MMP I
Solution Sheet 8

HS 21
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Exercise 1 [Bound oscillators (3 points)]

$$\begin{cases} m\ddot{x} = -\alpha x - \kappa(x-y) \\ m\ddot{y} = -\alpha y - \kappa(y-x) \end{cases} \quad \text{with } \alpha = \tfrac{mg}{l}$$

$$\ddot{x} = -\frac{g}{l}x - \frac{\kappa}{m}(x-y) \tag{1}$$
$$\ddot{y} = -\frac{g}{l}y - \frac{\kappa}{m}(y-x) \tag{2}$$

$$(1)+(2) \to \frac{d^2}{dt^2}(x+y) = -\frac{g}{l}(x+y) = -\omega^2(x+y)$$
$$(1)-(2) \to \frac{d^2}{dt^2}(x-y) = -\frac{g}{l}(x-y) - \frac{2\kappa}{m}(x-y) = -\left(\frac{g}{l}+\frac{2\kappa}{m}\right)(x-y) = -\Omega^2(x-y)$$
$$\text{with :} \qquad \omega = \sqrt{\frac{g}{l}}, \qquad \Omega = \sqrt{\frac{g}{l}+\frac{2\kappa}{m}}$$

2 harmonic oscillators:

$$(x+y) = A\cos(\omega t) + B\sin(\omega t)$$
$$(x-y) = A'\cos(\Omega t) + B'\sin(\Omega t)$$
$$\to \quad x = \frac{A}{2}\cos(\omega t) + \frac{B}{2}\sin(\omega t) + \frac{A'}{2}\cos(\Omega t) + \frac{B'}{2}\sin(\Omega t)$$
$$y = \frac{A}{2}\cos(\omega t) + \frac{B}{2}\sin(\omega t) - \frac{A'}{2}\cos(\Omega t) - \frac{B'}{2}\sin(\Omega t)$$

let's use the initial conditions

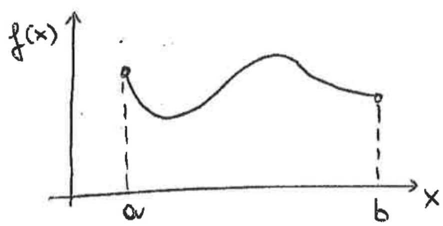
$$t = 0 \;\to\; x(0) = 0 \qquad\qquad x'(0) = 1$$
$$y(0) = 0 \qquad\qquad y'(0) = 0$$

$$x(0) = \frac{A}{2} + \frac{A'}{2} = 0$$

$$y(0) = \frac{A}{2} - \frac{A'}{2} = 0 \Rightarrow A' = A = 0$$
$$y'(0) = 0 = \frac{B\omega}{2} - \frac{B'\Omega}{2} = 0 \Rightarrow B\omega = B'\Omega$$
$$x'(0) = \frac{B\omega}{2} + \frac{B'\Omega}{2} = 1 \Rightarrow B\omega = 1 \rightarrow B = \frac{1}{\omega}, B' = \frac{1}{\Omega}$$

$$x = \frac{1}{2\omega}\sin(\omega t) + \frac{1}{2\Omega}\sin(\Omega t) \qquad \omega = \sqrt{\frac{g}{l}}$$
$$y = \frac{1}{2\omega}\sin(\omega t) - \frac{1}{2\Omega}\sin(\Omega t) \qquad \Omega = \sqrt{\frac{g}{l} + \frac{2\kappa}{m}}$$

Exercise 2 [Calculus of variations (6 points)]



$y = f(x)$; The endpoints are fixed: $f(a) = \alpha$, $f(b) = \beta$

Find $f(x)$ for which the surface of the body made by revolution of the curve $y = f(x)$ around x is minimal.

$\Rightarrow$ This is equivalent to making the integral I (that defines the surface) minimal.

In general:

An integral
$$I = \int_{x_1}^{x_2} F(x, y, y')dx$$
is stationary when $\delta I = 0$. This can be expressed as a condition on $F(x, y, y')$, i.e. on the argument of the integral: $\delta x = 0 \rightarrow$ the endpoints are fixed.
$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right) dx = 0$$
For the second term $\frac{\partial F}{\partial y'}\delta y$ we have:
$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y' dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d\delta y}{dx} dx$$

(integration by parts)

$$= \frac{\partial F}{\partial y'}\delta y\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y dx$$

$$= \frac{\partial}{\partial y'}F\left(x_2, y(x_2), y'(x_2)\right)\underbrace{\delta y'(x_2)}_{=0} - \frac{\partial}{\partial y'}F\left(x_1, y(x_1), y'(x_1)\right)\underbrace{\delta y'(x_1)}_{=0} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y dx$$

(the endpoints are fixed)

$$\rightarrow \delta I = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\delta y dx = 0 \leftrightarrow \underline{\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0}$$

(Euler-Lagrange equation)

If F is not an explicit function of x (i.e. $\frac{\partial F}{\partial x} = 0$), this explanation in equivalent to

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0 \leftrightarrow y'\frac{\partial F}{\partial y'} - F = const$$

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{x} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} \rightarrow y'\frac{\partial F}{\partial y} = \frac{dF}{dx} - \frac{\partial F}{\partial y'}\frac{dy'}{dx}$$

Using this result and multiplying the E-L-equation by y':

$$y'\frac{\partial F}{\partial y} - y'\frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

$$\rightarrow \frac{dF}{dx} - \underbrace{\frac{\partial F}{\partial y'}\frac{dy'}{dx} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)}_{=-\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)} = 0$$

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = 0 \rightarrow \underline{F - y'\frac{\partial F}{\partial y'} = const}$$

$\rightarrow$ Finding $F(x, y, y')$ such that $I = \int_{x_1}^{x_2} F(x, y, y')dx$ is stationary is equivalent to finding $F(x, y, y')$ such that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

Or, if $F = F(y, y') \leftrightarrow \frac{\partial F}{\partial x} = 0$, such that

$$F - y'\frac{\partial F}{\partial y'} = const$$

Going back to the original problem, the surface integral can be written as

$$S[f] = \underbrace{\int_0^{2\pi} d\phi}_{\text{rotation}} \quad \underbrace{L[f]}_{\text{line integral along } f(x)} \quad = 2\pi \int_a^b f(\gamma(t))||\dot{\gamma}(t)||dt$$

where $\gamma : [a, b] \to \mathbb{R}^2$ is a piecewise smooth curve, here $\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and thus its derivative with respect to t is $\dot{\gamma}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$. Applying the euclidean norm we get $||\dot{\gamma}(t)|| = \sqrt{\dot{x}^2 + \dot{y}^2}$ and the integral becomes

$$S[f] = 2\pi \int_a^b f(x(t))\sqrt{\dot{x}^2 + \dot{y}^2} dt = 2\pi \int_a^b f(x(t)) \left| \frac{dx}{dt} \right| \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)^2} dt$$

with $\frac{dx}{dt}(t_0) \neq 0$ for a $t_0 \in [a, b]$. Using the inverse function theorem we can find a local representation $y(x)$ around $x_0 = x(t_0)$ and $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = y'(x)$, because we are considering a path $(x, f(x))$ with a mapping $x \mapsto f(x)$ the assumption $\frac{dx}{dt} \neq 0$ is justified for all points. Moreover, the function is C^1 on all points in the interval except finitely many, so we may split the path into a finite number of paths at those non-continuously differentiable points and integrate them individually. With $\frac{dx}{dt} > 0$ (otherwise $t \to -t$) and substitution we get

$$S[f] = 2\pi \int_a^b y(x)\sqrt{1 + y'^2(x)} dx = 2\pi \int_a^b \underbrace{f(x)\sqrt{1 + f'^2(x)}}_{F = F(f, f')} dx \, .$$

$\to F$ doesn't explicitly depend on x

$$\delta S = 0 \leftrightarrow F - f' \frac{\partial F}{\partial f'} = const$$

$$\begin{aligned}
&f(x)\sqrt{1 + (f'(x))^2} - f'(x) f(x) \frac{1}{2} 2 f'(x) \frac{1}{\sqrt{1 + (f'(x))^2}} = C \\
&f(x)\,(1 + \cancel{(f'(x))^2}) - f(x) \cancel{(f'(x)^2)} = C\sqrt{1 + (f'(x))^2} \\
&f(x) = C\sqrt{1 + (f'(x))^2}
\end{aligned}$$

This differential equation can be solved by using the hint given in the exercise or with separation of variables. Lets use the hint, squaring the equation yields:

$$f^2 = C^2(1 + \left(f'(x)\right)^2)$$

Since $\cosh^2(x) - 1 = \sinh^2(x)$ and $\frac{\partial}{\partial x}\cosh(x) = \sinh(x)$ we spot that for $C = 1$ the solution is given by $f(x) = \cosh(x)$. Because we have a differential equation we need to add an integration constant that preserves the functional form of the solution such that the property given in the hint is still valid. Therefore we modulate the argument by a linear combination $Ax + B$. In order to generalize for a generic C without changing the functional form, we allow for a rescaling by a constant factor K, thus $f(x) = K\cosh(Ax + B)$. Plugging this ansatz into the differential equation we have:

$$K^2 \cosh^2(Ax + B) = C^2 + C^2 K^2 A^2 \sinh^2(Ax + B)$$

$$K^2\left(\cosh^2(Ax+B) - C^2A^2\sinh^2(Ax+B)\right) = C^2$$

We demand that $C^2A^2 = 1$ for the parenthesis to vanish, thus $A = \frac{1}{C}$. Then we immediately see that $K = C$ and therefore

$$f(x) = C\cosh(\frac{x}{C} + B)\,.$$

The constants C and B are determined by the endpoints:

$$f(a) = \alpha = C\cosh\left(\frac{a}{C} + B\right)$$
$$f(b) = \beta = C\cosh\left(\frac{b}{C} + B\right)$$

Exercise 3 [Brachistochrone problem (5 points)]
Total energy: $E = T + V = \frac{1}{2}mv^2 - mgy = 0$ (Energy conservation: $E(y=0) = 0$)

$\Rightarrow v^2 = 2gy$

Total time from P_1 to P_2:

$$T[y] = \int_{t_1}^{t_2} dt = \int_{s(t_1)}^{s(t_2)} \frac{ds}{v} = \int_{x_1}^{x_2} \underbrace{\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}}_{F(y,y')} dx$$

Special case (ii) from the lecture: $F_x = 0 \Rightarrow F - y'F_{y'} = C$. (first integral)

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - y'\frac{y'}{\sqrt{1+y'^2}\sqrt{2gy}} = c \qquad |\cdot\sqrt{1+y'^2}$$
$$\frac{1}{\sqrt{2gy}}(1+y'^2-y'^2) = c\sqrt{1+y'^2} \qquad |\wedge 2$$
$$1+y'^2 = \frac{1}{2gc^2}\frac{1}{y} = \frac{k}{y}$$
$$\text{where: } k := \frac{1}{2gc^2}$$
$$\Rightarrow y' = \pm\sqrt{\frac{k}{y} - 1} = \pm\sqrt{\frac{k-y}{y}}$$

Separation:

$$\Rightarrow \pm\int\sqrt{\frac{y}{k-y}}dy = \int dx + \alpha$$

$$\Rightarrow \int \sqrt{\frac{y}{k-y}} dy = \pm(x+\alpha) = x+\alpha$$

We want the bead to go right, not left: + sign.

With $y = k\sin^2\frac{\phi}{2}$ and $dy = k\sin\frac{\phi}{2}\cos\frac{\phi}{2}d\phi$:

$\int \sqrt{\frac{y}{k-y}}dy = \int k\sin\frac{\phi}{2}\cos\frac{\phi}{2}\sqrt{\frac{k\sin^2\frac{\phi}{2}}{k(1-\sin^2\frac{\phi}{2})}}d\phi = k\int \sin^2\frac{\phi}{2}d\phi = \frac{k}{2}[\phi - \sin\phi]$

$\Rightarrow x(\phi) = \frac{k}{2}[\phi - \sin\phi] - \alpha = \frac{k}{2}[\phi - \sin\phi] + x_1$ where $x_1 := x(0)$

and

$y(\phi) = k\sin^2\frac{\phi}{2} = \frac{k}{2}[1-\cos\phi]$

here we used: $\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = 1 - 2\sin^2\frac{\phi}{2}$

The resulting curve is a so called cycloid, one that can be seen by observing a certain point of a rolling circle with $r = \frac{k}{2}$.