uzh | eth | zürich

Applications of General Relativity
Sheet 2

Spring 2017
Prof. P. Jetzer

Y. Boetzel, M. Haney
Website: `http://www.physik.uzh.ch/en/teaching/PHY519/`

Issued: 03.03.2017
Due: 10.03.2017

Exercise 1 [Time delay in gravitational lensing and FERMATs principle]
In this exercise we will rederive the deflection of light and the corresponding time delay. In contrast to the lecture, we will consider extended mass distributions. On the way you will become familiar with some standard concepts of gravitational lensing.

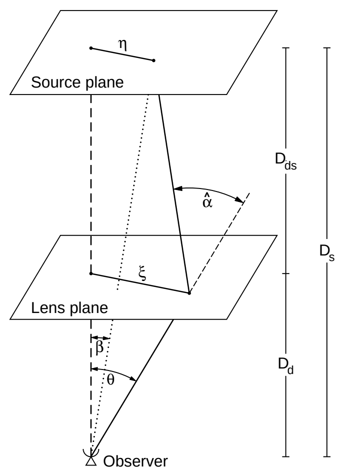

Figure 1: Configuration of the source-lens-observer system.

In the weak field limit the metric can be decomposed into a background MINKOWSKI metric $\eta_{\mu\nu}$ and a small perturbation $|h_{\mu\nu}| \ll 1$ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In contrast to the NEWTONian limit, the weak field approximation can be applied to describe relativistic test particles. The perturbed metric in the transverse gauge reads as

$$\mathrm{d}s^2 = \left(1 + 2\frac{\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - 2\frac{\Phi}{c^2}\right) \mathrm{d}\boldsymbol{x}^2. \tag{1}$$

For slowly moving, quasistatic $|\boldsymbol{v}| \ll c$ perfect fluid sources the energy momentum tensor in the fluid restframe reads as $T_{\mu\nu} = \rho u_\mu u_\nu$. Hence, the metric perturbation obeys the usual POISSON equation $\boldsymbol{\nabla}^2\Phi = 4\pi G\rho$ and can thus be calculated using the usual GREEN function of the POISSON equation

$$\Phi(\boldsymbol{r}) = -G \int \mathrm{d}^3 x \frac{\rho(\boldsymbol{x})}{|\boldsymbol{r} - \boldsymbol{x}|}, \tag{2}$$

where we have neglected retardation effects. We are now considering photon geodesics traversing this metric. Since the total deflection is usually small, the deflection angle can be calculated by an integration along the unperturbed lightpath

$$\hat{\boldsymbol{\alpha}} = \frac{2}{c^2} \int_{\mathrm{s}}^{\mathrm{o}} \mathrm{d}z \, \boldsymbol{\nabla}_\perp \Phi. \tag{3}$$

We will make use of the *thin lens approximation*: the mass distribution causing the deflection has negligible extent along the line of sight and can thus be described by a surface mass density $\rho(\boldsymbol{x}) = \Sigma(\boldsymbol{\xi})\delta^{(\mathrm{D})}(z)$ in the lens plane. In the thin lens approximation the photons emerging from a source at position $\boldsymbol{\eta}$ in the source plane can be associated with an image at position $\boldsymbol{\xi}$ in the lens plane (see Fig. 1 for the configuration).

(i) Show that the extended mass distribution in the lens plane leads to a total deflection angle

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2} \int \mathrm{d}^2\xi' \frac{\boldsymbol{\xi} - \boldsymbol{\xi}'}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2} \Sigma(\boldsymbol{\xi}'). \tag{4}$$

(ii) Derive the *lensing potential* $\psi(\boldsymbol{\xi})$ defined by

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \boldsymbol{\nabla}_{\boldsymbol{\xi}} \psi(\boldsymbol{\xi}). \tag{5}$$

(iii) Show that the Shapiro delay can be written as

$$c\,\delta t = \frac{D_\mathrm{s} D_\mathrm{d}}{2 D_\mathrm{ds}} \left(\frac{\boldsymbol{\xi}}{D_\mathrm{d}} - \frac{\boldsymbol{\eta}}{D_\mathrm{s}} \right)^2 - \psi(\boldsymbol{\xi}) + \text{const.}, \tag{6}$$

where the constant is the same for all rays.

(iv) Fermats principle asserts that for actual light paths the arrival time is stationary under variations of the deflection point $\boldsymbol{\xi}$, leading to the constraint $\boldsymbol{\nabla}_{\boldsymbol{\xi}} \delta t = 0$. Use this principle to derive the *lens equation*

$$\boldsymbol{\eta} = \frac{D_\mathrm{s}}{D_\mathrm{d}} \boldsymbol{\xi} - D_\mathrm{ds} \hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}). \tag{7}$$

Exercise 2 [$f(R)$ modified gravity]
In this Exercise we will derive the Friedmann equations governing the evolution of a Universe described by the $f(R)$ modified action theory of gravity. For simplicity we will set $c \equiv 1$.

(i) Consider a gravity theory with metric $g_{\mu\nu}$ and action

$$S = \frac{1}{16\pi} \int f(R) \sqrt{-g}\, d^4x, \tag{8}$$

where $f(R)$ is an arbitrary function of the scalar curvature R. By varying the action, derive the following equations of motion:

$$f_{,R} R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu f + (f_{,R})^{;\alpha}_{;\alpha} \delta^\mu_\nu - (f_{,R})^{;\mu}_{;\nu} = 0\ , \tag{9}$$

where $f_{,R} \equiv \frac{\partial f}{\partial R}$.

(ii) Derive the modified Friedmann equations for $f(R)$ gravity. Start with the action given in Equation (8), this time including the matter Lagrangian, and derive the equations of motion.

You may assume a flat Universe with a FRW-metric $g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$ in cartesian coordinates $x^\mu = (t, x, y, z)$.

Expressing your results in terms of the function $F(R)$, where $f(R) = R + F(R)$ you should obtain

$$H^2 + \frac{F}{6} - \frac{\ddot{a}}{a}F_{,R} + H\dot{F}_{,R} = \frac{8\pi\rho}{3} \tag{10}$$

and

$$\frac{\ddot{a}}{a} + \frac{1}{6}F - H^2 F_{,R} + \frac{1}{2}H\dot{F}_{,R} + \frac{1}{2}\ddot{F}_{,R} = -4\pi(p + \frac{\rho}{3}). \tag{11}$$