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Applications of General Relativity

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Solutions 6

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Exercise 1 [Gravitational field of a moving particle] (5 points)

We are considering a particle of mass M moving with constant velocity $\boldsymbol{V}$. In the particles rest frame Σ' we have

$$h'_{\mu\mu}(\boldsymbol{r}) = -\frac{2G}{c^2}\int \mathrm{d}^3r'\,\frac{\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{2GM}{c^2}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_M|} \tag{1}$$

and $h'_{0i}=0$. Here $\boldsymbol{r}_M$ is the general initial position of the mass M. If we now transfer to a general frame Σ, the metric changes according to $g_{\mu\nu} = \alpha^\kappa_\mu\,\alpha^\rho_\nu\,g'_{\kappa\rho}$. Since we are working at linear order in h and the mass M is moving on a straight line, the transformation is nothing but a Lorentz transformation along the particle velocity, $\alpha^\mu_\nu = \Lambda^\mu_\nu + \mathcal{O}(h)$. For the transformation matrix we have:

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & \gamma V_x/c & \gamma V_y/c & \gamma V_z/c \\ \gamma V_x/c & 1+(\gamma-1)\frac{V_x^2}{V^2} & (\gamma-1)\frac{V_xV_y}{V^2} & (\gamma-1)\frac{V_xV_z}{V^2} \\ \gamma V_y/c & (\gamma-1)\frac{V_xV_y}{V^2} & 1+(\gamma-1)\frac{V_y^2}{V^2} & (\gamma-1)\frac{V_yV_z}{V^2} \\ \gamma V_z/c & (\gamma-1)\frac{V_xV_z}{V^2} & (\gamma-1)\frac{V_yV_z}{V^2} & 1+(\gamma-1)\frac{V_z^2}{V^2} \end{pmatrix} \simeq \begin{pmatrix} 1 & V_x/c & V_y/c & V_z/c \\ V_x/c & 1 & 0 & 0 \\ V_y/c & 0 & 1 & 0 \\ V_z/c & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where we neglected terms $\mathcal{O}(V^2/c^2)$, i.e. set $\gamma = 1$. Note that we only need to transform the perturbation since $\Lambda^\kappa_\mu\Lambda^\rho_\nu\eta_{\kappa\rho} = \eta_{\mu\nu}$ and are thus left with:

$$h_{\mu\nu} = \Lambda^\kappa_\mu\,\Lambda^\rho_\nu\,h'_{\kappa\rho}. \tag{3}$$

Evaluating the above expression we obtain for the metric perturbation in Σ:

$$h_{\mu\mu} = h'_{\mu\mu} + \mathcal{O}\left(V^2/c^2\right), \qquad h_{0i} = 2\frac{V_i}{c}h'_{00} + \mathcal{O}\left(V^2/c^2\right). \tag{4}$$

The mass M is moving on a straight line in Σ, thus its position as a function of time can be written as:

$$\boldsymbol{r}_M(t) = \boldsymbol{r}_{M,0} + \boldsymbol{V}t. \tag{5}$$

The gravitomagnetic potential is:

$$\boldsymbol{h}(\boldsymbol{r}) = h_{0i}(\boldsymbol{r}) = -\frac{4GM}{c^3}\frac{\boldsymbol{V}}{|\boldsymbol{r}-\boldsymbol{r}_M(t)|}, \tag{6}$$

and for the gravitomagnetic field we then have:

$$\boldsymbol{\Omega}(\boldsymbol{r}) = -\frac{2}{c}\boldsymbol{\nabla}_{\boldsymbol{r}}\times\boldsymbol{h}(\boldsymbol{r}) = \frac{2GM}{c^2}\frac{\boldsymbol{V}\times(\boldsymbol{r}-\boldsymbol{r}_M(t))}{|\boldsymbol{r}-\boldsymbol{r}_M(t)|^3}. \tag{7}$$

The four-velocity of the test mass m is:

$$u^\mu = \dot{x}^\mu \equiv \gamma(c, \mathbf{v}) \sim (c, \mathbf{v}) + \mathcal{O}(v^2/c^2) \tag{8}$$

since where are also considering $|\mathbf{v}| \ll c$. Evaluating the geodesic equation we obtain, with $\mathrm{d}t = \gamma \mathrm{d}\tau \sim \mathrm{d}\tau$:

$$\begin{aligned}\frac{\mathrm{d}v^i}{\mathrm{d}t} &= -\gamma^i_{\alpha\beta} u^\alpha u^\beta = -\Gamma^i_{00} u^0 u^0 - 2\Gamma^i_{0j} u^0 u^j = \\ &= -c^2 \Gamma^i_{00} - 2\Gamma^i_{0j} c v^j.\end{aligned} \tag{9}$$

For the Christoffel symbols we have:

$$\Gamma^i_{00} = \frac{1}{2}\eta^{ij}\left(\partial_0 h_{j0} + \partial_0 h_{0j} - \partial_j h_{00}\right) = \partial_0 h^i_0 - \frac{1}{2}\partial^i h_{00} \tag{10}$$

$$\Gamma^i_{0j} = \frac{1}{2}\eta^{ik}\left(\partial_0 h_{jk} + \partial_j h_{0k} - \partial_k h_{0j}\right) = \frac{1}{2}\partial_0 h^i_j + \frac{1}{2}\eta^{ik}\left(\partial_j h_{0k} - \partial_k h_{0j}\right) \tag{11}$$

In contrast to the lecture, we now have a time dependent metric perturbation. The time dependence arises via $\boldsymbol{r}_M(t)$, such that the time derivative can be rewritten as:

$$\frac{\partial h_{\mu\nu}}{\partial t} = -\boldsymbol{V} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} h_{\mu\nu}. \tag{12}$$

Hence we have for the equation of motion of the test mass m:

$$\begin{aligned}\frac{\mathrm{d}v_i}{\mathrm{d}t} &= -c^2 \partial_0 h_{i0} + \frac{c^2}{2}\partial_i h_{00} - c v^j \partial_0 h_{ij} - c v^j \left(\partial_j h_{0i} - \partial_i h_{0j}\right) \\ &= -c^2 \partial_0 h_{i0} + \frac{c^2}{2}\partial_i h_{00} - c v^j \partial_0 h_{ij} + 2\epsilon_{ikj}\Omega^k v^j\end{aligned} \tag{13}$$

Neglecting $\partial_0 h_{i0} = \mathcal{O}(V^2/c^2)$ we finally have:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -GM\frac{(\boldsymbol{r} - \boldsymbol{r}_M(t))}{|\boldsymbol{r} - \boldsymbol{r}_M|^3} - \frac{2GM}{c^2}\frac{(\boldsymbol{r} - \boldsymbol{r}_M)\cdot \boldsymbol{V}}{|\boldsymbol{r} - \boldsymbol{r}_M(t)|^3}\boldsymbol{v}. + 2\boldsymbol{\Omega} \times \boldsymbol{v} \tag{14}$$

We see that the gravitomagnetic forces due to moving masses are of order $\mathcal{O}(vV/c^2)$.

Exercise 2 [Particles in the field of a gravitational wave] (3 points)

We start from the equation of an ellipse centered on the origin in polar coordinates:

$$r(\varphi) = \frac{b}{\sqrt{1 - \epsilon^2 \cos^2(\varphi)}}. \tag{15}$$

If we assume $\epsilon \ll 1$, we have:

$$r^2(\varphi) = b^2\left(1 + \frac{\epsilon^2}{2}\big(1 + \cos(2\varphi)\big)\right). \tag{16}$$

Our goal is to bring this in the form of the distortion of a circle due to an incoming gravitational wave:

$$r^2(\varphi) = R^2\big(1 - 2h\cos(\omega t)\cos(2\varphi)\big), \tag{17}$$

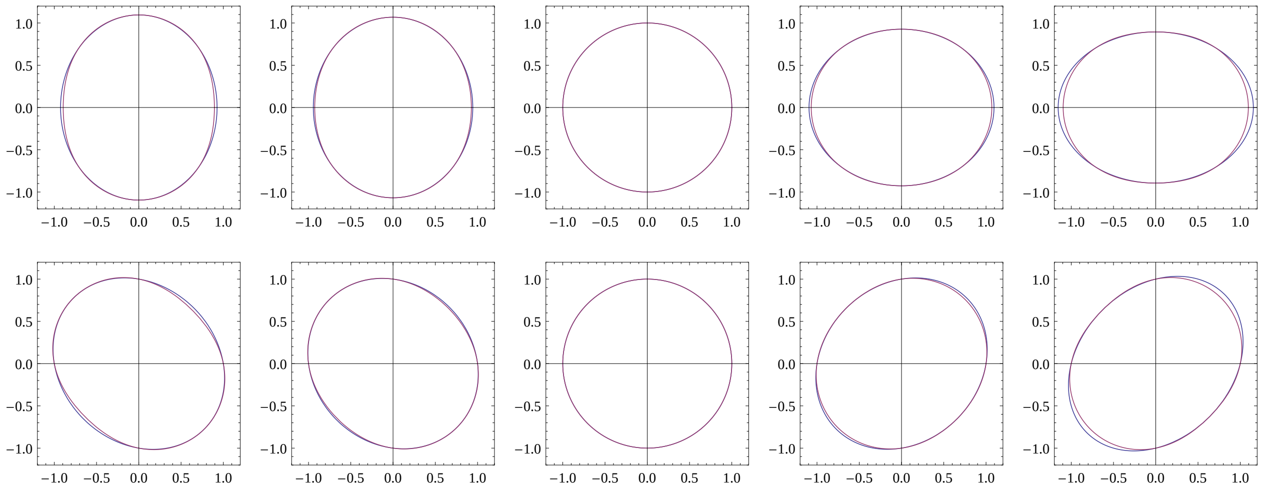

Figure 1: Time dependence of the elliptical distortion for $\omega t = 0, \pi/4, \pi/2, 3\pi/4, \pi$. *Top row:* h_+ polarization *Bottom row:* $h_\times$ polarization

which can be achieved by setting:

$$R^2 = b^2\left(1 + \frac{\epsilon^2}{2}\right) = \text{const.}, \qquad \epsilon^2 = -4h\cos(\omega t). \tag{18}$$

Using $b^2 = R^2(1 - \epsilon^2/2)$ and $b^2 - a^2 = a^2\epsilon^2$ we obtain:

$$a = R\big(1 - h\cos(\omega t)\big) \qquad b = R\big(1 + h\cos(\omega t)\big) \tag{19}$$

Equations (18) and (19) describe the time dependence of the ellipticity and the semi-minor and -major axis. This is shown in Figure 1. It might seem troublesome that for $-\pi/2 \leq \omega t \leq \pi/2$ we have $b > a$ and $\epsilon^2 < 0$. But this can be interpreted as a phase shift by $\pi/2$ which means that the ellipse is rotated by $90°$ with respect to its standard orientation. For the second polarization we have to add a phase factor of $-\pi/4$ to the ellipse in equation (15).

Exercise 3 [Gravitational Bremsstrahlung] (6 points)

The parabolic orbit of the mass m scattering on $M \gg m$ is described by the parametric form:

$$\boldsymbol{r}(\varphi) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2b}{1 + \cos(\varphi)} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}, \tag{20}$$

where the time dependence is described by:

$$\dot{\varphi} = \sqrt{\frac{GM}{8b^3}}\,[1 + \cos(\varphi)]^2 \tag{21}$$

Thus we have for the quadrupole tensor:

$$I_{ij}(t) = \int \mathrm{d}^3x'\,\rho(\boldsymbol{x}')x'_i x'_j = \frac{4b^2 m}{(1 + \cos(\varphi))^2} \begin{pmatrix} \cos^2(\varphi) & \sin(\varphi)\cos(\varphi) & 0 \\ \sin(\varphi)\cos(\varphi) & \sin^2(\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

where we have used $\rho(\boldsymbol{x}') = M\delta(\boldsymbol{x}') + m\delta\big(\boldsymbol{x}' - \boldsymbol{r}(\varphi)\big)$. Note that the angular position φ is not integrated over since it is given by the Newtonian equations of motion for m and φ represents just a parameter that substitutes time. Now we can use that for any function $f(\varphi)$ and $\dot\varphi = \dot\varphi(\varphi)$ we have:

$$\begin{aligned}\frac{\mathrm{d}^2 f(\varphi)}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2\varphi}{\mathrm{d}^2 t}\frac{\mathrm{d}f}{\mathrm{d}\varphi} + \frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = \\ &= \dot\varphi\frac{\mathrm{d}\dot\varphi}{\mathrm{d}\varphi}\frac{\mathrm{d}f}{\mathrm{d}\varphi} + \dot\varphi^2\frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2} = \\ &= \frac{1}{2}\frac{\mathrm{d}\dot\varphi^2}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}\varphi} + \dot\varphi^2\frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2}.\end{aligned} \tag{23}$$

With the above relation and Eq. (21) the second derivative of the quadrupole tensor simplifies to:

$$\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = \frac{GM}{8b^3}\left(-2\sin(\varphi)\big(1+\cos(\varphi)\big)^3\frac{\mathrm{d}I_{ij}}{\mathrm{d}\varphi} + \big(1+\cos(\varphi)\big)^4\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}\varphi^2}\right). \tag{24}$$

For the first and second term in the above equation we obtain:

$$\begin{aligned}&-2\sin(\varphi)\big(1+\cos(\varphi)\big)^3\frac{\mathrm{d}I_{ij}}{\mathrm{d}\varphi} = \\ &= 4b^2 m\begin{pmatrix} 4\sin^2(\varphi)\cos(\varphi) & -2\sin(\varphi)\left(\cos(2\varphi)+\cos(\varphi)\right) & 0 \\ -2\sin(\varphi)\left(\cos(2\varphi)+\cos(\varphi)\right) & -4\sin^2(\varphi)(\cos(\varphi)+1) & 0 \\ 0 & 0 & 0\end{pmatrix},\end{aligned} \tag{25}$$

and:

$$\begin{aligned}&\big(1+\cos(\varphi)\big)^4\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = \\ &4b^2 m\begin{pmatrix} 2\left[\cos(\varphi)\left(\cos^2(\varphi)-2\right)-\cos(2\varphi)\right] & 2\sin(\varphi)\left[\left(\cos^2(\varphi)-2\right)-\cos(\varphi)\right] & 0 \\ 2\sin(\varphi)\left[\left(\cos^2(\varphi)-2\right)-\cos(\varphi)\right] & 2\left(2+3\cos(\varphi)-\cos^3(\varphi)\right) & 0 \\ 0 & 0 & 0\end{pmatrix}.\end{aligned} \tag{26}$$

We have (with the right c):

$$\gamma_{ij} = \frac{2G^2 mM}{rbc^4}\begin{pmatrix} -\left(\cos(2\varphi)+\cos^3(\varphi)\right) & -\sin(\varphi)\left(1+\cos(\varphi)\right)^2 & 0 \\ -\sin(\varphi)\left(1+\cos(\varphi)\right)^2 & \cos(\varphi)\left(1+\cos(\varphi)\right)^2 & 0 \\ 0 & 0 & 0\end{pmatrix} \tag{27}$$

We immediately see that the metric perturbation in the trace reversed Lorentz gauge is not traceless. We then have:

$$\gamma = \eta^{ij}\gamma_{ij} = \left(\cos(2\varphi)+\cos^3(\varphi)\right) - \cos(\varphi)\left(1+\cos(\varphi)\right)^2, \tag{28}$$

and finally the result for h_{ij}:

$$\begin{aligned}&h_{ij} = \gamma_{ij} - \frac{1}{2}h\eta_{ij} = \\ &= \frac{G^2 mM}{rbc^4}\begin{pmatrix} 1-\cos(\varphi)-4\cos^2(\varphi)-2\cos^3(\varphi) & -2\sin(\varphi)\left(1+\cos(\varphi)\right)^2 & 0 \\ -2\sin(\varphi)\left(1+\cos(\varphi)\right)^2 & -\left(1-\cos(\varphi)-4\cos^2(\varphi)-2\cos^3(\varphi)\right) & 0 \\ 0 & 0 & 0\end{pmatrix}\end{aligned} \tag{29}$$

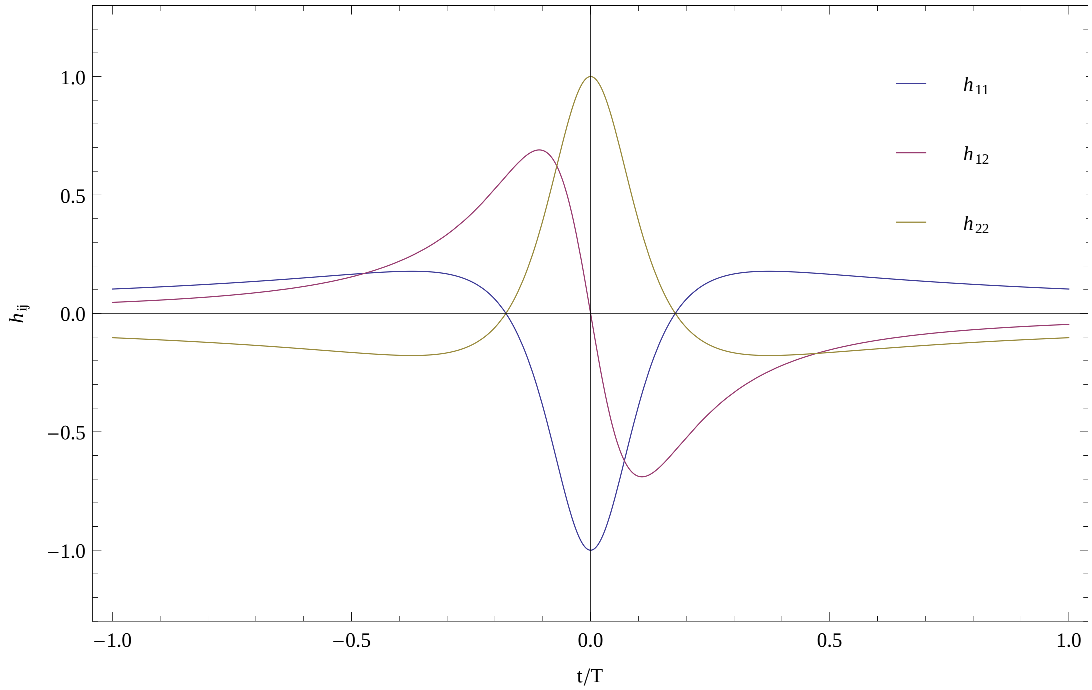

Figure 2: Time dependence of the h_{11}, h_{22} and h_{12} components normalized to $h_{11}(\varphi = 0)$. The time is expressed in terms of $T_{\rm o}$

Figure 2 shows the evolution with time of the components of h_{ij}. The timescale of the transit is given by the orbital time $T_{\rm o} = 2\pi\sqrt{\frac{b^3}{GM}}$, whereas the amplitude scales as $T_{\rm gw}^2/T_{\rm o}^2$, where we defined:

$$T_{\rm gw} = \sqrt{\frac{Gmb^2}{rc^4}}, \tag{30}$$

and the time dependence of the orbit was obtained solving the implicit equation for $\varphi(t)$:

$$t(\varphi) = \sqrt{\frac{2b^3}{GM}}\left(\tan\left(\frac{\varphi}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\varphi}{2}\right)\right). \tag{31}$$