# 3) The SUSY algebra

Premise: A "no-go" theorem by Coleman & Mandula states that under general assumptions (locality, causality, positive energy states, ...) the only possible symmetries of the S-matrix, beside C, P, T are

- Poincaré symmetries, with generators $P_\mu$, $M_{\mu\nu}$
- Internal symm., " " $B_k$ that must commute with the Poincaré generators:

Full algebra

Poincaré
$$\left\{\begin{array}{l} [P_\mu, P_\nu] = 0 \\ \boxed{[M_{\mu\nu}, M_{\rho\sigma}] = -i g_{\mu\rho} M_{\nu\sigma} - i g_{\nu\sigma} M_{\nu\rho} + i g_{\mu\sigma} M_{\nu\rho} + i g_{\nu\rho} M_{\mu\sigma}} \\ [M_{\mu\nu}, P_\rho] = -i g_{\rho\mu} P_\nu + i g_{\rho\nu} P_\mu \end{array}\right.$$

Lorentz

$$\boxed{[B_i, B_j] = i f_{ij}{}^{k} B_k}$$

Internal symm.

$$[P_\mu, B_k] = [M_{\mu\nu}, B_k] = 0$$

orthogonality condition

$$\Downarrow$$

$$\boxed{ISO(1,3) * G_{int}}$$

In other words, the generators of the internal symm. must be Lorentz scalars

See e.g. Weinberg's book III chapt. 24 for a proof

Haag, Lopuszanski & Sohnius have shown that the Coleman-Mandula th. can be evaded if we assume that the symmetry is based on a "Graded Lie Algebra" rather than on simple Lie Algebra $\rightarrow$ anticommutators

$\Rightarrow$ Extension (decomposition) of the Lorentz group by means of appropriate spinor generators $Q^I_\alpha$ & $(\bar{Q}^I)^{\dot{\alpha}}$ (susy Generators)

$$[P_\mu, Q^I_\alpha] = \emptyset$$

$$[P_\mu, \bar{Q}^{I\dot{\alpha}}] = \emptyset$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \quad \leftarrow Q^I \text{ transf. as } (\tfrac{1}{2}, 0)$$

$$[M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \quad \leftarrow \bar{Q}^I \text{ " " } (0, \tfrac{1}{2})$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \times \delta^{IJ}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI}$$

$$\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*$$

$\rightarrow$ since $Q \sim (\tfrac{1}{2}, 0)$ & $\bar{Q} \sim (0, \tfrac{1}{2})$ $\Rightarrow$ $Q \times \tilde{Q} \sim (\tfrac{1}{2}, \tfrac{1}{2})$

$\Rightarrow$ The $Q$ $(\bar{Q})$ algebra closes with $P_\mu \sim (\tfrac{1}{2}, \tfrac{1}{2})$

$\Downarrow$

- The commutator of two susy transformations is a translation
- Local susy transformations $\leftrightarrow$ local transations

$\updownarrow$

gravity

The $Z^{IJ}$ are called "central charges", they commute with all the generators of the full algebra

The define the dimensionality of the SUSY generators

$N=1$ SUSY $\rightarrow$ no $I,J$ index $\rightarrow$ no central charges

$N=2$ " $\rightarrow$ $I,J=1,2$ $\rightarrow$ 1 central charge $Z^{12}$

In principle there is no limit to $N$, but large $N$ implies increasing n° of particles with higher spin in the same multiplet

$\rightarrow$ $N \leq 4$ for theories with spin $\leq 1$

$N \leq 8$ " " " " $\leq 2$

N.B. The central charges can be non-trivial operators, combinations of the internal gen. of the algebra $(B_K)$ but they must commute with all generators

$$[Z^{IJ}, B_K] = 0 \qquad [Z^{IJ}, Z^{KL}] = 0$$

$$[Z^{IJ}, P_\mu] = 0 \qquad [Z^{IJ}, M_{\mu\nu}] = 0$$

$$[Z^{IJ}, Q^K_\alpha] = 0 \qquad [Z^{IJ}, \bar{Q}^K_{\dot\alpha}] = 0$$

## 4) Representations of the SUSY algebra

The Poincaré algebra has 2 Casimir operators

$$P^2 = P_\mu P^\mu \qquad W^2 = W_\mu W^\mu$$

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$$ (generalised ang. momentum)

Pauli-Lubanski vector

$\downarrow$

representations classified in terms of their eigenvalues

$\downarrow$

IRREP = single particle states

A) MASSIVE PARTICLE

$$P^2 = m^2 \qquad W_\mu\big|_{rest} = (0, \underbrace{\tfrac{1}{2}\epsilon_{i0jk}\, m M^{jk}}_{J_i}) \rightarrow W^2 = -m^2 \vec{J}^2$$

$$W^2 = -m^2 j(j+1) \qquad j = 0, \tfrac{1}{2}, \ldots$$

B) MASSLESS PARTICLES

$$P^2 = 0 \qquad W^2 = 0 \qquad W_\mu\big|_{rest} = M_{12} P_\mu$$

helicity

In SUSY $[Q, M_{\mu\nu}] \neq 0 \iff$ irreducible representations cannot be spin eigenstates, while $P^2$ remains a casimir operator

$\Downarrow$

$$\boxed{\text{degenerate multiplets with different spins}}$$

(*) In SUSY the energy of any state is always positive

proof:

$$\sum_{\alpha=\dot\alpha=1}^{2} \left( \langle \Phi | \{ Q^I_\alpha , \bar Q^I_{\dot\alpha} \} | \phi \rangle \right) = 2 \sum_{\alpha=\dot\alpha=1}^{2} \sigma^\mu_{\alpha\dot\alpha} \langle \phi | P_\mu | \phi \rangle \, \delta^{II}$$

$$= 2 \underset{2\delta^{\mu 0}}{\underset{\parallel}{\mathrm{Tr}\,(\sigma^\mu_{\alpha\dot\alpha})}} \langle \phi | P_\mu | \phi \rangle = 4 \langle \phi | P_0 | \phi \rangle$$

$$= 4 E_\phi$$

$$\sum_{\alpha=\dot\alpha=1}^{2} \left( \langle \phi | Q^I_\alpha (Q^I_\alpha)^+ + (Q^I_\alpha)^+ Q^I_\alpha | \phi \rangle \right) =$$

$$= \sum_{\alpha=\dot\alpha=1} \left( | \langle \phi | Q^I_\alpha |^2 + | \langle \phi | (Q^I_\alpha)^+ |^2 \right) \geq 0$$

(*) A SUSY multiplet contains an equal number of bosonic & fermionic d.o.f.

proof:

Let's consider the operator $(-1)^{\hat N_F}$ ; $(-1)^{\hat N_F} | \phi \rangle = (-1)^{n_F} \cdot | \phi \rangle$

n. fermionic dof

Since $Q^I_a$ & $\bar Q^I_{\dot\alpha}$ create/anihilate fermions,

$$Q^I_a (-1)^{\hat N_F} | \phi \rangle = - (-1)^{\hat N_F} Q^I_a | \phi \rangle \qquad (\Delta)$$

Let's consider now the trace over a finite-dim. representation

$$\mathrm{Tr} \left[ (-1)^{\hat N_F} \{ Q^I_\alpha , \bar Q^J_{\dot\beta} \} \right] = \mathrm{Tr} \left[ (-1)^{\hat N_F} Q^I_\alpha \bar Q^J_{\dot\beta} + (-1)^{\hat N_F} \bar Q^J_{\dot\beta} Q^I_\alpha \right]$$

$$= 0$$

because of $(\Delta)$ + ciclyc property of the trace

$$= \mathrm{Tr}\left[ (-1)^{\hat{N}_F} \, 2\, \sigma^{\mu}_{\alpha\dot{\beta}} \, P_\mu \, \delta^{IJ} \right] \Big|_{I=J} = \mathrm{Tr}\left[(-1)^{\hat{N}_F}\right] * 2\, \sigma^{\mu}_{\alpha\dot{\beta}} \, P_\mu$$

(N.B.: this is a number for fixed $\alpha, \dot{\beta}$)

$$\Rightarrow \quad \mathrm{Tr}\left[ (-1)^{\hat{N}_F} \right] = 0 \qquad (\text{for any } P_\mu \neq 0)$$

$$\hookrightarrow \quad \sum_B \langle B | (-1)^{\hat{N}_F} | B \rangle + \sum_F \langle F | (-1)^{\hat{N}_F} | F \rangle = 0$$

$$\Rightarrow \quad \boxed{n_B - n_F = 0}$$