MMP I
Exercise Sheet 3

HS 21
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Exercise 1 [Properties of Fourier transform (5 points)]
Consider the functions $f, g, \hat{f}, \hat{g} \in L^1(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{C}$, $\lambda \in \mathbb{R}/\{0\}$ and the two operators D_λ and T_y are defined as

$$\begin{aligned}(D_\lambda f)(x) &= f(\lambda x)\\ (T_\alpha f)(x) &= f(x+\alpha), \quad \alpha \in \mathbb{R}^n.\end{aligned}$$

and demonstrate the following properties of the Fourier transform in $\mathbb{R}^n$:

a) $\widehat{(\alpha f + \beta g)} = \alpha\hat{f} + \beta\hat{g}$,

b) $\overline{\hat{f}(k)} = \hat{\overline{f}}(-k)$,

c) $\widehat{(D_\lambda f)}(k) = |\lambda|^{-n}\hat{f}(\frac{1}{\lambda}k)$,

d) $\widehat{T_\alpha f}(k) = \hat{f}(k)e^{ik\alpha}$,

e) $\int_{\mathbb{R}^n} \hat{f}g = \int_{\mathbb{R}^n} f\hat{g}$.

Exercise 2 [Properties of Fourier transform (4 points)]
Show that:

a) $\hat{f}(k) = \hat{\overline{f}}(-k) \leftrightarrow f(x) \in \mathbb{R}$

b) $\widehat{f'}(k) = ik\hat{f}(k)$, for $f(x)$ compactly supported.

c) $\widehat{xf}(k) = i\frac{d}{dk}\hat{f}(k)$.

– please turn over –

Exercise 3 [Fourier transform (5 points)]
Calculate the Fourier transform for the following functions in $\mathbb{R}$:

a) Step function:

$$f(x) = \begin{cases} 1, & \text{for} \quad -3 \leq x \leq -1 \\ 1, & \text{for} \quad 1 \leq x \leq 3 \\ 0, & \text{for} \quad x = 0 \quad \text{elsewhere} \end{cases}$$

b) $f(x) = e^{-a^2x^2}$.
What can you say if $a = 1/\sqrt{2}$? [*Hint:* Use $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$]

c)

$$f(x) = \begin{cases} e^{-ax}, & \text{for} \quad x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Which is the Fourier transform of $x^2 f(x)$?