ORDINARY DIFFERENTIAL EQUATIONS

Differential equations are equations that relate a function with its derivatives. The "unknown" is thus a function of one (or more) variables.

Ordinary diff. eq. (ODE) → equation that involves derivatives with respect to only one variable

EXAMPLE : $\ddot{x} = -g$ free fall

$\ddot{x} = -g - \lambda \dot{x}$ with friction

Partial diff. eq. (PDE) → diff. eq. that involves partial derivatives

EXAMPLE : $\dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{1}{a^2} \dfrac{\partial u(x,t)}{\partial t} = 0$

heat equation

In this Chapter we focus on ODE.

General form of n-order ODE :

$F(t, x, x', x'' \ldots x^{(n)}) = 0$ implicit form

$x^{(n)} = f(t, x, x' \ldots x^{(n-1)})$ explicit form

A solution of the differential equation in a intervall I is a function $\phi(t)$ such that

$$F(t, \phi(t), \phi'(t) \ldots \phi^{(n)}(t)) = 0$$

$$\forall t \in I$$

$$\phi^{(n)} = f(t, \phi(t) \ldots \phi^{(n-1)}(t))$$

EXAMPLE

Newton equation $m\ddot{x} = F(t)$ → force

↳ we look for the "position" $x(t)$

In this case it is relatively easy: we need to integrate 2 times with respect to t → two integration constants corresponding to the initial conditions $x(0)$, $\dot{x}(0)$.

In general the problem can be quite complicated. The majority of ODE in physics are at most of the second order. This means that they contain x, $\dot{x}$ and $\ddot{x}$ (or y, y', y'') but not higher derivatives. The general case can be complicated and requires a numerical solution.

In the following we discuss a few specific cases in which an analytic solution can be found.

$\boxed{\text{FIRST ORDER ODE}}$

$$y' = f(x,y) \quad \text{or} \quad F(x,y,y') = 0$$

Let us discuss the cases in which one (or more) solutions can be found.

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \qquad \text{CAUCHY PROBLEM}$$

$f: I \times D \to \mathbb{R}$

$I \subset \mathbb{R} \quad D \subset \mathbb{R}$

$x_0 \in I$, $y_0 \in D$

Under which conditions a solution to the problem exists?

LOCAL EXISTENCE THEOREM (PEANO)

We consider the Cauchy problem $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ $\quad f: I \times D \to \mathbb{R}$, $x_0 \in I$, $y_0 \in D$

If f is continuous over $I \times D$

$\Rightarrow \exists\, \delta > 0$ and $\varphi(x) \;\Big|\; \varphi'(x) = f(x, \varphi(x)) \quad \forall x \in [x_0 - \delta, x_0 + \delta]$

with $\varphi(x_0) = y_0$

Note that the theorem states that the solution exists but not that it is unique

EXAMPLE

$$\begin{cases} y' = \sqrt[3]{y} \\ y(0) = 0 \end{cases}$$

the function $f(x,y) = \sqrt[3]{y}$ is continuous over $\mathbb{R}$

$\varphi_1(x) = 0$ is a solution

But also $\varphi_2(x) = \begin{cases} 0 & x \leq 0 \\ -\sqrt{\left(\frac{2}{3}x\right)^3} & x > 0 \end{cases}$ is solution!

EXISTENCE AND UNICITY THEOREM (CAUCHY)

We consider the Cauchy problem $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ $\quad f: I \times D \to \mathbb{R}$, $x_0 \in I$, $y_0 \in D$

If f is continuous over $I \times D$ and $\dfrac{\partial f}{\partial y}$ is continuous in a neighbourhood of (x_0, y_0) $\Rightarrow \exists\, \delta > 0$ and $\varphi(x) \;\Big|\; \varphi'(x) = f(x, \varphi(x)) \quad \forall x \in [x_0 - \delta, x_0 + \delta]$

$\varphi(x_0) = y_0$

and the solution is UNIQUE

Note that in the previous example $\dfrac{\partial f}{\partial y}$ was not continuous in $x_0 = 0$

as $\dfrac{\partial f}{\partial y} = \dfrac{1}{3} y^{-\frac{2}{3}}$

We now discuss a few specific cases in which we can find an analytic solution.

Separation of variables

$y' = f(x,y)$ suppose $f(x,y) = g(x)\,h(y)$ $\Rightarrow$ $\dfrac{dy}{h(y)} = g(x)\,dx$

the general solution can be found by

$$\int \frac{dy}{h(y)} = \int g(x)\,dx \quad \rightarrow$$ indefinite integral: one constant of integration $\rightarrow$ initial condition

EXAMPLE

$$\begin{cases} x y' - y = 3 \\ y(1) = 2 \end{cases} \qquad y' = \frac{3+y}{x} \qquad \int \frac{dy}{3+y} = \int \frac{dx}{x} \Rightarrow \ln(3+y) = \ln x + c$$

$$\Rightarrow y = Ax - 3$$

setting $y(1)=2$ $\Rightarrow$ $A = 5$

Alternatively we can obtain the same result through a definite integral

$$\int_{\underset{y(1)}{2}}^{y} \frac{dy}{3+y} = \int_1^x \frac{dx}{x} \Rightarrow \ln\frac{y+3}{5} = \ln x \Rightarrow y = 5x - 3$$

EXAMPLE

$y' = f(y)$ $\Rightarrow$ special case with $g(x)=1$ $\quad y(x_0) = y_0$

$$\int_{y_0}^{y} \frac{dy}{f(y)} = \int_{x_0}^{x} dx = x - x_0$$

Suppose for example $f(y) = y^2$

$$\int_{y_0}^{y} \frac{1}{y^2}\,dy = \frac{1}{y_0} - \frac{1}{y} = x - x_0$$

HYPERBOLE

EXAMPLE

$y' = f(y/x)$ in this case the variables cannot be trivially separated

but we can set $v = y/x$

$\Rightarrow \quad y' = v'x + v = f(v) \quad \Rightarrow \quad v' = \dfrac{f(v)-v}{x}$ separable!

Let us consider for example

$$y' = \frac{2xy}{x^2+y^2} = \frac{2\,y/x}{1+y^2/x^2} \qquad f(v) = \frac{2v}{1+v^2}$$

$$v' = \frac{f(v)-v}{x} = \frac{\frac{2v}{1+v^2} - v}{x} = \frac{v(1-v^2)}{x(1+v^2)} \Rightarrow \int \frac{dv}{v(1-v^2)}(1+v^2) = \int \frac{dx}{x}$$

$$\int \frac{1+v^2}{v(1-v^2)} = \int \frac{1}{v} + \frac{2v}{1-v^2} = \ln\frac{1}{1-v^2} + \ln v = \ln x + \text{const}$$

$$\Rightarrow \frac{v}{1-v^2} = cx \qquad \frac{y/x}{1-y^2/x^2} = cx \qquad y = c(x^2-y^2) \quad \text{HYPERBOLE}$$

It is important to note that if the ODE is non linear there can be additional singular solutions

EXAMPLE

$y' = \sqrt{1-y^2}$ find all the solutions

$$\int \frac{dy}{\sqrt{1-y^2}} = \int dx \Rightarrow \text{ArcSin}\,y + c = x \Rightarrow y = \sin(x-c)$$

But $y=1$ and $y=-1$ are also solutions! These solutions, expected from the EXISTENCE THEOREM ($\partial f/\partial y$ is divergent at $y \to \pm 1$) are lost with the standard separation of variables

Linear first order ODE

$$y' = a(x)\,y + b(x)$$

$$\begin{cases} y' = a(x)\,y + b(x) \\ y(x_0) = y_0 \end{cases}$$

$a(x), b(x)$ continuous

$\Rightarrow \exists!$ solution!

(thanks to the CAUCHY theorem)

consider first the case $b(x) = 0$ $\rightarrow$ homogeneous 1-st order ODE

$$y' = a(x)\,y \qquad \frac{y'}{y} = a(x) \qquad \ln y = A(x) + \text{const}$$

separable!

$A(x)$: primitive of $a(x)$

$$y_H = C\,e^{A(x)}$$

C: integration constant

If we require a solution such that

$y(x_0) = y_0$ we can fix the integration constant

$$y_0 = C\,e^{A(x_0)} \quad \Rightarrow \quad C = \frac{y_0}{e^{A(x_0)}} \qquad y_H = y_0\,e^{A(x) - A(x_0)}$$

We now move to consider the case $b \neq 0$ $\rightarrow$ inhomogeneous 1-st order ODE

We need to find one particular solution of the inhomogeneous ODE, y_P

The general solution of the problem can indeed be found as $y = y_H + y_P$

Thanks to the linearity of the equation we can write

$$y' = y_H' + y_P' = a(x)\,y_H + a(x)\,y_P + b(x) = a(x)\,y + b(x)$$

A particular solution can be found by using an ansatz:

$$y_P = C(x)\,e^{A(x)}$$

$$y_P' = C'(x)\,e^{A(x)} + \cancel{C(x)\,e^{A(x)}\,a(x)} = \cancel{a(x)\,C(x)\,e^{A(x)}} + b(x)$$

$$\Rightarrow \quad C'(x) = b(x)\,e^{-A(x)}$$

$$C(x) = \int b(x)\,e^{-A(x)} + \text{const}$$

The general solution is thus

$$y(x) = e^{A(x)} \left[\int^x b(t) e^{-A(t)} dt + \text{const} \right]$$

Example

$y' = \tan x \, y + \sin 2x \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$a(x) = \tan x \qquad b(x) = \sin 2x$

$A(x) = \int a(x) dx = -\log(\cos x) \qquad \Rightarrow \quad y_H(x) = C e^{-\log(\cos x)} = \frac{C}{\cos x}$

$C'(x) = \sin 2x \cos x = 2 \sin x \cos^2 x \qquad \Rightarrow \quad C(x) = -\frac{2}{3}\cos^3(x) + C_0$

$\Rightarrow$ the general solution is $\quad y(x) = \frac{1}{\cos x}\left(-\frac{2}{3}\cos^3 x + C_0\right)$

$$= \underbrace{-\frac{2}{3}\cos^2 x}_{\text{special solution}} + \underbrace{\frac{C_0}{\cos x}}_{\text{solution of the homogeneous eq.}}$$

Bernoulli differential equation

$y' + p(x) y = q(x) y^m \qquad$ non linear and non separable!

set $\quad z(x) = y^{1-m}(x) \qquad z' = \frac{dz}{dy} y' = (1-m) y^{-m} y'$

If we multiply the equation by $(1-m) y^{-m}$ we get

$$(1-m) y^{-m} y' + p(x)(1-m) y^{-m+1} = q(x)(1-m)$$

$$z' + (1-m) p(x) z = (1-m) q(x)$$

$\Rightarrow$ we get a linear differential equation for z!

Ricceti differential equation

$$y' = f(x)\, y^2 + g(x)\, y + h(x)$$ non linear and non-separable

Suppose we know one solution $\phi_0(x)$ $\Rightarrow$ look for solutions of the form

$$y = \phi_0 + \frac{1}{u}$$ to be determined

$$y = \phi_0 + \frac{1}{u}$$

$$y' = \phi_0' - \frac{u'}{u^2}$$ let us replace into the original equation

$$\cancel{\phi_0'} - \frac{u'}{u^2} = f(x)\left(\cancel{\phi_0^2} + \frac{1}{u^2} + 2\phi_0/u\right) + g(x)\left(\cancel{\phi_0} + \frac{1}{u}\right) \cancel{+h(x)}$$

$$\Rightarrow \quad u' = -(2f\phi_0 + g)\,u - f$$ linear inhomogeneous equation for $u(x)$

EXAMPLE

$$y' = y^2 + \frac{y}{x} + \frac{1}{x^2}$$ look for a special solution of the form $y = \frac{a}{x}$

$$-\frac{a}{x^2} = \frac{a^2}{x^2} + \frac{a}{x^2} + \frac{1}{x^2}$$

$$\Rightarrow \quad (a+1)^2 = 0 \quad \Rightarrow \quad \phi_0(x) = -\frac{1}{x}$$

$$y = -\frac{1}{x} + \frac{1}{u} \quad \Rightarrow \quad u' = \frac{u}{x} - 1$$ $u_H = cx$ solution of the homogeneous eq.

$$\Rightarrow \quad u = c(x)\, x \qquad c'x + \cancel{c} = \cancel{c} - 1 \qquad c' = -\frac{1}{x}$$

$$c(x) = -\ln|x| + \text{const}$$

$$\Rightarrow \quad u(x) = -x\,(\ln|x| + \text{const})$$

Exact differential equations

Let us consider a first-order differential equation of the form

$$P(x,y) + y' Q(x,y) = 0$$

The equation can be formally rewritten as

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

Suppose that there exists a function f such that $(P,Q) = \nabla f$.

We say that the field $F = (P,Q)$ is <u>conservative</u> and that f plays the role of a <u>potential</u>. The functions $y(x)$ such that $f(x,y(x)) = \text{const}$ provide a solution of the differential equation. Indeed

$$df = \frac{\partial f}{\partial x}(x,y(x)) + y'(x)\frac{\partial f}{\partial y}(x,y(x)) = 0$$

$$= P(x,y) + y' Q(x,y)$$

A condition for the field F to be conservative is that, being P and Q of class C^1, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In this case the differential equation is said to be EXACT.

The solutions are said to be INTEGRAL CURVES of the diff. equation

EXAMPLE

$$\overbrace{(e^y + y e^x)}^{P}\,dx + \overbrace{(e^x + x e^y)}^{Q}\,dy = 0$$

$$\frac{\partial P}{\partial y} = e^y + e^x = \frac{\partial Q}{\partial x} \implies \text{the equation is exact}$$

Look for f such that $(P,Q) = \nabla f$

$f_x = e^y + y e^x \quad \rightarrow \quad f(x,y) = x e^y + y e^x + \varphi(y)$

$f_y = x e^y + e^x + \varphi'(y) = e^x + x e^y \quad \Rightarrow \quad \varphi(y) = \text{const}$

$\Rightarrow$ the curves $x e^y + y e^x = \text{const}$ are the integral curves of the differential equation

Integrating factor

Let us consider a differential equation of the form $P(x,y)\,dx + Q(x,y)\,dy = 0$ which is NOT EXACT. A factor $m(x,y) \neq 0$ is called integrating factor if

$$\big(m(x,y)\,P(x,y)\big)dx + \big(m(x,y)\,Q(x,y)\big)dy = 0$$

is exact.

Example

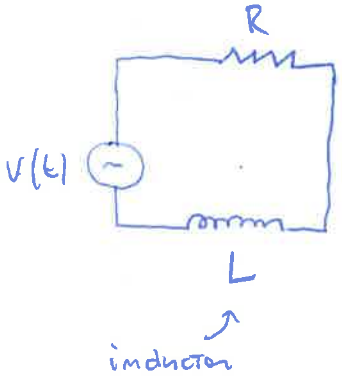
$y\,dx + 2x\,dy = 0$ is not exact since $P_y = 1 \quad Q_x = 2$

for $x > 0$ we can choose $m(x,y) = \frac{1}{\sqrt{x}}$

$$\frac{y}{\sqrt{x}}\,dx + 2\sqrt{x}\,dy = 0 \qquad \text{is now exact}$$

$f(x,y) = 2y\sqrt{x} = \text{const}$ are the integral curves

EXAMPLE: RL circuit



$$\underset{\text{flux}}{\Phi_B} = \underset{\text{inductance}}{L}\ \underset{\text{current}}{I} \Rightarrow V_L = L\frac{dI}{dt} \qquad \text{FARADAY LAW}$$

$\Rightarrow$ the current can be obtained by solving the differential equation

$$L\frac{dI}{dt} + RI(t) = V(t)$$

This is a linear inhomogeneous equation

$\Rightarrow$ we first solve the homogeneous eq.

$$L\frac{dI}{dt} = -RI \qquad \frac{dI}{I} = -\frac{R}{L}dt$$

$$\Rightarrow I_H = I_0 e^{-\frac{R}{L}t}$$

We now find a particular solution

by variation of constants $I_0 \to I_0(t)$ $\qquad I_p(t) = I_0(t)e^{-\frac{R}{L}t}$

$$I_p'(t) = I_0'(t)e^{-\frac{R}{L}t} + \left(-\frac{R}{L}\right)I_0(t)e^{-\frac{R}{L}t}$$

Plug it into the equation

$$I_0'(t)e^{-\frac{R}{L}t} \cancel{-\frac{R}{L}I_0(t)e^{-\frac{R}{L}t}} = \cancel{-\frac{R}{L}I_0(t)e^{-\frac{R}{L}t}} + \frac{1}{L}V(t)$$

$$I_0'(t) = \frac{V_0}{L}e^{\frac{R}{L}t}$$

$$I_0(t) - I_0(0) = \frac{V_0}{R}\left(e^{\frac{R}{L}t} - 1\right) \Rightarrow I_p(t) = \frac{V_0}{R}\left(1 + ce^{-\frac{R}{L}t}\right)$$

Complete solution $I = I_H + I_p$

Assuming for example $I(0) = 0$ we get $\boxed{I(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$