$$\begin{aligned} \tan(\frac{y}{2}) = t \Rightarrow dt &= \frac{d}{dy} \frac{\sin(\frac{y}{2})}{\cos(\frac{y}{2})} dy = \frac{1 + \tan^2(\frac{y}{2})}{2} dy = \frac{1 + t^2}{2} dy \\ \Rightarrow dy &= \frac{2dt}{1 + t^2} \end{aligned}$$

we find:

$$\int \frac{dy}{\cos(y)-\sin(y)} = \int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}} = 2\int \frac{dt}{1-t^2-2t} = -2\int \frac{dt}{(1+t)^2-2}$$
$$= 2\int \frac{da}{2-a^2} = \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1-t}{\sqrt{2}-1-t}\right) = \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1+\tan(\frac{y}{2})}{\sqrt{2}-1-\tan(\frac{y}{2})}\right)$$
$$\text{with } \int \frac{da}{b^2-a^2} = \frac{1}{2b}\ln\left(\frac{b+a}{b-a}\right)$$

Set (1) = (2):

$$\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1+\tan(\frac{y}{2})}{\sqrt{2}-1-\tan(\frac{y}{2})}\right) = \sin(x)-\cos(x)+C \quad //\exp$$
$$\to \frac{\sqrt{2}+1+\tan(\frac{y}{2})}{\sqrt{2}-1-\tan(\frac{y}{2})} = e^{\sqrt{2}(\sin(x)-\cos(x))}\underbrace{e^{\sqrt{2}C}}_{:=\widetilde{C}}$$
$$\Rightarrow \sqrt{2}+1+\tan(\frac{y}{2}) = (\sqrt{2}-1)e^{\sqrt{2}(\sin(x)-\cos(x))}\widetilde{C} - \tan(\frac{y}{2})e^{\sqrt{2}(\sin(x)-\cos(x))}\widetilde{C}$$
$$\Rightarrow \tan(\frac{y}{2}) = \frac{-1-\sqrt{2}+(\sqrt{2}-1)e^{\sqrt{2}(\sin(x)-\cos(x))}\widetilde{C}}{1+e^{\sqrt{2}(\sin(x)-\cos(x))}\widetilde{C}}$$
$$\Rightarrow \underline{\underline{y(x) = 2\arctan\left(\frac{-1-\sqrt{2}+(\sqrt{2}-1)e^{\sqrt{2}(\sin(x)-\cos(x))}\widetilde{C}}{1+e^{\sqrt{2}(\sin(x)-\cos(x))}\widetilde{C}}\right)}}$$

with $\widetilde{C}$ determined, by demanding, that the solution passes through the origin:

$$y(0) \overset{!}{=} 0 \;\Rightarrow\; \tan(\frac{0}{2}) = \frac{-1-\sqrt{2}+(\sqrt{2}-1)e^{-\sqrt{2}}\widetilde{C}}{1+e^{-\sqrt{2}}\widetilde{C}}$$
$$\Leftrightarrow -1-\sqrt{2}+(\sqrt{2}-1)e^{-\sqrt{2}}\widetilde{C} = 0$$
$$\Rightarrow \underline{\widetilde{C} = \frac{1+\sqrt{2}}{(\sqrt{2}-1)e^{-\sqrt{2}}} = \frac{1+\sqrt{2}}{-1+\sqrt{2}}e^{-\sqrt{2}} = 23.9737...}$$

Exercise 2 [Making yogurt (2 points)]

Newton's law of cooling: $\frac{dT}{dt} = k(T - T_{\text{ext}})$ with

$$t_0 = 0 \qquad T_0 = 85^\circ\text{ C}$$
$$t_1 = 5\,\text{min} \qquad T_1 = 65^\circ\text{ C}$$
$$t_2 =? \qquad T_2 = 45^\circ\text{ C}$$
$$T_{\text{ext}} = 21^\circ\text{ C}$$

$$\begin{aligned}
\frac{dT}{T - T_{\text{ext}}} &= kdt \\
\int_{T_0}^{T} \frac{d\tilde{T}}{\tilde{T} - T_{\text{ext}}} &= \int_{t_0}^{t} kd\tilde{t} \\
\ln\left(\frac{T - T_{\text{ext}}}{T_0 - T_{\text{ext}}}\right) &= k(t - t_0) = kt \\
\Rightarrow \quad \underline{T - T_{\text{ext}}} &\underline{= (T_0 - T_{\text{ext}})e^{k(t-t_0)}}
\end{aligned}$$

With $t = t_1$, $T = T_1$ and the values above we get:

$$65 - 21 = (85 - 21)e^{5k}$$

$$\underline{k = \ln\left(\frac{44}{64}\right)\frac{1}{5} \approx -0.075}$$

Using $T_2 = 45^\circ$ C and solving for t_2 we get:

$$45 - 21 = (85 - 21)e^{-0.075t_2}$$

$$\underline{t_2 = \ln\left(\frac{24}{64}\right)\frac{1}{-0.075} \approx 13\,\text{minutes}}$$

The milk will reach $T_2 = 45^\circ$ C after about 13 minutes.

Exercise 3 [Parachutist (7 points [2, 1.5, 2.5, 1])]

a)

$$a(t) = \frac{dv}{dt} = g - \frac{k_1}{m}v^2 = g\left(1 - \frac{k_1}{mg}v^2\right), \qquad \frac{dv}{dt} \geq 0 \quad \forall t$$

Use the substitution: $x = \sqrt{\frac{k_1}{mg}}v, \quad dx = \sqrt{\frac{k_1}{mg}}dv$

$$\begin{aligned}
&\Rightarrow \frac{dx}{dt} = \sqrt{\frac{k_1 g}{m}}(1 - x^2) = \frac{dx}{dt} = \lambda(1 - x^2), \qquad \text{with} \quad \lambda = \sqrt{\frac{k_1 g}{m}} \\
&\Rightarrow \frac{1}{1 - x^2}dx = \lambda dt
\end{aligned}$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{1-\tilde{x}^2} d\tilde{x} = \frac{1}{2}\left(\int_{x_0}^{x} \frac{1}{1-\tilde{x}} d\tilde{x} + \int_{x_0}^{x} \frac{1}{1+\tilde{x}} d\tilde{x}\right)$$

$$\text{with } \frac{1}{1-\tilde{x}} + \frac{1}{1+\tilde{x}} = \frac{2}{1-\tilde{x}^2}$$

$$= \frac{1}{2}\left(-\ln(1-x) + \ln(1-x_0) + \ln(1+x) - \ln(1+x_0)\right)$$

$$= \frac{1}{2}\ln\left(\left(\frac{1+x}{1-x}\right)\left(\frac{1-x_0}{1+x_0}\right)\right) = \lambda t$$

$$\Rightarrow \left(\frac{1+x}{1-x}\right)\underbrace{\left(\frac{1-x_0}{1+x_0}\right)}_{:=C} = e^{2\lambda t}$$

$$\Rightarrow C + Cx = e^{2\lambda t} - xe^{2\lambda t}$$

$$\Rightarrow x = \frac{e^{2\lambda t} - C}{e^{2\lambda t} + C} = \frac{e^{2\lambda t} - 1}{e^{2\lambda t} + 1}$$

$$\text{with } C = \frac{1-x_0}{1+x_0} = 1, \quad \text{since} \quad x_0 = x(v(t=0)=0) = 0$$

$$\Rightarrow v(t) = \frac{g}{\lambda}\frac{e^{2\lambda t} - 1}{e^{2\lambda t} + 1} = \frac{g}{\lambda}\tanh(\lambda t)$$

$$\text{with } \tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow \underline{\underline{v(t) = \sqrt{\frac{mg}{k_1}}\tanh\left(\sqrt{\frac{k_1 g}{m}}t\right)}}$$

Taking the limit $t \to \infty$ we find $v_s = \sqrt{\frac{mg}{k_1}}$ as $\lim_{t\to\infty} \tanh(\lambda t) = 1$. Thus the velocity converges to a stationary velocity v_s which is given at equal forces (without parachute):

$$a(t) = g - \frac{k_1}{m}v_s^2 = 0 \Leftrightarrow v_s = \sqrt{\frac{mg}{k_1}}$$

b)

$$v_s = \sqrt{\frac{gm}{k_1}} \approx \underline{52.4\,\text{m/s} \approx 190\,\text{km/h}}$$

$t(v) =? \to$ use $t(v(t)) = t$

$$\to \text{artanh}\left(\sqrt{\frac{k_1}{mg}}v(t)\right) = \text{artanh}\left(\tanh(\sqrt{\frac{k_1 g}{m}}t)\right) = \sqrt{\frac{k_1 g}{m}}t$$

$$\Rightarrow \underline{t(v) = \sqrt{\frac{m}{k_1 g}}\text{artanh}\left(\sqrt{\frac{k_1}{mg}}v\right)}$$

$$\Rightarrow t(v_s) = \sqrt{\frac{m}{k_1 g}}\,\mathrm{artanh}\,(1) = \underline{\underline{\infty}}$$

$$t(0.99 v_s) = \sqrt{\frac{m}{k_1 g}}\,\mathrm{artanh}\,(0.99) \approx \underline{\underline{14\,\mathrm{s}}}$$

The parachutist reaches 99% of the stationary velocity after falling for 14 s. It (theoretically) still takes infinite time to reach 100%.

c)

$$\frac{dv}{dt} = g - \frac{k_2}{m} v$$

Lets make the Ansatz

$$v(t) = c(t)\, e^{A(t)} = c(t)\, e^{-\frac{k_2}{m} t} \quad \text{with} \quad A(t) = \int^t dt' \left(-\frac{k_2}{m}\right)$$

plug it into the differential equation and solve for $c(t)$:

$$c'(t)\, e^{-\frac{k_2}{m} t} - c(t) \frac{k_2}{m} e^{-\frac{k_2}{m} t} = g - c(t) \frac{k_2}{m} e^{-\frac{k_2}{m} t}$$

$$\Rightarrow \quad c'(t) = g\, e^{\frac{k_2}{m} t}$$

$$\Rightarrow \quad c(t) = g \int^t dt' e^{\frac{k_2}{m} t'} = \frac{mg}{k_2} e^{\frac{k_2}{m} t} + c_0$$

The general solution for $v(t)$ is

$$v(t) = \frac{mg}{k_2} + c_0\, e^{-\frac{k_2}{m} t}$$

c_0 is determined by the initial condition $v(0) = v_{s_1}$, thus $c_0 = v_{s_1} - \frac{mg}{k_2}$. Therefore the solution for the velocity once the parachute has opened is

$$v(t) = \left(\sqrt{\frac{mg}{k_1}} - \frac{mg}{k_2}\right) e^{-\frac{k_2}{m} t} + \frac{mg}{k_2}$$

By taking the limit $t \to \infty$ we find $v_{s_2} = \frac{mg}{k_2}$. Thus

$$v(t) = (v_{s_1} - v_{s_2})\, e^{-\frac{k_2}{m} t} + v_{s_2}$$

Checking the limits we find that they are indeed as expected:

$$v(t \to 0) = v_{s_1}, \qquad v(t \to \infty) = v_{s_2}$$

The stationary velocity under the parachute evaluates to: $v_{s_2} = \frac{mg}{k_2} \approx \underline{\underline{6.2\,\mathrm{m/s}}}$

d)

$$y(t) = \int_0^t v(t') dt' + \underbrace{y_0}_{=0} = \int_0^t \left((v_{s1} - v_{s2}) e^{-\frac{k_2}{m} t'} + v_{s2}\right) dt'$$

$$= \left(-\frac{m}{k_2}(v_{s1} - v_{s2}) e^{-\frac{k_2}{m} t'} + v_{s2} t'\right)_0^t$$

$$= \underline{\underline{-\frac{m}{k_2}(v_{s1} - v_{s2}) e^{-\frac{k_2}{m} t} + \frac{m}{k_2}(v_{s1} - v_{s2}) + v_{s2} t}}$$