Effective Field Theories for Particle Physics

Lecture Notes, UZH / ETHZ, Fall Semester 2021

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February 11, 2022

Abstract

This course provides an introduction to effective field theories (EFTs) and dispersion theory in particle physics. We will start by introducing the core concepts of constructing EFTs and apply them to the low-energy description of the weak interaction and the effective description of heavy physics beyond the Standard Model (SM). We will discuss chiral perturbation theory (χPT), the low-energy effective theory of quantum chromodynamics (QCD). We will briefly discuss the application of this concept to describe a class of theories beyond the SM in which the Higgs particle arises as a composite state of a new confining sector.

In addition, the course provides an introduction to the concepts of dispersion theory and its interplay with EFTs. We will discuss how to make use of the constraints from unitarity of the S-matrix and analyticity of scattering amplitudes, in order to extend the range of validity of the theoretical description compared to pure EFT methods.

Contents

Organization

Dates, contacts

- Lecture: Tuesday, 13:45–15:30, room HCI H 8.1 at ETH
- Exercises: Tuesday, 15:45–16:30, room HCI H 8.1 at ETH
- Lecturer: Peter Stoffer, stoffer@physik.uzh.ch, office Y36 H68 and PSI
- Teaching assistants:
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- Lecture website: `https://www.physik.uzh.ch/en/teaching/PHY578.html`

Schedule

date	subject	exercise sheet
21.09.2021	Introduction, general principles	Sheet 1
28.09.2021	Operator bases, EOM	Sheet 1
05.10.2021	Renormalization, running, tree-level matching	Sheet 2
12.10.2021	One-loop matching	Sheet 3
19.10.2021	Fermi theory, Fierz relations	Sheet 4
26.10.2021	Operator mixing, schemes and evanescents	Sheet 5
02.11.2021	Chiral symmetry, SSB, Goldstone's theorem	Sheet 6
09.11.2021	CCWZ, explicit symmetry breaking, external fields	Sheet 7
16.11.2021	Chiral power counting, mesonic Lagrangian, χPT at NLO	Sheet 8
23.11.2021	Loops in chiral perturbation theory	Sheet 9
30.11.2021	SMEFT	Sheet 10
07.12.2021	EWχL/HEFT	Sheet 11
14.12.2021	S-matrix theory	Sheet 12
21.12.2021	Dispersion relations and χPT, constraining EFT parameters	Q&A

Lecture notes and literature

These lecture notes partially draw on the QFTIII lecture notes by Gino Isidori (lectures held in 2016 and 2019) [1], the Les Houches lecture notes by Aneesh Manohar [2], and EFT lecture notes by Thomas Becher and Martin Hoferichter [3]. Many more EFT lectures notes are available on arXiv—suggested references are [4–9].

1 Introduction

The Standard Model (SM) of particle physics is an extremely successful theory making accurate predictions over a wide range of energies—from phenomena at the eV scale as in atomic physics to high-energy processes in colliders at scales of hundreds of GeV. The spectrum of the SM contains particles with very different masses: e.g., the up quark has a mass of about $m_u = 2.2\,\text{MeV}$ (in $\overline{\text{MS}}$ at $\mu = 2\,\text{GeV}$), while the $\overline{\text{MS}}$ mass of the top quark is about $162.5\,\text{GeV}$. In order to make accurate predictions of processes at very low energies, we do not need to take into account the entire SM, but it is sufficient to work only with the light particles and to use quantum electrodynamics (QED) and quantum chromodynamics (QCD), considering the effects of the heavy electroweak particles as a perturbative correction. This approach is an example of an effective field theory (EFT).

The concept is very general: consider an apple falling from a tree. In order to compute its velocity when hitting the ground, we will simply use

$$mgh = \frac{mv^2}{2} \quad \Rightarrow \quad v = \sqrt{2gh}\,. \tag{1.1}$$

Of course, the gravitational potential is not linear in h. However, the height of the tree is small compared to the radius of the earth, which is the scale over which the gravitational force changes. Therefore the above result is accurate up to corrections of $\mathcal{O}(h/R)$, which is $\sim 10^{-6}$ for a typical apple tree on earth. The linear gravitational potential can be regarded as an effective theory for the more complete Newtonian theory of gravity, which in turn is an effective theory of General Relativity.

EFTs are general quantum field theories (QFTs) based on quantum mechanics and relativity but without the restriction of renormalizability (in the traditional sense). They are valid only in a certain energy range and achieve an effective description of a more fundamental underlying QFT. EFTs are widely used in modern particle physics, for several reasons. The full theory will typically involve several mass scales. The EFT approach allows to split the calculation into pieces involving fewer mass scales, thereby simplifying calculations significantly. The calculation in the full theory will lead to logarithms of ratios of the different mass scales, $\log(m^2/M^2)$. If m is much smaller than M, the logarithm will become a large number. These logs accompany the expansion parameters of perturbation theory (the interaction couplings of the theory)—if the logs become too large, they spoil the convergence of the perturbative expansion. As we will see, EFTs provide a means to resum these large logs, leading to an improved perturbative expansion. Another reason for using EFTs might be that the fundamental theory cannot be solved easily, such as QCD at low energies. In some cases the underlying theory is not known at all: historically, the Fermi theory was known long before the theory of electroweak interaction; EFT methods were applied to hadronic processes already in the sixties before QCD was developed. In both cases, EFTs helped to develop the underlying theory. Today, we may regard the SM as an EFT of an even more general theory of particle physics.

2 General concepts of EFTs

2.1 Principles of EFTs

In general, the construction of an EFT is based on the following three core principles.

1. **Degrees of freedom**

   In a first step, we need to determine the degrees of freedom that are relevant to describe the physical system we are interested in. These degrees of freedom are the building blocks of the Lagrangian and the effective action. In the simplest example, we keep the fields that are describing light particles, while dropping the fields that represent heavy particles. However, the determination of the relevant degrees of freedom can be more involved. An EFT description of strong interaction at low energies is based on hadronic degrees of freedom, which are completely different from the fields in the QCD Lagrangian, the quarks and massless gluons. This leads to chiral perturbation theory (χPT). Other examples with non-trivial field content are heavy quark effective theory (HQET) [10, 11] and soft-collinear effective theory (SCET) [12].

2. **Symmetries**

   In a second step, it is crucial to know the symmetries of the problem at hand, which constrain possible interactions. In case that the underlying theory is not known, one will write down all possible interactions that are compatible with the assumed symmetries. Many different types of symmetries may occur: global symmetries, gauged symmetries, accidental ones, symmetries that are broken spontaneously or anomalously. In the case of χPT, chiral symmetry of the QCD Lagrangian is essential in the construction of the Lagrangian of the EFT, which consists of different fields but respects the same symmetries. Sometimes, certain symmetries only hold in the limit that is studied with the EFT. In such a case, the symmetry is hidden in the full theory, but it becomes manifest in the EFT. An example of such a symmetry emerging in the expansion is heavy-quark spin symmetry in HQET. Other subtleties can happen with symmetries, as in χPT, where the usual mesonic Lagrangian turns out to possess an additional symmetry that is not present in QCD—the effect of anomalies in QCD needs to be addressed separately.

3. **Power counting**

   An EFT turns a certain type of approximation into a systematic expansion. In order to do so, we need to determine the parametric limit that we are considering: what is the parameter that we consider as small? E.g., this can be a light mass, a low velocity or similar. In the end, we need to determine a dimensionless parameter that describes what is small and we perform an expansion in powers of this small dimensionless quantity. This could be the ratio of a light to a heavy mass or the ratio of a low velocity to the speed of light. The counting of powers of the small parameter is the systematic expansion of the EFT. We apply it to fields and the different terms of the Lagrangian, as well as to the amplitudes. Observing consistently the power counting is as important as the counting in terms of coupling constants in a perturbative expansion. It allows us to handle the effective action containing an infinite number of terms, because it guarantees that only a finite number of them enters at each order in the power counting.

EFTs are full-fledged QFTs and it is important to note that one can perform calculations with them without knowledge of the underlying fundamental theory—this is one of the main advantages of EFTs. There is an aspect of EFTs that at first sight might seem confusing: they involve an infinite number of interaction terms that are non-renormalizable in the traditional sense. However, EFTs *are* renormalizable order by order in the power counting and they are as consistent as renormalizable theories as long as one applies them in their domain of validity and is interested only in corrections of finite order in the expansion parameter.

Sometimes, the literature distinguishes between top-down and bottom-up EFTs. The top-down perspective usually means that one knows the underlying theory and one is able to perturbatively calculate the parameters of the EFT in terms of the parameters of the fundamental theory, a procedure called *matching*. The bottom-up perspective applies if the underlying theory is unknown or if the parameters of the EFT encode non-perturbative dynamics (such as in χPT). This distinction is rather something about the perspective of the physicist than a property of the EFT and it can change over time.

When speaking about the high-energy or short-distance region, we often use the term *ultraviolet* or UV, while the low-energy or long-distance region is called *infrared* or IR. Alternatively, UV and IR are also called *hard* and *soft*.

Within its domain of validity, the EFT approximates the full theory in a systematic expansion. Consider the generating functional for Green's functions with an external sources J_i coupled to some operators $\mathcal{O}_i$:

$$Z[J] = \int \mathcal{D}\Phi \exp\left(iS_{\rm UV}[\Phi] + i \int d^D x \, J_i(x) \mathcal{O}_i[\Phi(x)] \right) . \tag{2.1}$$

Depending on the context, the operators $\mathcal{O}_i$ could denote light elementary fields, or some composite operator, such as quark currents. Green's functions of these operators are obtained by taking functional derivatives:

$$\langle 0 | T\{\mathcal{O}_1(x_1) \ldots \mathcal{O}_m(x_m)\} | 0 \rangle = \frac{1}{Z[0]} \frac{-i\delta}{\delta J_1(x_1)} \cdots \frac{-i\delta}{\delta J_m(x_m)} Z[J] \bigg|_{J=0} . \tag{2.2}$$

For the EFT to be an approximation of the full theory, we require

$$Z[J] = Z_n[J] + \mathcal{O}(\delta^{n+1}) = \int \mathcal{D}\phi \exp\left(iS_n^{\rm eff}[\phi, J] \right) + \mathcal{O}(\delta^{n+1}) \tag{2.3}$$

at the order n in the expansion in the small parameter δ. This is the matching condition, requiring that 1PI Green's functions in the EFT agree with the corresponding ones in the fundamental theory up to the considered order in the expansion. (In fact, a weaker matching condition is sufficient: it is enough to require that on-shell S-matrix elements agree between the two theories.) The fields ϕ in the EFT are different from the fields Φ in the UV theory. In special cases, the degrees of freedom in the EFT can be identified with a subset of fields or Fourier modes of the fields in the UV theory. In this case, one can split up the functional integral into an integral over soft and hard modes:

$$Z[J] = \int \mathcal{D}\Phi_S \mathcal{D}\Phi_H \exp\left(iS_{\rm UV}[\Phi_S, \Phi_H] + i \int d^D x \, J_i(x) \mathcal{O}_i[\Phi_S(x)] \right) . \tag{2.4}$$

We are interested only in Green's functions of operators that depend on the soft modes. Hence, one can define the Wilsonian effective action

$$\int \mathcal{D}\Phi_H \exp\left(iS_{\rm UV}[\Phi_S, \Phi_H] \right) =: \exp\left(iS[\Phi_S] \right) , \tag{2.5}$$

which is obtained by performing the path integral over the hard modes—they are *integrated out*. The resulting quantity encodes interactions at the hard scale and therefore is non-local at the short-distance length scale $\Delta x = 1/\Lambda$, where Λ is the hard momentum scale. However, given that we are interested in soft processes only, corresponding to wave-lengths much larger than Δx, the non-local hard interactions can be expanded around $\Delta x = 0$. After the expansion, these interaction terms have the form of local operators. This expansion of the non-local hard interaction into a tower of local operators goes under the name of *operator-product expansion* (OPE) and amounts to writing the effective action in terms of an effective Lagrangian that contains an infinite tower of local operators:

$$S^{\rm EFT} = \int d^D x \, \mathcal{L}_{\rm EFT}(x) , \quad \mathcal{L}_{\rm EFT}(x) = \sum_i C_i(\mu) \mathcal{O}_i(x) . \tag{2.6}$$

The coefficients $C_i(\mu)$ are known as *Wilson coefficients* and depend on the renormalization scale μ.

Integrating out certain degrees of freedom directly in the path integral is not always possible and sometimes it is not the appropriate way of thinking. In the case of low-energy QCD, the matching to χPT is non-perturbative and does not consist of integrating out a part of the field content. Even when the EFT contains degrees of freedom that are already described by the UV theory and often denoted by the same symbol, the fields in the EFT and in the UV theory are different, e.g., they have a different UV structure, as the EFT is only designed to reproduce the IR singularity structure of the full theory.

In order to get familiar with the application of all these concept, in the following we will study a very simple toy example of an EFT. This will allow us to learn the basic techniques, before we study the realistic but somewhat more complicated cases, such as the Fermi theory, χPT, and SMEFT.

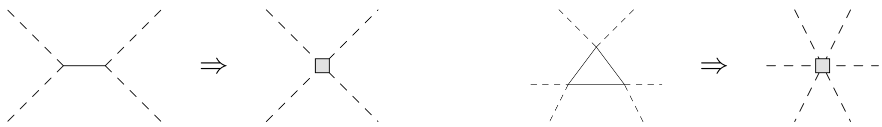


Figure 1: Examples of contributions to light-particle processes due to tree-level and one-loop effects of the heavy particle: propagators of the light (heavy) particle are denoted by dashed (solid) lines. In the EFT, these contributions are described in terms of local operators.

2.2 A scalar toy model

Consider a toy model of a QFT with two real scalar fields [13]:

$$\begin{aligned}\mathcal{L}_{\mathrm{UV}} = &\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\phi^2 - \frac{M^2}{2}\Phi^2 \\ &- \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2\Phi - \frac{\lambda_2}{4}\phi^2\Phi^2 - \frac{\lambda_3}{3!}M\Phi^3 - \frac{\lambda_4}{4!}\Phi^4 \\ &+ \mathcal{L}_{\mathrm{ct}}\,. \end{aligned} \tag{2.7}$$

We are assuming that there is a $\mathbb{Z}_2$ symmetry $\phi \mapsto -\phi$, which only permits terms of even power in ϕ.[1] The Lagrangian is already written in terms of renormalized fields and parameters and $\mathcal{L}_{\mathrm{ct}}$ is the counterterm Lagrangian.

Let us now assume that $M \gg m$ and that we are considering processes in a restricted energy range well below the production threshold of the heavy particle Φ. In this case, we can only observe processes with light scalar particles as external states. Furthermore, we are considering the amplitudes in a kinematic region well below the poles and cuts due to the heavy particle. This means that virtual effects due to the heavy scalar can be systematically approximated by local operators, as illustrated in Fig. 1. The low-energy scattering processes of the light scalar particle can be described in terms of an EFT containing only the light scalar as degree of freedom.

Let us consider the three EFT core principles in this particular example.

1. *Degrees of freedom*: at low energies, only one type of particles can be produced, which in the UV theory is described by the field ϕ. In the EFT, we only need one scalar field to describe the light particle type—for simplicity, we denote it by the same symbol ϕ. Note, however, that the meaning of the field changes when going to the EFT.

2. *Symmetries*: the UV theory is assumed to be invariant under the $\mathbb{Z}_2$ symmetry $\phi \mapsto -\phi$. This symmetry will directly translate to the EFT, which will only contain operators with an even power of fields ϕ. The EFT Lagrangian will contain all possible operators fulfilling this constraint: as we will see, the presence of one operator of mass dimension larger than 4 will automatically induce an infinite tower of operators. The EFT Lagrangian (in 4 space-time dimensions) will have the form

$$\mathcal{L}_{\mathrm{EFT}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\phi^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{C_6}{\Lambda^2}\frac{\phi^6}{6!} + \ldots\,, \tag{2.8}$$

where the ellipsis stands for the infinite tower of all Lorentz-scalar operators with an even number of fields ϕ. We have introduced a parameter Λ of mass dimension $[\Lambda] = 1$: appropriate powers of Λ are used to make the Wilson coefficients C_i of all the effective operators dimensionless.

[1] One often introduces a term linear in the field, $Y\Phi$, which can be used to cancel tadpoles and to obtain a vanishing vacuum expectation value (vev) $\langle 0|\Phi(x)|0\rangle = 0$. Since this renormalization condition for a UV counterterm depends on IR scales, we refrain from doing this and we will use modified minimal subtraction ($\overline{\mathrm{MS}}$) throughout. The consequence is that tadpole diagrams need to be included explicitly. A linear term $Y\Phi$, generated through a shift in Φ, can still be used to absorb the UV divergence of the one-point function. Alternatively, instead of using a linear counterterm, mass renormalization is enough to render all S-matrix elements finite, leaving the one-point function $\langle 0|\Phi(x)|0\rangle$ divergent.

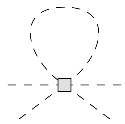


Figure 2: One-loop contribution to $\phi\phi \to \phi\phi$ scattering due to a ϕ^6 operator. The loop graph is quadratically divergent in a cutoff scheme.

3. *Power counting*: in order to make sense of the EFT, we need a systematic way to handle the infinite tower of operators and to apply an ordering principle to them. In our example, the parameter Λ will be used for this task. Consider an amplitude $\mathcal{A}$ that is normalized to be dimensionless. From diagrams with an insertion of an effective operator of mass dimension d, we will obtain a contribution to the amplitude in $D = 4$ space-time dimensions of order

$$\mathcal{A} \propto \left(\frac{p}{\Lambda}\right)^{d-4} , \tag{2.9}$$

which follows from dimensional analysis: the operator has a coefficient $\propto 1/\Lambda^{d-4}$ and in order to obtain a dimensionless amplitude, a factor of p^{d-4} will be provided by kinematic factors, made of external momenta or mass factors. The EFT expansion is now implemented by considering the ratio $\delta = p/\Lambda$ as the EFT expansion parameter: we are interested in processes, where the external momenta as well as the mass of the light particle are small compared to the scale Λ, which denotes the UV scale, where the underlying dynamics starts to show up. It is related to the mass M of the heavy particle, but we should consider it as an auxiliary parameter: note that the EFT does not depend on the Wilson coefficients C_d and Λ separately, but only on the products $C_d\Lambda^{4-d}$.

Tree-level diagrams with insertions of a set of operators will scale as

$$\mathcal{A} \propto \delta^n , \quad \delta = \frac{p}{\Lambda} , \quad n = \sum_i (d_i - 4) . \tag{2.10}$$

An important point will be to understand why the power-counting formula Eq. (2.10) holds in general, i.e., not only for tree-level diagrams, but also at loop level. As soon as we have established this, it is clear how the power counting allows us to organize the EFT calculation: the leading contributions are given by graphs with vertices from the $d \leq 4$ Lagrangian, i.e., the renormalizable part. Graphs with an insertion of a dimension $d = 6$ operator scale as δ^2. Graphs with two insertions of dimension-6 operators scale as δ^4, as do graphs with a single insertion of a dimension-8 operator. The EFT contains an infinite number of operators and parameters (Wilson coefficients), but to a fixed order in δ, only a finite number of them contributes.

The validity of Eq. (2.10) is not obvious at the loop level, since one has to integrate over all loop momenta $-\infty < \ell < \infty$, including regions where ℓ/Λ is not small. If we were using a naive cutoff Λ_c for the loop integrals that is of the order Λ, the power-counting formula would indeed break down. Consider the graph shown in Fig. 2, which contributes at one loop to $\phi\phi \to \phi\phi$ scattering due to an insertion of the ϕ^6 operator. In $D = 4$ dimensions and with a cutoff Λ_c, we would calculate the integral

$$i\mathcal{A} = i\frac{C_6}{2\Lambda^2} \int_{\Lambda_c} \frac{d^4\ell}{(2\pi)^4} \frac{i}{\ell^2 - m_\phi^2} \approx i\frac{C_6}{32\pi^2} \frac{\Lambda_c^2}{\Lambda^2} , \tag{2.11}$$

neglecting the mass m_ϕ. This is a contribution of $\mathcal{O}(\delta^0)$ instead of $\mathcal{O}(\delta^2)$, violating the power counting. However, the right conclusion is not that the power counting breaks down and EFTs do not work, but rather that we are using a bad regulator: hard cutoffs come with all sorts of problems, they violate gauge and chiral symmetry, and they are not used in connection with EFTs. Instead, the problems are avoided by using dimensional regularization (DR). In DR with $D = 4-2\varepsilon$ space-time dimensions, the graph in Fig. 2 is given by

$$i\mathcal{A} = i\mu^{4\varepsilon}\frac{C_6}{2\Lambda^2}\int \frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2 - m_\phi^2}\,, \tag{2.12}$$

where we have introduced a renormalization scale μ to keep C_6 dimensionless away from $D = 4$ space-time dimensions (the amplitude now has dimension 2ε). Using standard techniques for DR, we obtain

$$\begin{aligned}\frac{\mathcal{A}}{\mu^{2\varepsilon}} &= \mu^{2\varepsilon}\frac{C_6}{\Lambda^2}\frac{m_\phi^{D-2}}{2^{D+1}\pi^{D/2}}\Gamma\left(1-\frac{D}{2}\right)\\ &= -\frac{C_6}{32\pi^2}\frac{m_\phi^2}{\Lambda^2}\left[\frac{1}{\varepsilon} + \log\left(\frac{\bar\mu^2}{m_\phi^2}\right) + 1 + \mathcal{O}(\varepsilon)\right]\,, \end{aligned}\tag{2.13}$$

where the $\overline{\text{MS}}$ scale $\bar\mu$ is defined by

$$\mu^2 = \bar\mu^2\frac{e^{\gamma_E}}{4\pi}\,, \tag{2.14}$$

with the Euler–Mascheroni constant γ_E. There are a few important points to note.

- The divergences of the loop graphs show up as poles in $1/\varepsilon$. They can be reabsorbed by local counterterms.
- The $\overline{\text{MS}}$ renormalization scale $\bar\mu$ only enters through logarithms, which are generated by expanding μ^ε terms around $\varepsilon = 0$.
- The dependence on Λ only comes from the insertion of effective vertices. Therefore, Λ only appears with negative powers—the loops do not generate factors of Λ in the numerator. The only scales in the problem that can make the result dimensionless are the IR scales of the EFT. Hence, the loop graphs obey the same power-counting formula (2.10) as tree-level graphs. In our case, the loop is or order $m_\phi^2/\Lambda^2 = \delta^2$.

Also remember that in DR, scaleless integrals vanish,

$$I_a = \mu^{2\varepsilon}\int\frac{d^D\ell}{(2\pi)^D}(\ell^2)^a = 0\,. \tag{2.15}$$

This can be shown in different ways. E.g., split the integral into two pieces:

$$\begin{aligned}I_a &= i(-1)^a\frac{\mu^{2\varepsilon}}{(2\pi)^D}\Omega_D\int_0^\infty d\ell_E \ell_E^{D-1+2a}\\ &= i\frac{\mu^{2\varepsilon}}{(4\pi)^{D/2}}\frac{2(-1)^a}{\Gamma(D/2)}\left[\int_0^\Lambda d\ell_E\ell_E^{D-1+2a} + \int_\Lambda^\infty d\ell_E\ell_E^{D-1+2a}\right]\\ &= i\frac{\mu^{2\varepsilon}}{(4\pi)^{D/2}}\frac{2(-1)^a}{\Gamma(D/2)}\left[\frac{\Lambda^{2a+D}}{2a+D} - \frac{\Lambda^{2a+D}}{2a+D}\right] = 0. \end{aligned}\tag{2.16}$$

In the first term, we have assumed $D+2a > 0$, whereas in the second term, we have assumed $D+2a < 0$. Both terms are then continued analytically to the same D. For $a = -2$, both terms are separately divergent. Sometimes, it is useful to keep track of the two divergences by denoting $D = 4-2\varepsilon_{\text{UV}}$ for $\varepsilon_{\text{UV}} > 0$ and $D = 4-2\varepsilon_{\text{IR}}$ for $\varepsilon_{\text{IR}} < 0$:

$$I_{-2} = i\frac{\mu^{2\varepsilon}}{(4\pi)^{D/2}}\frac{1}{\Gamma(D/2)}\left[-\frac{1}{\varepsilon_{\text{IR}}} + \frac{1}{\varepsilon_{\text{UV}}}\right] = 0\,, \tag{2.17}$$

by analytic continuation to $\varepsilon_{\text{UV}} = \varepsilon_{\text{IR}} = \varepsilon$. For power-law divergent scaleless integrals, the limits $D \to 4$ of both terms exist and the integral vanishes without producing any $1/\varepsilon$ poles.

2.3 Constructing the EFT

2.3.1 Operator basis

Having established the power counting of the EFT, we see that in our scalar toy example, the operators are simply ordered by their canonical mass dimension, so that the EFT Lagrangian consists of an infinite tower of operators with increasing mass dimension. The effective operators are local operators built from the scalar field ϕ and partial derivatives. The symmetry under $\phi \mapsto -\phi$ only permits even powers of ϕ. Lorentz invariance also requires an even number of derivatives, which are contracted pairwise. Therefore, the Lagrangian only contains operators of even mass dimension and it can be written as

$$\mathcal{L}_{\mathrm{EFT}} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m_\phi^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \sum_{\substack{d\geq 6 \\ d \text{ even}}} \sum_{i=1}^{n_d} \frac{C_i^{(d)}}{\Lambda^{d-4}} \mathcal{O}_i^{(d)} \,, \tag{2.18}$$

where n_d denotes the number of operators with mass dimension d. (In renormalized perturbation theory, there will also be a counterterm Lagrangian and explicit factors of the renormalization scale μ, which are introduced to make the quartic coupling λ and the Wilson coefficients $C_i^{(d)}$ dimensionless also away from $D = 4$ space-time dimensions.)

The EFT allows us to work at a given accuracy in the expansion in powers of $\delta = p/\Lambda$. Only a finite number of terms in the Lagrangian contribute to the amplitudes at a fixed order in δ. Observables therefore only depend on a finite number of parameters, the mass m_ϕ, the quartic coupling λ, as well as the Wilson coefficients $C_i^{(d)}/\Lambda^{d-4}$ up to a fixed dimension d. A finite number of observations is enough to determine these parameters—afterwards, the EFT predicts any other observable to a given accuracy. It is possible to use the EFT as a stand-alone field theory without reference to the underlying UV theory. Alternatively, if the UV theory is known, the parameters in the EFT can be related to the parameters of the fundamental theory, as will be discussed in Sect. 2.5. In either case, it is important that one knows the complete set of effective operators up to the desired order in the expansion, or at least all the operators that can contribute to the observables under consideration. Therefore, the first step in constructing the EFT consists in listing the set of operators. Ideally, one works with an *operator basis*, i.e., a set that is not only complete but also non-redundant. Otherwise, one is introducing additional parameters into the problem that cannot be determined, i.e., only certain linear combinations of the redundant parameters are fixed. Redundancies in the set of operators are not always immediately obvious. Often they are related to operators that do not contribute to observables: these are total-derivative operators and operators that vanish by the classical equations of motion (EOM). Total derivatives do not contribute to observables as we integrate over space-time to get the action and assume the fields to vanish at infinity.[2] The reason why operators that vanish by the classical EOM can be removed from the operator basis is a little more subtle and is due to the fact that we can perform non-linear field redefinitions, as we will discuss below.

Constructing the set of operators to a certain order in the power counting can be done by listing by hand an exhaustive set of operators that are compatible with the symmetries and afterwards removing total derivatives and EOM operators. For more complicated theories and/or to higher powers in the EFT expansion, this can become a daunting task. Using a method from invariant theory called Hilbert series, at least the counting of independent operators at a given order in the expansion can be automated [14–16], but this goes beyond the scope of this course.

Considering our scalar toy model of an EFT, we can easily list a complete set of operators at dimension six. The building blocks are the field ϕ and an even number of partial derivatives ∂^μ.

[2] Exceptions, where total derivatives are relevant concern topological effects, such as the QCD theta term.

Directly omitting operators that are total derivatives (e.g. $\Box(\phi^4)$), we find

$$\begin{aligned} \text{class } \phi^6 : &\quad \phi^6\,, \\ \text{class } \partial^2\phi^4 : &\quad \phi^3\Box\phi\,, \quad \phi^2(\partial_\mu\phi)(\partial^\mu\phi)\,, \\ \text{class } \partial^4\phi^2 : &\quad \phi\Box^2\phi\,, \quad (\partial_\mu\phi)(\partial^\mu\Box\phi)\,, \quad (\Box\phi)^2\,, \quad (\partial_\mu\partial_\nu\phi)(\partial^\mu\partial^\nu\phi)\,. \end{aligned} \tag{2.19}$$

The omission of total-derivative operators implies that we can make use of integration by parts, e.g.,

$$\partial_\mu(\phi^3\partial^\mu\phi) = 3\phi^2(\partial_\mu\phi)(\partial^\mu\phi) + \phi^3\Box\phi\,, \tag{2.20}$$

which means that all but the first operator in each line of Eq. (2.19) can be removed. The intermediate result for the EFT Lagrangian up to dimension six is then

$$\mathcal{L}_{\text{EFT}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\phi^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{C_{0,6}^{(6)}}{\Lambda^2}\frac{\phi^6}{6!} + \frac{C_{2,4}^{(6)}}{\Lambda^2}\frac{\phi^3\Box\phi}{3!} + \frac{C_{4,2}^{(6)}}{\Lambda^2}\phi\Box^2\phi + \mathcal{O}(\Lambda^{-4})\,. \tag{2.21}$$

2.3.2 Field redefinitions and equations of motion

If we were using the Lagrangian in Eq. (2.21) to compute observables, we would notice that up to $\mathcal{O}(\Lambda^{-2})$ we are not able to determine the three dimension-six Wilson coefficients independently. The reason is that Eq. (2.21) still contains redundant operators. Consider a nonlinear field redefinition

$$\phi(x) \mapsto \phi(x) + \frac{a}{\Lambda^2}\phi^3(x)\,. \tag{2.22}$$

Under this transformation, the Lagrangian maps to

$$\mathcal{L}_{\text{EFT}} \mapsto \mathcal{L}_{\text{EFT}} - \frac{a}{\Lambda^2}\phi^3\left(\Box + m_\phi^2 + \frac{\lambda}{3!}\phi^2\right)\phi + \mathcal{O}(\Lambda^{-4})\,. \tag{2.23}$$

We see that with the choice $a = C_{2,4}^{(6)}/3!$, the transformed Lagrangian does no longer contain the $\phi^3\Box\phi$ operator.

The reason why we are allowed to perform such a field redefinition is the following.

1. Consider the generating functional

$$Z[J] = \int \mathcal{D}\phi \exp\left\{ i\int d^Dx\ (\mathcal{L}[\phi(x)] + J(x)\phi(x)) \right\}\,, \tag{2.24}$$

which produces correlation functions of ϕ via

$$\langle 0|T\{\phi(x_1)\dots\phi(x_m)\}|0\rangle = \frac{1}{Z[J]}\frac{-i\delta}{\delta J(x_1)}\cdots\frac{-i\delta}{\delta J(x_m)}Z[J]\bigg|_{J=0}\,. \tag{2.25}$$

We first relabel the field ϕ, which is just an integration variable, by ϕ' and then perform the field redefinition:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \exp\left\{ i\int d^Dx\ (\mathcal{L}[\phi(x)] + J(x)\phi(x)) \right\} \\ &= \int \mathcal{D}\phi' \exp\left\{ i\int d^Dx\ \left(\mathcal{L}[\phi'(x)] + J(x)\phi'(x)\right) \right\} \\ &= \int \mathcal{D}\phi \det\left(\frac{\delta\phi'}{\delta\phi}\right) \exp\left\{ i\int d^Dx\ (\mathcal{L}[F[\phi(x)]] + J(x)F[\phi(x)]) \right\} \\ &= \int \mathcal{D}\phi \exp\left\{ i\int d^Dx\ \left(\mathcal{L}'[\phi(x)] + J(x)F[\phi(x)]\right) \right\}\,, \end{aligned} \tag{2.26}$$

where we denoted $\mathcal{L}[F[\phi(x)]] = \mathcal{L}'[\phi(x)]$. In the fourth line, we have used that in dimensional regularization, the determinant of the field redefinition is trivial,

$$\det\left(\frac{\delta\phi'}{\delta\phi}\right) = 1\,, \tag{2.27}$$

as will be shown below. The generating functional $Z[J]$ produces Green's functions of ϕ using the Lagrangian $\mathcal{L}$ or Green's functions of the interpolating field $F[\phi]$ using the modified Lagrangian $\mathcal{L}'$, which are the same. In contrast, with the modified generating functional

$$Z'[J] = \int \mathcal{D}\phi \exp\left\{i \int d^Dx\ \big(\mathcal{L}'[\phi(x)] + J(x)\phi(x)\big)\right\}\,, \tag{2.28}$$

we can compute correlation functions of ϕ with the modified Lagrangian $\mathcal{L}'$. The field redefinition therefore leads to different Green's functions: either correlation functions of ϕ or correlation functions of $F[\phi]$, but both computed with the same Lagrangian $\mathcal{L}'$.

The determinant of the field redefinition can be shown to be unity in DR: we consider field redefinitions of the form

$$\phi'(x) = F[\phi(x)] = \phi(x) + \frac{1}{\Lambda^n} f[\phi(x)]\,, \tag{2.29}$$

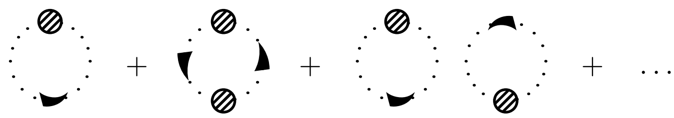
hence

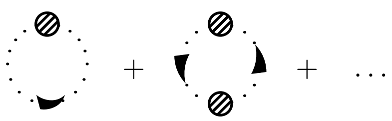
$$\frac{\delta\phi'(x)}{\delta\phi(y)} = \delta^{(D)}(x-y) + \frac{1}{\Lambda^n} f'[\phi(x)]\delta^{(D)}(x-y)\,. \tag{2.30}$$

The determinant can be computed by introducing Grassmann fields c and $\bar{c}$:

$$\begin{aligned}\det\left(\frac{\delta\phi'}{\delta\phi}\right) &= \int \mathcal{D}\bar{c} \int \mathcal{D}c\ \exp\left\{i\int d^Dx \int d^Dy\, \bar{c}(x)\left(i\frac{\delta\phi'(x)}{\delta\phi(y)}\right)c(y)\right\} \\ &= \int \mathcal{D}\bar{c} \int \mathcal{D}c\ \exp\left\{i\int d^Dx\, \bar{c}(x)\, i\left[1 + \frac{1}{\Lambda^n} f'[\phi(x)]\right]c(x)\right\}\,.\end{aligned} \tag{2.31}$$

The term suppressed by $1/\Lambda^n$ in the EFT power counting is treated as a perturbation, resulting in Feynman diagrams consisting of fermionic loops with insertions of the source $f'[\phi(x)]$:

$\det\left(\frac{\delta\phi'}{\delta\phi}\right) = 1 +$ 

$= \exp\Big[$  $+ \ \ldots \Big]\,.$ (2.32)

However, the "ghost" propagator is just a constant, since there is no kinetic term and no mass scale. Therefore, all diagrams lead to scaleless integrals, which vanish in DR, hence $\det\left(\frac{\delta\phi'}{\delta\phi}\right) = 1$.

2. According to the first point, performing a field redefinition amounts to computing Green's functions of different interpolating fields with the *same action*. However, Green's functions are not physical observables. The central point is that identical S-matrix elements can be obtained using different interpolating fields. Note that the fields ϕ and $\phi' = \phi + \frac{a}{\Lambda^2}\phi^3$ have the same quantum numbers. The S-matrix elements depend on the particle states $\langle p|$ and are obtained from Green's functions through the LSZ formula. In the case of scalar particles, we can extract S-matrix elements from Green's functions

$$\tilde{G}(k_1,\ldots,k_n) = \int d^4x_1 \ldots d^4x_n \exp\Big(-i\sum_{i=1}^{n} x_i\cdot k_i\Big)\langle 0|T\{\phi(x_1)\ldots\phi(x_n)\}|0\rangle \tag{2.33}$$

with the reduction formula for m incoming and n outgoing particles:

$$\begin{aligned}&\tilde{G}(p_1,\dots,p_m,-q_1,\dots,-q_n)\\&=\prod_{i=1}^{m}\left(\frac{i\sqrt{\mathcal{R}}}{p_i^2-m_\phi^2+i\epsilon}\right)\prod_{j=1}^{n}\left(\frac{i\sqrt{\mathcal{R}}}{q_j^2-m_\phi^2+i\epsilon}\right)\,{}_{\text{out}}\langle q_1,\dots,q_n|p_1,\dots,p_m\rangle_{\text{in}}+\text{less singular}\,,\end{aligned} \tag{2.34}$$

provided that the interpolating field $\phi(x)$ fulfills

$$\langle p|\phi(x)|0\rangle\neq 0\,. \tag{2.35}$$

Under the field redefinition Eq. (2.29), which changes the field only by effects suppressed in the power counting, this condition is still fulfilled,

$$\langle p|F[\phi(x)]|0\rangle\neq 0\,. \tag{2.36}$$

Although the Green's functions and the pole residue $\mathcal{R}$ of the propagator change under the field redefinition, these two effects compensate each other. The same physical S-matrix elements can be extracted from correlation functions of different interpolating fields. Particles and fields are not the same thing.

We conclude that we can simplify the EFT Lagrangian by applying field redefinitions. In renormalizable theories, field redefinitions are limited to linear transformations

$$\phi_i\mapsto\phi_i'=A_{ij}\phi_j\,, \tag{2.37}$$

in order to preserve renormalizability. In an EFT, we have more freedom and can use non-linear field redefinitions that respect the power counting.

Consider again the field redefinition in Eq. (2.29). The shifted term is proportional to the leading-order classical EOM:

$$\left(\Box+m_\phi^2+\frac{\lambda}{3!}\phi^2\right)\phi\overset{\text{EOM}}{=}0\,. \tag{2.38}$$

Of course, quantum fields do in general not obey the classical EOM. However, field redefinitions of the form

$$\phi\mapsto\phi+\frac{1}{\Lambda^n}f[\phi] \tag{2.39}$$

transform the Lagrangian as

$$\mathcal{L}_{\text{EFT}}\mapsto\mathcal{L}_{\text{EFT}}-\frac{1}{\Lambda^n}f[\phi]\underbrace{\left(\Box+m_\phi^2+\frac{\lambda}{3!}\phi^2\right)\phi}_{\text{EOM}}+\mathcal{O}\left(\Lambda^{-(n+2)}\right)\,. \tag{2.40}$$

This implies that field redefinitions can be used to shift the effects of operators that are proportional to the leading-order classical EOM to higher orders in the EFT expansion. This can be done iteratively, bringing the Lagrangian to a form that does not contain any EOM operators. In this sense, EOM operators can be considered as redundant. However, note that the elimination of EOM operators induces a shift at higher orders in the power counting, which needs to be taken into account, depending on the order one is interested in. While single insertions of EOM operators do not contribute to S-matrix elements, double insertions of EOM operators can contribute to them—nevertheless, they can be removed from the operator basis by a field redefinition, as double insertions of EOM operators can be described by single insertions of higher-dimension operators.

In our scalar toy example, we find that field redefinitions can be used to remove all but one operator at dimension six, and we write the Lagrangian as:[3]

$$\mathcal{L}_{\text{EFT}}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m_\phi^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4+\frac{C^{(6)}}{\Lambda^2}\frac{\phi^6}{6!}+\mathcal{O}(\Lambda^{-4})\,. \tag{2.41}$$

[3] In principle, we should distinguish it from Eq. (2.21), since the performed field redefinitions change the parameters of the Lagrangian.

2.4 Renormalization-group equations

Renormalization of an EFT is similar to the procedure in a renormalizable QFT (i.e., with interaction terms of mass dimension ≤ 4). The important difference is that we have to apply the EFT power counting in the renormalization procedure. We have seen in Sect. 2.2 that we need to work with DR in order to respect the power-counting formula Eq. (2.10) at the loop level. If we want to renormalize the EFT at $\mathcal{O}(\delta^2) = \mathcal{O}((p/\Lambda)^2)$, we need to take into account one insertion of the dimension-six operator. Loop diagrams with two insertions of $\mathcal{O}^{(6)}$ will in general also be divergent. The divergence can be compensated by a local counterterm. By power counting, the double-insertion of $\mathcal{O}^{(6)}$ and the resulting divergence are of order $\mathcal{O}(\delta^4)$, which is the same order as a single insertion of a dimension-8 operator. Although the presence of one operator beyond dimension four implies an infinite tower of operators, power counting allows us to deal only with a finite number of them at a given order in δ.

When switching to renormalized perturbation theory in $D = 4 - 2\varepsilon$ dimensions, the EFT Lagrangian reads

$$\mathcal{L}_{\mathrm{EFT}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\phi^2}{2}\phi^2 - \frac{\mu^{2\varepsilon}\lambda}{4!}\phi^4 + \frac{\mu^{4\varepsilon}C^{(6)}}{\Lambda^2}\frac{\phi^6}{6!} + \mathcal{L}_{\mathrm{ct}} + \mathcal{O}(\Lambda^{-4})\,, \tag{2.42}$$

where all quantities now denote renormalized field and parameters and we have introduced factors of the renormalization scale μ to keep the renormalized couplings λ and $C^{(6)}$ dimensionless in D dimensions. As usual, the renormalized parameters implicitly depend on the scale μ. In loop diagrams, explicit factors of μ only enter through the couplings. When we expand around $D = 4$ dimensions, μ explicitly appears only in logarithms of the form $\log(p^2/\mu^2)$ and $\log(m_\phi^2/\mu^2)$. Observables do not depend on the artificial scale μ: the explicit logarithmic dependence on μ is compensated by the implicit μ-dependence of the renormalized parameters. In order not to spoil perturbation theory, we want the logarithms to stay small, i.e., we should choose a value of μ close to the low scales m_ϕ and p, the external momenta.

On the other hand, if we are interested in expressing the results in terms of the parameters of the underlying UV theory, we need to calculate the renormalized EFT parameters λ, $C^{(6)}$, ... (i.e., the Wilson coefficients) in terms of the renormalized UV parameters. This matching calculation will be discussed in Sect. 2.5. We will see that the scale appearing in this calculation is the heavy mass scale M, which sits at the UV scale. In order to avoid large logs in this calculation, we should calculate the Wilson coefficients at $\mu = M$. The dependence of the EFT parameters on the scale is described by the renormalization-group equations (RGEs). Solving the RGEs allows us to relate the Wilson coefficients at the low-energy scale $\mu \approx m$ to the values of the Wilson coefficients at the matching scale $\mu \approx M$. The effect that the parameters depend on the scale μ is called *running*. In general, the value of a parameter at $\mu \approx m$ does not only depend on its value at $\mu \approx M$, but also on the value of the other parameters at the UV scale. This interplay of Wilson coefficients is called *mixing*. Applying this procedure effectively avoid large logarithms in the calculation and leads to so-called *RG-improved perturbation theory*. What happens is that the solution of the RGE resums large logarithms to all orders, as will be discussed below. This strategy is sketched in Fig. 3.

When we compute a loop diagram with an insertion of an operator $\mathcal{O}_A^{(d)}$ that comes with a coefficient $\propto 1/\Lambda^{d-4}$, we will generate divergent terms with a power-counting factor $1/\Lambda^{d-4}$. If no other scale appears, this divergence needs to be compensated by a counterterm that corresponds to a local operator of mass dimension d, e.g., $\mathcal{O}_B^{(d)}$. If there are multiple operators at the same mass dimension, $\mathcal{O}_A^{(d)}$ and $\mathcal{O}_B^{(d)}$ can be different. The counterterm is generated by splitting the bare coefficient $\tilde{C}_B^{(d)}$ into a renormalized part $C_B^{(d)}(\mu)$ and a divergent counterterm proportional to $C_A^{(d)}(\mu)/\varepsilon$. Hence, the RGE for $C_B^{(d)}(\mu)$ will contain a piece

$$\mu\frac{d}{d\mu}C_B^{(d)}(\mu) = \frac{d}{d\log\mu}C_B^{(d)}(\mu) = \ldots + a \times C_A^{(d)}(\mu) + \ldots\,. \tag{2.43}$$

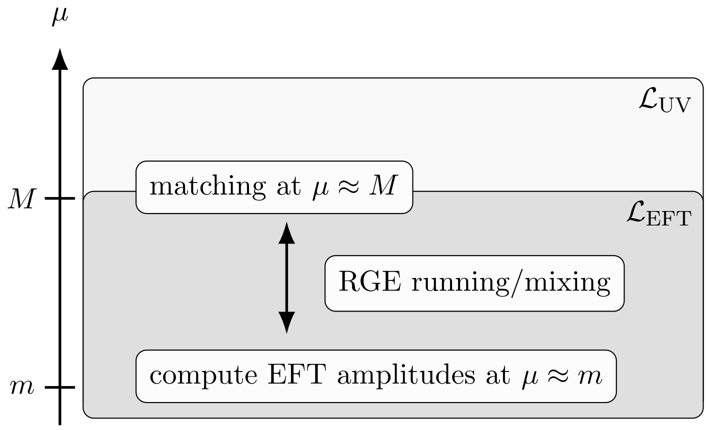
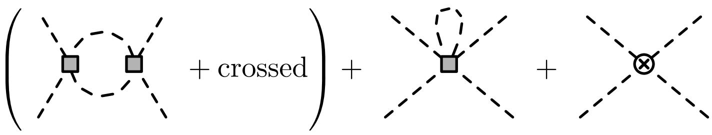


Figure 3: Sketch of the general strategy of an EFT, used to resum large logarithms.

The prefactor a will involve loop factors $1/(4\pi)^2$ and factors of the dimensionless coupling $\lambda(\mu)$. In case that there are light scales in the problem (such as in our toy example the mass m_ϕ), the loop diagram can produce light scales in the numerator. The required counterterm then stems from an operator of lower dimension. Hence, the RGE for the coefficients of lower-dimension operators can contain pieces proportional to the coefficients of higher-dimension operators, together with suppression factors $(m/\Lambda)^n$. E.g., the RGE for λ can be extracted from the calculation of the $2 \to 2$ scattering

$i\mathcal{A}(\phi\phi \to \phi\phi) =$ 

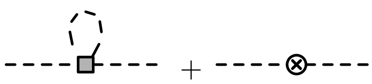
$+ \mathcal{O}(\Lambda^{-4}) + \mathcal{O}(\text{2-loop})\,,$ (2.44)

leading to

$$\beta(\lambda(\bar{\mu}), \varepsilon) = \bar{\mu}\frac{d}{d\bar{\mu}}\lambda(\bar{\mu}) = -2\varepsilon\lambda(\bar{\mu}) + 3\frac{\lambda^2(\bar{\mu})}{16\pi^2} - \frac{C^{(6)}(\bar{\mu})}{16\pi^2}\frac{m_\phi^2(\bar{\mu})}{\Lambda^2} + \mathcal{O}(\Lambda^{-4}) + \mathcal{O}(\text{2-loop})\,. \tag{2.45}$$

We see that due to the self-loop diagram with a ϕ^6 insertion, the RGE for λ contains a term that depends on the Wilson coefficient $C^{(6)}$. In contrast to this *down-mixing*, which only happens in the presence of light mass scales, an *up-mixing* is not possible for single-operator insertions [17], but only for insertions of multiple operators. In general, the mixing structure has to follow the power counting. E.g., at $\mathcal{O}(\Lambda^{-4})$ the RGE of a dimension-8 operator coefficient can contain terms that are either linear in dimension-8 operator coefficients or quadratic in dimension-6 operator coefficients, but no terms linear in dimension-6 operator coefficients. To higher accuracy in the power counting, the RGE of a dimension-8 operator coefficient can also contain even higher powers of dimension-6 and -8 operator coefficients, as well as coefficients of operators beyond dimension 8. However, these contributions only appear in the presence of light mass scales in the EFT and they are suppressed by appropriate factors of m_ϕ/Λ.

The mass parameter m_ϕ also depends on the renormalization scale: the RG running is determined from the one-particle-irreducible (1PI) two-point function:

$-i\Sigma(p^2) =$ 

$+\ \mathcal{O}(\Lambda^{-4}) + \mathcal{O}(\text{2-loop})\,,$ (2.46)

so that the resummed propagator given by

$$\tilde{D}(p^2) = \frac{i}{p^2 - m_\phi^2 - \Sigma(p^2)} = \frac{i\mathcal{R}}{p^2 - m_{\phi,\mathrm{ph}}^2} + (\text{regular at } p^2 = m_{\phi,\mathrm{ph}}^2)\,, \tag{2.47}$$

with $m_{\phi,\text{ph}}$ the physical pole mass. The RGE for the $\overline{\text{MS}}$ mass reads

$$\bar{\mu}\frac{d}{d\bar{\mu}}m_\phi^2(\bar{\mu}) = \gamma_{m_\phi}(\lambda(\bar{\mu}))m_\phi^2(\bar{\mu}) = \frac{\lambda(\bar{\mu})}{16\pi^2}m_\phi^2(\bar{\mu})\,. \tag{2.48}$$

When working at $\mathcal{O}(\delta^2)$ in the power counting, we find that the RGE for $C^{(6)}$ depends on the renormalized coupling λ. We can first solve the RGE for λ ignoring the $\mathcal{O}(\delta^2)$ correction and use this solution in the RGE for $C^{(6)}$—the $\mathcal{O}(\delta^2)$ corrections in λ produce corrections of relative order δ^2 to $C^{(6)}$, which is an effect of $\mathcal{O}(\delta^4)$ in the power counting, since $C^{(6)}$ is already parametrizing $\mathcal{O}(\delta^2)$ effects in amplitudes.

Neglecting the $\mathcal{O}(\delta^2)$ corrections, the solutions to the RGEs for λ and m_ϕ can be calculated by separation of variables and are given by

$$\begin{aligned}
\lambda(\bar{\mu}) &= \frac{\lambda(\bar{\mu}_0)}{1-\frac{3\lambda(\bar{\mu}_0)}{32\pi^2}\log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)}\,,\\
m_\phi^2(\bar{\mu}) &= m_\phi^2(\bar{\mu}_0)\exp\left(\int_{\lambda(\bar{\mu}_0)}^{\lambda(\bar{\mu})} d\lambda'\,\frac{\gamma_{m_\phi}(\lambda')}{\beta(\lambda')}\right) = m_\phi^2(\bar{\mu}_0)\left(\frac{\lambda(\bar{\mu})}{\lambda(\bar{\mu}_0)}\right)^{1/3}\\
&= \frac{m_\phi^2(\bar{\mu}_0)}{\left(1-\frac{3\lambda(\bar{\mu}_0)}{32\pi^2}\log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)\right)^{1/3}}\,,
\end{aligned} \tag{2.49}$$

relating the running parameters at the scale $\bar{\mu}$ to their values at a scale $\bar{\mu}_0$. Let us expand these solutions again around $\lambda(\bar{\mu}_0) = 0$:

$$\begin{aligned}
\lambda(\bar{\mu}) &= \lambda(\bar{\mu}_0)\left[1+\frac{3\lambda(\bar{\mu}_0)}{32\pi^2}\log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)+\ldots\right]\,,\\
m_\phi^2(\bar{\mu}) &= m_\phi^2(\bar{\mu}_0)\left[1+\frac{\lambda(\bar{\mu}_0)}{32\pi^2}\log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)+\ldots\right]\,.
\end{aligned} \tag{2.50}$$

This is a good approximation provided that $\left|\frac{\lambda(\bar{\mu}_0)}{16\pi^2}\log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)\right| \ll 1$. However, even if the coupling λ is small, this quantity can become large if the scales $\bar{\mu}$ and $\bar{\mu}_0$ are very different, i.e., when the logarithm becomes large. In this case, Eq. (2.49) is still a good approximation (provided that $\frac{\lambda(\bar{\mu}_0)}{16\pi^2} \ll 1$), but for $\left|\lambda(\bar{\mu}_0)\log\left(\bar{\mu}^2/\bar{\mu}_0^2\right)\right| \gtrsim 16\pi^2$, the fixed-order result in Eq. (2.50) is no longer valid and ordinary perturbation theory breaks down. The solution of the RGEs in Eq. (2.49) achieves a resummation of terms inside the brackets of Eq. (2.50) of the form

$$\left[\frac{\lambda(\bar{\mu}_0)}{16\pi^2}\log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)\right]^n \tag{2.51}$$

for all $n \geq 0$, known as *leading logs*. The solution of the two-loop RGEs would allow us to resum terms of the form

$$\frac{\lambda(\bar{\mu}_0)}{16\pi^2}\left[\frac{\lambda(\bar{\mu}_0)}{16\pi^2}\log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)\right]^n\,, \tag{2.52}$$

which are the *next-to-leading logs*. This RGE improvement of perturbation theory needs to be applied in situations where large logarithms arise.

In summary, the EFT framework allows us to resum large logarithms in the presence of widely separated scales by improving perturbation theory through the following multi-step procedure:

- calculation of the Wilson coefficients renormalized at the UV matching scale;
- evolution of the Wilson coefficients down to the IR scale by solving the RGEs—this leads to a resummation of logarithms;
- calculation of low-energy observables in terms of EFT parameters renormalized at the IR scale.

2.5 Tree-level matching

Matching denotes the procedure of expressing the parameters of the EFT, renormalized at the matching scale $\bar{\mu}_0$, in terms of the parameters of the UV theory, renormalized as well at $\bar{\mu}_0$. In order for the EFT to be an effective description of the underlying fundamental theory, both theories need to reproduce the same S-matrix elements for processes involving only light particles—up to a certain accuracy in the EFT power counting. The generic procedure to perform the matching is therefore to equate the S-matrix elements in both theories and to solve these matching equations for the EFT parameters. These calculations can be performed in a diagrammatic way, which we will discuss here. Alternatively, the matching can also be performed by *integrating out* the fields for the heavy particles directly in the path integral.

2.5.1 On-shell method

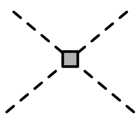
The matching of the EFT to the UV theory can be done in several ways. In the on-shell method, we calculate on-shell S-matrix elements in both theories and equate them to each other.

The mass parameter can be obtained by considering the physical pole mass in both theories. At tree level, this is trivial since there is no self-energy contribution and the propagator is given by the tree-level propagator, leading directly to

$$m_\phi = m\,. \tag{2.53}$$

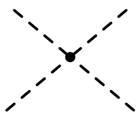
The quartic coupling is determined through the four-point function. The $2 \to 2$ on-shell scattering amplitudes in the EFT and UV theory are

$$i\mathcal{A}_{\mathrm{EFT}}(\phi\phi \to \phi\phi) =$$

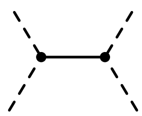


$$+ \mathcal{O}(\delta^4) + \mathcal{O}(\text{1-loop}) = -i\lambda + \mathcal{O}(\delta^4) + \mathcal{O}(\text{1-loop})\,,$$

$$i\mathcal{A}_{\mathrm{UV}}(\phi\phi \to \phi\phi) =$$



$$+ \Bigg($$



$$+ \text{crossed}\Bigg) + \mathcal{O}(\text{1-loop})$$

$$= -i\lambda_0 + (-i\lambda_1 M)\left(\frac{i}{s-M^2} + \frac{i}{t-M^2} + \frac{i}{u-M^2}\right)(-i\lambda_1 M)\,, \tag{2.54}$$

where s, t, u are the usual Mandelstam variables, fulfilling (on-shell) $s+t+u = 4m^2$. In the EFT expansion, the external momenta and the light mass are counted as $p_i^\mu/\Lambda = \mathcal{O}(\delta)$ and $m/\Lambda = \mathcal{O}(\delta)$, while the heavy mass is $M/\Lambda = \mathcal{O}(1)$. Hence, we can expand the propagators of the heavy particle as

$$\frac{i}{s-M^2} = \frac{-i}{M^2}\left(1 + \frac{s}{M^2} + \mathcal{O}(\delta^4)\right) \tag{2.55}$$

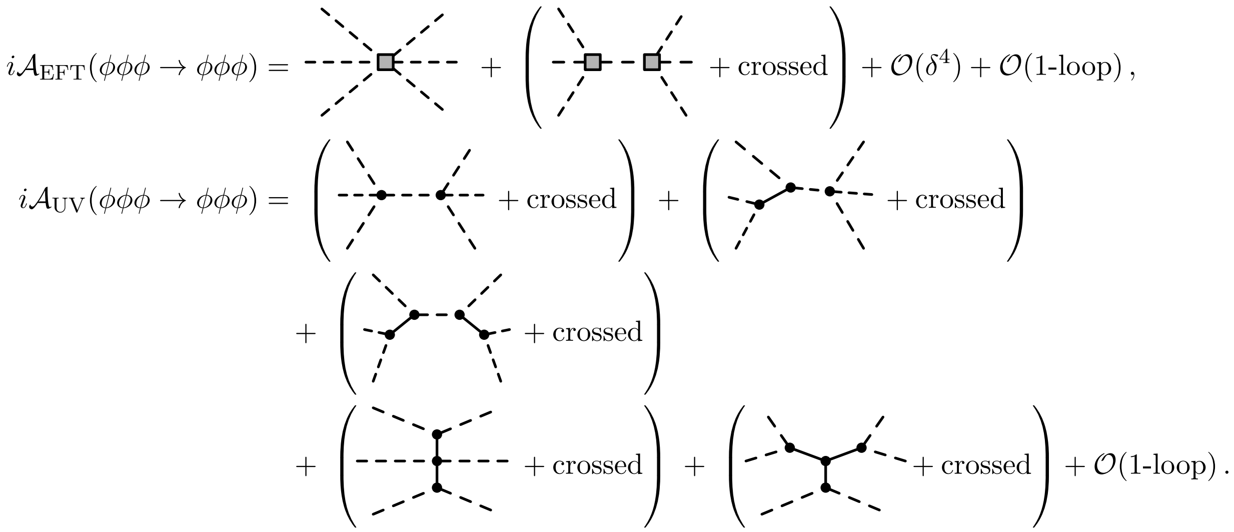
and the UV amplitude becomes

$$\begin{aligned} i\mathcal{A}_{\mathrm{UV}}(\phi\phi \to \phi\phi) &= -i\lambda_0 + i\lambda_1^2\left(3 + \frac{s+t+u}{M^2}\right) + \mathcal{O}(\delta^4) + \mathcal{O}(\text{1-loop}) \\ &= -i\lambda_0 + i\lambda_1^2\left(3 + \frac{4m^2}{M^2}\right) + \mathcal{O}(\delta^4) + \mathcal{O}(\text{1-loop})\,, \end{aligned} \tag{2.56}$$

hence

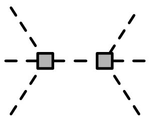
$$\lambda = \lambda_0 - \lambda_1^2\left(3 + \frac{4m^2}{M^2}\right) + \mathcal{O}(\delta^4) + \mathcal{O}(\text{1-loop})\,. \tag{2.57}$$

For the matching of the Wilson coefficient $C^{(6)}$, we need to consider the six-point function:



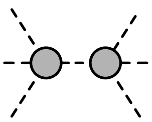
(2.58)

We see that the number of diagrams in particular in the UV theory is exploding. The diagrams in the third line of the UV contribution give a direct contribution to the contact term in the EFT, depending on λ_2 and λ_3. In contrast, when solving the equation $\mathcal{A}_{\mathrm{EFT}} = \mathcal{A}_{\mathrm{UV}}$ for $C^{(6)}$, the diagrams in the first two lines of the UV contribution partially cancel with the second EFT diagram and its crossed versions, where for the quartic coupling we need to insert the matching for λ, Eq. (2.57). In the EFT, we find for this set of diagrams



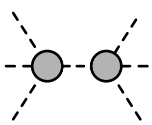
$$= \left[-i\lambda_0 + i\lambda_1^2\left(3 + \frac{4m^2}{M^2}\right)\right] \frac{i}{p_{123}^2 - m^2} \left[-i\lambda_0 + i\lambda_1^2\left(3 + \frac{4m^2}{M^2}\right)\right] + \ldots . \tag{2.59}$$

The corresponding set of diagrams in the UV theory, expanded around $\delta = 0$, amounts to



$$= \left[-i\lambda_0 + i\lambda_1^2\left(3 + \frac{3m^2 + p_{123}^2}{M^2}\right)\right] \frac{i}{p_{123}^2 - m^2} \left[-i\lambda_0 + i\lambda_1^2\left(3 + \frac{3m^2 + p_{123}^2}{M^2}\right)\right] + \ldots , \tag{2.60}$$

where the blobs denote the UV four-point function. Since the internal line is not on shell, there is not an exact cancellation between the EFT and UV diagrams. However, the difference does not have a pole at $p_{123}^2 = m^2$ and is given by



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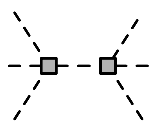


$$= \frac{1}{M^2}\left(2i\lambda_1^2(\lambda_0 - 3\lambda_1^2) + \mathcal{O}(\delta^2)\right) . \tag{2.61}$$

From the $\frac{1}{2} \times 5 \times 4 = 10$ permutations of the external legs, we obtain a contribution to the Wilson coefficient of

$$\frac{C^{(6)}}{\Lambda^2} = \frac{20\lambda_1^2(\lambda_0 - 3\lambda_1^2)}{M^2} + \ldots , \tag{2.62}$$

where the ellipsis denotes terms involving λ_2 and λ_3 from the last line of Eq. (2.58).

2.5.2 Off-shell method

Performing the matching on shell has the advantage that one can work directly with the non-redundant operator basis in the EFT. The disadvantage is that for the higher n-point functions the number of diagrams increases rapidly and one has to take into account the complete set of diagrams, including the ones with sub-amplitudes corresponding to correlation functions of a lower number of light fields. This can be avoided by performing the matching off shell, which however requires field redefinitions to arrive at the given EFT operator basis. The idea is not to match S-matrix elements, but rather off-shell n-point functions of the light fields that are one-particle irreducible with respect to cutting propagators of only the light particle, called 1LPI (one-light-particle irreducible).

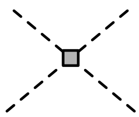
At tree level, the matching of the mass parameter is again trivial. Next, we will match the off-shell four-point function:

$$i\mathcal{A}_{\mathrm{UV}}(\phi\phi \to \phi\phi) = -i\lambda_0 + i\lambda_1^2 \left(3 + \frac{p_1^2 + p_2^2 + p_3^2 + p_4^2}{M^2} \right) + \mathcal{O}(\delta^4) + \mathcal{O}(\text{1-loop}) \,, \tag{2.63}$$

where we do not use $p_i^2 = m^2$. Rather, we interpret the momentum-dependent piece as the contribution from the dimension-six operator

$$\mathcal{O}_{2,4}^{(6)} = \frac{\phi^3 \Box \phi}{3!} \,, \tag{2.64}$$

which has the vertex rule



$$= -i\frac{C_{2,4}^{(6)}}{\Lambda^2}(p_1^2 + p_2^2 + p_3^2 + p_4^2) \,. \tag{2.65}$$

From the matching of the four-point function, we then obtain

$$\begin{aligned} \lambda &= \lambda_0 - 3\lambda_1^2 + \ldots \,, \\ \frac{C_{2,4}^{(6)}}{\Lambda^2} &= -\frac{\lambda_1^2}{M^2} + \ldots \,, \end{aligned} \tag{2.66}$$

however not in the reduced basis Eq. (2.41), but in an EFT that still contains the redundant operator $\mathcal{O}_{2,4}^{(6)}$. We switch back to the reduced basis by performing a field redefinition—at the considered order in the expansion, i.e., up to neglected dimension-8 effects, this amounts to applying the classical leading-order EOM to the redundant operator,

$$\mathcal{O}_{2,4}^{(6)} \mapsto -\frac{1}{3!}\phi^3 \left(m_\phi^2 \phi + \frac{\lambda}{3!}\phi^3 \right) \,. \tag{2.67}$$

After the field redefinition, the (redefined) coefficients of reduced basis are

$$\begin{aligned} \lambda &= \lambda_0 - 3\lambda_1^2 - \frac{\lambda_1^2}{M^2} 4m_\phi^2 = \lambda_0 - \lambda_1^2 \left(3 + \frac{4m^2}{M^2} \right) + \ldots \,, \\ \frac{C^{(6)}}{\Lambda^2} &= \frac{20\lambda_1^2 \lambda}{M^2} = \frac{20\lambda_1^2(\lambda_0 - 3\lambda_1^2)}{M^2} + \ldots \,, \end{aligned} \tag{2.68}$$

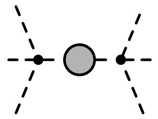
where we made use of the matching of m_ϕ and in the second line of the matching result for λ. We see that after the field redefinition, the matching result is identical to the one obtained in the on-shell matching. The final step is the matching of the 1LPI six-point function, which only involves the diagrams in the last line of Eq. (2.58) and provides the contributions to $C^{(6)}$ depending on $\lambda_{2,3}$.

2.6 Matching at one loop

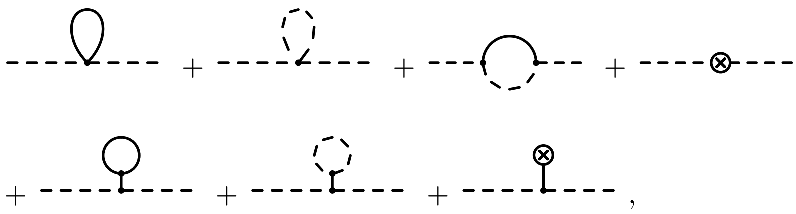
Matching at tree level basically consists of a Taylor expansion of the UV amplitudes. The matching calculation becomes more involved at loop level. Due to multiparticle intermediate states in loop diagrams, the analytic structure of the amplitudes involves branch cuts. The cut structure of the full amplitudes due to light intermediate particles needs to be reproduced by the EFT, since the EFT has to encode the IR dynamics of the full theory. In contrast, branch cuts due to heavy intermediate states only start at invariant masses of the order of the UV scale. When considering low-energy processes, we are in a kinematic regime far away from these cuts. Therefore, the contribution of heavy particles in loops can be described in terms of local effective interactions. In order to see how this works, we will first perform a "naive" matching calculation at one loop and simply equate the amplitudes in the EFT and the UV theory. In a second step, we will discuss a trick that significantly reduces the complexity of the matching calculation.

2.6.1 Naive matching procedure

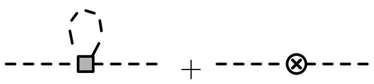
As an example, we consider the matching of the mass parameter of the scalar EFT to the UV toy model. For simplicity, we will perform the matching on shell. When including the higher-dimensional operators, off-shell matching is more efficient, as it will capture, e.g., effects of the EOM operator $\phi\Box^2\phi$, which through field redefinitions contributes to the dimension-6 operator ϕ^6. In the on-shell method, this is reproduced by the diagram


. (2.69)

We proceed with the general recipe of equating the observables in both UV theory and EFT, in this case the pole mass. In the full theory, the self-energy is given by the following diagrams:

$-i\Sigma_\phi^{\mathrm{UV}}(p^2) =$

, (2.70)

where the counterterm diagrams only absorb the $1/\varepsilon$ poles in the $\overline{\mathrm{MS}}$ scheme. (Since we do not cancel the finite parts of the tadpoles by a linear counterterm, we have to include them in addition to the standard 1PI diagrams.) In the EFT, we only have the diagrams

$-i\Sigma^{\mathrm{EFT}}(p^2) =$

. (2.71)

To one-loop accuracy, the physical pole mass is given by

$$m_{\mathrm{ph}}^2 = m^2(\bar{\mu}) + \Sigma_\phi^{\mathrm{UV}}(m^2, \bar{\mu}) = m_\phi^2(\bar{\mu}) + \Sigma^{\mathrm{EFT}}(m_\phi^2, \bar{\mu})\,. \tag{2.72}$$

Inside the one-loop diagrams, we can use the tree-level mass and employ the matching relation $m_\phi^2 = m^2$ at tree-level accuracy. The matching at one loop then becomes

$$m_\phi^2(\bar{\mu}) = m^2(\bar{\mu}) + \Sigma_\phi^{\mathrm{UV}}(m^2, \bar{\mu}) - \Sigma^{\mathrm{EFT}}(m^2, \bar{\mu})\,. \tag{2.73}$$

This equation should be understood as expanded in the light scale, i.e., as a polynomial in m^2 of the order that corresponds to the desired accuracy in the power counting. Since the EFT needs to reproduce the analytic IR structure of the UV theory, all contributions that are non-analytic in the light scale have to cancel in the difference of UV and EFT loops in Eq. (2.73).

The explicit calculation of the EFT loop diagram leads to the following result:

$$\Sigma^{\mathrm{EFT}}(p^2,\bar{\mu}) = -\frac{\lambda m_\phi^2}{32\pi^2}\left(1-\log\left(\frac{m_\phi^2}{\bar{\mu}^2}\right)\right) . \tag{2.74}$$

For the purpose of the one-loop matching, we can insert the tree-level matching for the EFT parameters, i.e., up to dimension six

$$\begin{aligned} m_\phi &= m \,, \\ \lambda &= \lambda_0 - \lambda_1^2\left(3+\frac{4m^2}{M^2}\right) . \end{aligned} \tag{2.75}$$

The diagrams in the UV theory give an on-shell result of

$$\begin{aligned} \Sigma_\phi^{\mathrm{UV}}(m^2,\bar{\mu}) = &\frac{\left(\lambda_1^2-\lambda_0\right)m^2 - \left(4\lambda_1^2-\lambda_3\lambda_1+\lambda_2\right)M^2}{32\pi^2} \\ &+ \frac{\left(\lambda_0-\lambda_1^2\right)m^4 + 2\lambda_1^2 m^2 M^2 + \lambda_1^2 M^4\left(\sigma_m(M^2)-1\right)}{32\pi^2 m^2}\log\left(\frac{m^2}{\bar{\mu}^2}\right) \\ &+ \frac{M^2\left((\lambda_2-\lambda_1\lambda_3)m^2 - \lambda_1^2 M^2\sigma_m(M^2) + \lambda_1^2 M^2\right)}{32\pi^2 m^2}\log\left(\frac{M^2}{\bar{\mu}^2}\right) \\ &- \frac{\lambda_1^2 M^4\sigma_m(M^2)}{16\pi^2 m^2}\log\left(\frac{1+\sigma_m(M^2)}{2}\right) , \end{aligned} \tag{2.76}$$

where $\sigma_m(M^2) = \sqrt{1-4m^2/M^2}$. Expanding in m^2/M^2, this leads to

$$\begin{aligned} \Sigma_\phi^{\mathrm{UV}}(m^2,\bar{\mu}) = &-\frac{1}{32\pi^2}\left[\lambda_0 m^2 + \left(2\lambda_1^2-\lambda_3\lambda_1+\lambda_2\right)M^2 + \frac{10\lambda_1^2}{3}\frac{m^4}{M^2}\right] \\ &+ \frac{1}{32\pi^2}\left[\lambda_0 - \lambda_1^2\left(3+\frac{4m^2}{M^2}\right)\right] m^2\log\left(\frac{m^2}{\bar{\mu}^2}\right) \\ &+ \frac{1}{32\pi^2}\left[2\lambda_1^2 m^2 + \left(2\lambda_1^2-\lambda_3\lambda_1+\lambda_2\right)M^2 + 4\lambda_1^2\frac{m^4}{M^2}\right]\log\left(\frac{M^2}{\bar{\mu}^2}\right) . \end{aligned} \tag{2.77}$$

We see that the $\log(m^2)$ dependence cancels exactly between $\Sigma_\phi^{\mathrm{UV}}$ and Σ^{EFT}. If we choose the matching scale $\bar{\mu} = M$, then all the logs drop out and we obtain for the one-loop matching of the mass

$$m_\phi^2(M) = m^2(M) - \frac{1}{32\pi^2}\left[3\lambda_1^2 m^2 + \left(2\lambda_1^2-\lambda_3\lambda_1+\lambda_2\right)M^2 + \frac{22\lambda_1^2}{3}\frac{m^4}{M^2}\right] . \tag{2.78}$$

2.6.2 Simplified matching procedure: expanding loops

The matching procedure as explained so far is rather cumbersome at the loop level: we have discussed that the EFT reproduces the analytic structure in the light scales of UV theory, such as poles and branch cuts. Since this structure is explicitly reproduced by the loops, the matching equations for the EFT parameters depend on the light scales only polynomially. In contrast, there is a non-analytic dependence on the UV scales, but these logarithms are small (or vanish) if the matching scale is chosen at the UV scale. In order to extract this polynomial in m^2, we calculated in Eq. (2.76) all UV diagrams, including the bulb diagram

$$i\mu^{2\varepsilon}\int\frac{d^D\ell}{(2\pi)^D}\frac{i}{\ell^2-M^2}\frac{i}{(\ell+p)^2-m^2} , \tag{2.79}$$

which depends on two mass scales and the squared momentum p^2. We subtracted the EFT loop diagram and expanded in m^2/M^2. There is in fact a shortcut to this procedure: since we know that the result is a polynomial in the IR scales, we can expand all the *integrands* in these IR scales before doing the integration. For each integral, such an expansion will distort the IR structure and lead to different IR singularities. However, since the analytic structure in the IR scales agrees between UV and EFT amplitudes, this distortion will affect both in the same way and cancel again in the difference, hence the matching equations for the EFT parameters, which are polynomials in the IR scales, remain unaffected by this manipulation. This trick simplifies the matching calculation significantly: when we expand the integrands in the IR scales, the modified EFT loop diagrams (which only depend on IR scales) become scaleless integrals and vanish identically in DR. The same applies to diagrams in the UV theory with only light particles in the loop, e.g., the second and sixth diagram in Eq. (2.70). The bulb diagram (third diagram in Eq. (2.70)) after the modification becomes an integral that only depends on the heavy mass M, i.e., a single-scale integral—note that we expand both in the mass m and in the external momentum p^μ. In general, integrals with fewer scales are easier to calculate than multi-scale integrals.

It is interesting to observe what happens with the divergences of the loops in this simplified matching procedure. Consider the one-loop matching (2.73). The original self-energies of both EFT and UV theory are finite, as the UV divergences of the loop diagrams are cancelled by the corresponding counterterms in each theory. Note that the UV divergences and counterterms in the EFT are different from the ones in the full theory:

$$m_\phi^2(\bar\mu) = m^2(\bar\mu) + \underbrace{\Sigma_\phi^{\text{UV,loops}} + \Sigma_\phi^{\text{UV,ct}}}_{\text{finite}} - \underbrace{\left(\Sigma^{\text{EFT,loops}} + \Sigma^{\text{EFT,ct}}\right)}_{\text{finite}} . \tag{2.80}$$

Now, we expand the integrands before integration in the IR scales: this does not change the UV divergences of the loops, but introduces IR divergences so that the EFT loop diagrams vanish in DR due to

$$\frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}} = 0 , \tag{2.81}$$

hence

$$m_\phi^2(\bar\mu) = m^2(\bar\mu) + \Sigma_{\phi,\text{ expanded}}^{\text{UV,loops}} + \Sigma_\phi^{\text{UV,ct}} - \left(\cancel{\Sigma_{\text{expanded}}^{\text{EFT,loops}}} + \Sigma^{\text{EFT,ct}}\right) . \tag{2.82}$$

If we denote the original UV singularities by

$$\begin{aligned} \Sigma_\phi^{\text{UV,loops}} &= \Sigma_{\phi,\text{ finite}}^{\text{UV,loops}} + \frac{A}{\varepsilon_{\text{UV}}} , \quad \Sigma_\phi^{\text{UV,ct}} = -\frac{A}{\varepsilon_{\text{UV}}} , \\ \Sigma^{\text{EFT,loops}} &= \Sigma_{\text{finite}}^{\text{EFT,loops}} + \frac{B}{\varepsilon_{\text{UV}}} , \quad \Sigma^{\text{EFT,ct}} = -\frac{B}{\varepsilon_{\text{UV}}} , \end{aligned} \tag{2.83}$$

the expansion of the integrands modifies the EFT loops to

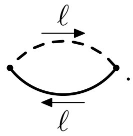
$$\Sigma_{\text{expanded}}^{\text{EFT,loops}} = -\frac{B}{\varepsilon_{\text{IR}}} + \frac{B}{\varepsilon_{\text{UV}}} = 0 . \tag{2.84}$$

As the IR modifications affect the EFT loops and the full theory loops in the same way, the expanded loops of the full theory are

$$\Sigma_{\phi,\text{ expanded}}^{\text{UV,loops}} = \Sigma_{\phi,\text{ expanded, finite}}^{\text{UV,loops}} - \frac{B}{\varepsilon_{\text{IR}}} + \frac{A}{\varepsilon_{\text{UV}}} . \tag{2.85}$$

Therefore, the UV divergences of the expanded loops are still cancelled by the (UV) counterterms of the full theory, while the IR divergences of the expanded loops are cancelled by the subtracted (UV) counterterms of the EFT. The finite remainder is the matching contribution.

Let us illustrate this method with a simplified example. Consider the bulb diagram, but with zero momentum flowing through (we ignore some external coupling constants):

$$I^{\rm UV} = \mu^{2\varepsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{i}{\ell^2 - M^2} \frac{i}{\ell^2 - m^2} \quad \sim$$  $$. \tag{2.86}$$

Since

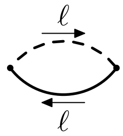
$$\frac{1}{\ell^2 - M^2} \frac{1}{\ell^2 - m^2} = \frac{1}{M^2 - m^2} \left(\frac{1}{\ell^2 - M^2} - \frac{1}{\ell^2 - m^2} \right) , \tag{2.87}$$

the integral can be easily calculated in terms of single-scale integrals:

$$I^{\rm UV} = -\frac{i}{16\pi^2} \left[\frac{1}{\varepsilon_{\rm UV}} + 1 - \frac{1}{M^2 - m^2} \left(M^2 \log\left(\frac{M^2}{\bar\mu^2}\right) - m^2 \log\left(\frac{m^2}{\bar\mu^2}\right) \right) \right] . \tag{2.88}$$

The divergence will be cancelled in the UV theory by an $\overline{\rm MS}$ counterterm:

$$I^{\rm UV} + I^{\rm UV}_{\rm ct} = -\frac{i}{16\pi^2} \left[1 - \frac{1}{M^2 - m^2} \left(M^2 \log\left(\frac{M^2}{\bar\mu^2}\right) - m^2 \log\left(\frac{m^2}{\bar\mu^2}\right) \right) \right]$$

$$\sim$$  $$+$$  $$. \tag{2.89}$$

The diagrams in the EFT correspond to a self-loop diagram, where the heavy propagator is shrunk to a local interaction, plus a tree-level local operator, which contains the one-loop matching correction, as well as the $\overline{\rm MS}$ counterterm:

 $$+$$  $$+$$  $$. \tag{2.90}$$

In the one-loop diagram, we can insert the tree-level matching result for the local operator, which is obtained by expanding the heavy propagator for $\ell^2 \ll M^2$:

$$\frac{i}{\ell^2 - M^2} = -i \left(\frac{1}{M^2} + \frac{\ell^2}{M^4} + \dots \right) , \tag{2.91}$$

hence

$$\begin{aligned} I^{\rm EFT} &= \mu^{2\varepsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{M^2} \left(1 + \frac{\ell^2}{M^2} + \frac{(\ell^2)^2}{M^4} + \dots \right) \frac{1}{\ell^2 - m^2} \\ &= \frac{i}{16\pi^2} \frac{m^2}{M^2} \left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right) \left[\frac{1}{\varepsilon_{\rm UV}} + 1 - \log\left(\frac{m^2}{\bar\mu^2}\right) \right] . \end{aligned} \tag{2.92}$$

Although in this example it is simple to sum up the series to

$$I^{\rm EFT} = \frac{i}{16\pi^2} \frac{m^2}{M^2 - m^2} \left[\frac{1}{\varepsilon_{\rm UV}} + 1 - \log\left(\frac{m^2}{\bar\mu^2}\right) \right] , \tag{2.93}$$

it is better to think in terms of the expanded form (2.92), since the EFT is an expansion in powers of $1/M^2$. Equating the UV and EFT amplitudes, we can solve for the local contribution, which contains the one-loop matching:

$$\begin{aligned} I^{\rm EFT}_{\rm matching} = I^{\rm UV} + I^{\rm UV}_{\rm ct} - \left(I^{\rm EFT} + I^{\rm EFT}_{\rm ct} \right) &= -\frac{i}{16\pi^2} \frac{M^2}{M^2 - m^2} \left(1 - \log\left(\frac{M^2}{\bar\mu^2}\right) \right) \\ &= -\frac{i}{16\pi^2} \left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right) \left(1 - \log\left(\frac{M^2}{\bar\mu^2}\right) \right) . \end{aligned} \tag{2.94}$$

It is easier to calculate the matching contribution by expanding the loop integrands before integration. This means that we expand the light-particle propagator for $m^2 \ll \ell^2$,

$$\frac{1}{\ell^2 - m^2} = \frac{1}{\ell^2}\left(1 + \frac{m^2}{\ell^2} + \frac{m^4}{(\ell^2)^2} + \dots\right) . \tag{2.95}$$

The expanded EFT integrals are given by

$$I^{\text{EFT}}_{\text{expanded}} = \mu^{2\varepsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{M^2}\left(1 + \frac{\ell^2}{M^2} + \frac{(\ell^2)^2}{M^4} + \dots\right) \frac{1}{\ell^2}\left(1 + \frac{m^2}{\ell^2} + \frac{m^4}{(\ell^2)^2} + \dots\right) . \tag{2.96}$$

They are integrated term by term—each of them is a scaleless integral and vanishes in DR, hence

$$\begin{aligned} I^{\text{EFT}}_{\text{matching}} &= I^{\text{UV}} + I^{\text{UV}}_{\text{ct}} - \left(I^{\text{EFT}} + I^{\text{EFT}}_{\text{ct}}\right) \\ &= I^{\text{UV}}_{\text{expanded}} + I^{\text{UV}}_{\text{ct}} - \left(I^{\text{EFT}}_{\text{expanded}} + I^{\text{EFT}}_{\text{ct}}\right) = I^{\text{UV}}_{\text{expanded}} + I^{\text{UV}}_{\text{ct}} - I^{\text{EFT}}_{\text{ct}} . \end{aligned} \tag{2.97}$$

The expanded UV integral reads

$$\begin{aligned} I^{\text{UV}}_{\text{expanded}} &= -\mu^{2\varepsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2 - M^2} \frac{1}{\ell^2}\left(1 + \frac{m^2}{\ell^2} + \frac{m^4}{(\ell^2)^2} + \dots\right) \\ &= -\frac{i}{16\pi^2}\left[\frac{1}{\varepsilon_{\text{UV}}} + 1 - \log\left(\frac{M^2}{\bar\mu^2}\right)\right] \\ &\quad - \frac{i}{16\pi^2}\frac{m^2}{M^2}\left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots\right)\left[\frac{1}{\varepsilon_{\text{IR}}} + 1 - \log\left(\frac{M^2}{\bar\mu^2}\right)\right] , \end{aligned} \tag{2.98}$$

where we integrated term by term. The first integral is UV divergent, while all the remaining ones are IR divergent. The UV divergence gets cancelled by the UV counterterm, while the IR divergences cancel against the subtracted EFT counterterm—the finite part reproduces the matching contribution.

From Eq. (2.97), we find a remarkable relation:

$$I^{\text{UV}} = I^{\text{UV}}_{\text{expanded}} + I^{\text{EFT}} , \tag{2.99}$$

i.e., the full integral I^{UV} is the sum of two expansions: first, we expand the integrand in the momentum region $\ell^2 \sim M^2$, $\ell^2 \gg m^2$, which is the *hard region*. Integrated term by term, this expansion does not reproduce the full integral. The EFT integral is given by an expansion of the original integral in the *soft region*, i.e., $\ell^2 \sim m^2$, $\ell^2 \ll M^2$. Again, integrated term by term, this expanded integral does not correspond to the original one. However, the sum of the two expanded integrals exactly reproduces the original integral I^{UV}, without double counting. The overlap of the two regions corresponds to $I^{\text{EFT}}_{\text{expanded}}$ and vanishes in DR. These observations are related to the *method of regions* [18], which can be used to simplify the calculation of loop integrals.

The full integral I^{UV} contains logarithms $\log(M^2/\bar\mu^2)$ and $\log(m^2/\bar\mu^2)$, or, equivalently, one of these two logs and $\log(M^2/m^2)$, which is a large logarithm for widely separated scales m and M. As we have seen in Sect. 2.4, the EFT framework allows us to resum these logs.

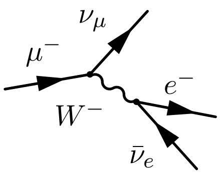
3 EFT below the electroweak scale

After having established the general concepts of EFTs in our scalar toy model, we now consider realistic cases of EFTs. In 1933/1934, Enrico Fermi introduced a four-fermion interaction to describe beta decay [19]. This Fermi theory also successfully describes muon decays and is historically one of the first EFTs. In this chapter, we discuss the EFT for the SM below the electroweak scale. We apply the general EFT concepts of the previous chapter and discuss some new specific features that appear in theories with (chiral) fermions.

3.1 Fermi theory of weak interaction

The weak force is successfully described by the SM of particle physics, which in the electroweak (EW) sector is a gauge theory based on the $SU(2)_L \times U(1)_Y$ symmetry group. The Higgs sector of the SM is responsible for spontaneous symmetry breaking and the generation of mass terms for fermions, EW gauge bosons, and the Higgs particle. Since the EW bosons are very heavy ($M_W \approx 80\,\text{GeV}$, $M_Z \approx 91\,\text{GeV}$, $M_h \approx 125\,\text{GeV}$), processes at energies much below this electroweak scale can be described by an EFT that only contains the light SM particles.

As an illustration, consider the decay of a muon, which at tree level in the SM is mediated by the exchange of a W boson:



The W boson couples only to left-chiral fermions through the interaction term

$$\mathcal{L}_{\text{EW}}^{\text{cc}} = -\frac{g_2}{\sqrt{2}} W_\mu^+ j_W^\mu + \text{h.c.}\,, \quad j_W^\mu = \sum_{\ell=e,\mu,\tau} \bar{\nu}_\ell \gamma^\mu P_L \ell + \text{quark currents}\,, \tag{3.1}$$

where g_2 is the $SU(2)_L$ gauge coupling. The chiral projectors are defined as

$$P_R = \frac{1+\gamma_5}{2}\,, \quad P_L = \frac{1-\gamma_5}{2}\,. \tag{3.2}$$

The SM tree-level amplitude for muon decay is given by

$$i\mathcal{M}(\mu^-(p_1) \to \nu_\mu(p_2) e^-(p_3) \bar{\nu}_e(p_4)) = \left(\frac{-ig_2}{\sqrt{2}}\right)^2 \left[\bar{u}_{\nu_\mu}(p_2)\gamma^\mu P_L u_\mu(p_1)\right] D_{\mu\nu}^W(p) \left[\bar{u}_e(p_3)\gamma^\nu P_L v_{\nu_e}(p_4)\right]\,, \tag{3.3}$$

with the W propagator in unitary gauge

$$D_{\mu\nu}^W(p) = \frac{-i}{p^2 - M_W^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_W^2}\right) \tag{3.4}$$

and $p = p_1 - p_2 = p_3 + p_4$. Since the W mass is much larger than the muon mass and hence much larger than all the momenta, we can expand the propagator for $p^2 \ll M_W^2$:

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{(p^2)^2}{M_W^4} + \ldots\right)\,. \tag{3.5}$$

This directly gives us the tree-level matching to the EFT below the weak scale: this theory does not contain any weak gauge bosons and muon decay is described by a higher-dimension effective operator:

$$\mathcal{L}_{\text{Fermi}} = \ldots + \frac{C}{\Lambda^2}(\bar{\nu}_\mu \gamma^\mu P_L \mu)(\bar{e}\gamma_\mu P_L \nu_e) + \text{h.c.} + \ldots\,. \tag{3.6}$$

This four-fermion contact interaction is a dimension-six term and reproduces the leading term of the SM decay amplitude if we impose the matching condition

$$\frac{C}{\Lambda^2} = -\frac{g_2^2}{2M_W^2} . \tag{3.7}$$

Traditionally, the EFT Lagrangian is written in terms of the Fermi constant G_F as

$$\mathcal{L}_{\text{Fermi}} = \ldots - \frac{4G_F}{\sqrt{2}} (\bar{\nu}_\mu \gamma^\mu P_L \mu)(\bar{e}\gamma_\mu P_L \nu_e) + \text{h.c.} + \ldots , \tag{3.8}$$

where

$$G_F = \frac{\sqrt{2}g_2^2}{8M_W^2} = \frac{1}{\sqrt{2}v^2} \tag{3.9}$$

is related to the Higgs vev, $v \approx 246\,\text{GeV}$.

From Eq. (3.5), we see that the expansion parameter of the theory is $\delta \sim p/M_W$. We have only used the leading term in the expansion—the higher corrections correspond to the effects of operators of dimension 8 and higher. We can identify the UV scale Λ either with M_W or with v, which only differ by the coupling g_2 and numerical factors, or we can keep Λ generic. Observables in the EFT depend only on G_F, but neither on C and Λ separately, nor on g_2 and M_W separately. By measuring the muon lifetime, we can determine $G_F \approx 1.166 \times 10^{-5}\,\text{GeV}^{-2}$. To dimension-six accuracy, the effective theory then becomes predictive. Deviations from the dimension-six prediction are described by dimension-8 effects and in muon decay are suppressed by $m_\mu^2/M_W^2 \approx 10^{-6}$. High-precision determinations of the Dalitz-plot distribution of muon decay can then be used to probe dimension-8 effects. The relative suppression determines the scale M_W.

3.2 LEFT operator basis

The Fermi theory can be extended to the most general low-energy EFT (LEFT) below the weak scale, which describes the indirect effects of the weak interaction (but also of heavy physics beyond the SM). Let us review the three core EFT principles for the LEFT.

1. *Degrees of freedom*: the relevant degrees of freedom are all SM particles with masses below the EW scale. These are the photon and the gluon, three generations of neutrinos, charged leptons, and down-type quarks, and two generations of up-type quarks. We exclude the $W^\pm$, Z, and h bosons and the heavy t quark. Quarks and charged leptons come as right- and left-chiral fields, but we only include left-chiral neutrino fields.

   Alternatively, one can also consider the LEFT below the b-quark threshold, which only contains two generations of down-type quarks.

2. *Symmetries*: the underlying UV theory is the SM with gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. Spontaneous symmetry breaking leaves the sub-group $SU(3)_c \times U(1)_{\text{em}}$, which is the gauge group of the LEFT. Furthermore, the SM contains accidental symmetries: as a consequence of the SM gauge symmetries and the restriction of renormalizability, the SM contains "by chance" a global $U(1)$ symmetry that implies baryon-number conservation, as well as three $U(1)$ symmetries that lead to lepton-family-number conservation (with a diagonal sub-group that corresponds to overall lepton-number conservation). The LEFT inherits these accidental symmetries when matched to the SM.

   When we extend the LEFT to describe effects beyond the SM, we will no longer impose the accidental SM symmetries, but restrict ourselves to the gauged $SU(3)_c \times U(1)_{\text{em}}$ symmetry group.

3. *Power counting*: the power-counting parameter is the ratio of a light mass scale or an external momentum divided by the electroweak scale v, or, equivalently, M_W (the numerical factors and the gauge coupling do not affect the overall power counting). In analogy to the scalar toy example, the operator basis is organized in powers of the canonical mass dimension of the operators.

The leading-order Lagrangian of the LEFT is given by the QED and QCD Lagrangian:

$$\mathcal{L}_{\mathrm{QED+QCD}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} + \theta_{\mathrm{QCD}}\frac{g^2}{32\pi^2}G^A_{\mu\nu}\widetilde{G}^{A\mu\nu} + \sum_{\psi=u,d,e,\nu}\bar{\psi}i\not{D}\psi - \left[\sum_{\psi=u,d,e}\bar{\psi}_R M_\psi \psi_L + \text{h.c.}\right], \tag{3.10}$$

where the covariant derivative is

$$D_\mu = \partial_\mu + igT^A G^A_\mu + ieQA_\mu \tag{3.11}$$

with $SU(3)_c$ color generators T^A and the charge matrix Q. The field-strength tensors are $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - gf^{ABC}G^B_\mu G^C_\nu$. The chiral components of the Dirac fields are given by $\psi_{L,R} = P_{L,R}\psi$ and ψ is a vector in flavor space of dimension n_ψ. We wrote a CP-violating QCD theta term, which involves the dual field-strength tensor

$$\widetilde{G}^{A\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}G^A_{\lambda\sigma}\,. \tag{3.12}$$

If the accidental symmetries of the SM are disregarded, we can write down a Majorana-mass term for the left-chiral neutrinos, which violates lepton number by $\Delta L = 2$ units.

The LEFT is defined by the leading-order Lagrangian plus a tower of higher-dimension effective operators:

$$\mathcal{L}_{\mathrm{LEFT}} = \mathcal{L}_{\mathrm{QED+QCD}} + \sum_{d\geq 5}\sum_{i=1}^{n_d} L_i^{(d)}\mathcal{O}_i^{(d)}\,, \tag{3.13}$$

where the Wilson coefficients are denoted by $L_i^{(d)}$ and scale as[4]

$$L_i^{(d)} \propto \frac{1}{v^{d-4}}\,. \tag{3.14}$$

It is important to note that the LEFT is a chiral gauge theory: although the leading-order QED and QCD Lagrangian is a vector theory (i.e., a non-chiral theory), where left- and right-chiral fermions interact in the same way with the gauge bosons, the higher-dimension operators distinguish between left- and right-chiral fermions: the LEFT inherits the chiral properties of the SM.

The construction of the operator basis proceeds as in the case of the scalar toy example: at a given mass dimension, we can write down an exhaustive list of Lorentz- and gauge-invariant operators consisting of the field variables and covariant derivatives. In a second step, one has to remove redundancies due to total-derivative and EOM operators. At dimension six, a new type of redundancy appears in four-fermion operators that we have not encountered in the scalar toy example and will be discussed in Sect. 3.3.

Schematically denoting the field-strength tensors by X and (anti-)fermions by ψ, the operator basis up to dimension six consists of the following types:

$$\begin{aligned} &\text{dimension 5:} \quad \psi^2 X\,, \\ &\text{dimension 6:} \quad X^3\,,\ \psi^4\,. \end{aligned} \tag{3.15}$$

[4] In contrast to chapter 2, we do not split off powers of the UV scale from the Wilson coefficients, which now have mass dimension $[L_i^{(d)}] = 4 - d$.

All other operator types can be reduced to total derivatives or EOM operators. The leading-order classical EOM read

$$\begin{aligned} i\not{D}\psi &= M_\psi \psi_L + M_\psi^\dagger \psi_R \,, \\ \partial_\mu F^{\mu\nu} &= e j^\nu_{\text{em}} = e \sum_{\psi=u,d,e} \bar\psi \gamma^\nu Q \psi \,, \\ (D_\mu G^{\mu\nu})^A &= g j^{A\nu} = g \sum_{\psi=u,d} \bar\psi T^A \gamma^\nu \psi \,. \end{aligned} \tag{3.16}$$

The dimension-five operators are photonic and gluonic dipole operators, e.g.,

$$\begin{aligned} \mathcal{O}_{e\gamma} &= \bar e_L \sigma^{\mu\nu} e_R F_{\mu\nu} \,, \\ \mathcal{O}_{uG} &= \bar u_L \sigma^{\mu\nu} T^A u_R G^A_{\mu\nu} \,, \end{aligned} \tag{3.17}$$

which in addition can carry fermion-flavor indices. There are only two X^3 operators: a CP-even and a CP-odd operator built from the QCD field-strength tensor:

$$\begin{aligned} \mathcal{O}_G &= f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu} \,, \\ \mathcal{O}_{\widetilde G} &= f^{ABC} \widetilde G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu} \,. \end{aligned} \tag{3.18}$$

The four-fermion operators ψ^4 can be classified in terms of the chiralities of the (anti-)fermions. In addition, we can split the four-fermion operators generated in the SM into leptonic operators (products of two leptonic bilinears), semi-leptonic operators (products of a leptonic with a quark bilinear), as well as non-leptonic operators (consisting of two quark bilinears). A complete and non-redundant operator basis for the LEFT up to dimension six can be found in Ref. [20]. The LEFT basis is already known up to dimension 9 [21–23]—at such high dimension, the number of independent operators explodes.

3.3 Fierz relations

When listing the four-fermion operators in the LEFT that mediate muon decay and only involve left-chiral fields, we could also write down an operator

$$\mathcal{O} = (\bar\nu_\mu \gamma^\mu P_L \nu_e)(\bar e \gamma_\mu P_L \mu) \tag{3.19}$$

in addition to the operator from Eq. (3.8),

$$\mathcal{O}' = (\bar\nu_\mu \gamma^\mu P_L \mu)(\bar e \gamma_\mu P_L \nu_e) \,. \tag{3.20}$$

However, it turns out that these two operators are equivalent due to another type of redundancy appearing in four-fermion operators, the *Fierz relations.* In general, a four-fermion operator can be written as

$$(\bar\psi_1 \Gamma^a \psi_2)(\bar\psi_3 \Gamma_a \psi_4) \,, \tag{3.21}$$

where Γ^a denotes a generic Dirac structure. In $D = 4$ space-time dimensions, a basis for the 4×4 matrices in Dirac space consists of 16 elements, which can be chosen as

$$\{\Gamma^a\} \;= \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, i\gamma^\mu\gamma_5, \gamma_5\} \,, \quad \mu > \nu \,, \tag{3.22}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The corresponding set in the dual space is given by

$$\{\Gamma_a\} \;= \{\mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}, i\gamma_\mu\gamma_5, \gamma_5\}. \tag{3.23}$$

These elements are normalized to

$$\mathrm{Tr}[\Gamma^a \Gamma_b] = 4\delta^a_b \,. \tag{3.24}$$

An arbitrary 4×4 matrix X can be decomposed as

$$X = \frac{1}{4}\mathrm{Tr}[X\Gamma^a]\Gamma_a \,. \tag{3.25}$$

By comparing the coefficients of X on both sides of this equation, one deduces the relation

$$\delta_{ii'}\delta_{jj'} = \frac{1}{4}\Gamma^a_{j'i'}\Gamma_{a,ij} \,. \tag{3.26}$$

By contracting this relation with two gamma matrices, one can derive the following generic Fierz identity:

$$\Gamma^a_{ij}\Gamma^b_{kl} = \sum_{c,d} \frac{1}{16}\mathrm{Tr}\{\Gamma^a\Gamma_c\Gamma^b\Gamma_d\}\Gamma^d_{il}\Gamma^c_{kj} \,. \tag{3.27}$$

The Dirac indices in this relation are often denoted by round and square brackets, leading to a more compact expression

$$(\Gamma^a) \otimes [\Gamma^b] = \sum_{c,d} \frac{1}{16}\mathrm{Tr}\{\Gamma^a\Gamma_c\Gamma^b\Gamma_d\}(\Gamma^d] \otimes [\Gamma^c) \,. \tag{3.28}$$

For arbitrary Dirac structures, one can now calculate the traces in Eq. (3.28) and derive relations between four-fermion operators, where the fermion fields are exchanged between the two bilinears. In the basis of chiral fermions, the following relations are obtained:

$$\begin{aligned}
(\gamma^\mu P_L) \otimes [\gamma_\mu P_L] &= -(\gamma^\mu P_L] \otimes [\gamma_\mu P_L) \,, \\
(\gamma^\mu P_L) \otimes [\gamma_\mu P_R] &= 2(P_R] \otimes [P_L) \,, \\
(\sigma^{\mu\nu} P_L) \otimes [\sigma_{\mu\nu} P_L] &= 8(P_L] \otimes [P_L) - 4(P_L) \otimes [P_L] \,, \\
(\sigma^{\mu\nu} P_L) \otimes [\sigma_{\mu\nu} P_R] &= 0 \,,
\end{aligned} \tag{3.29}$$

and analogous relations with flipped chiralities. Note that at the level of four-fermion operators, one will pick up an additional overall minus sign when reordering the fields due to the anticommutation relations of fermion fields. We therefore derive

$$\mathcal{O} = \mathcal{O}' \tag{3.30}$$

for the two operators in Eqs. (3.19) and (3.20). We also find that we can remove from the LEFT operator basis all leptoquark operators, which consist of an anti-lepton–quark bilinear and an anti-quark–lepton bilinear: the Fierz identities transform those operators into semileptonic operators.

When applying Fierz relations to four-quark operators, one needs to take into account that the quark fields also carry (contracted) color indices. When reordering the Dirac indices of the quark fields, one has to reshuffle the color indices by using the $SU(N_c)$ Fierz relation

$$T^A_{\alpha\beta}T^A_{\gamma\delta} = \frac{1}{2}\delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{2N_c}\delta_{\alpha\beta}\delta_{\gamma\delta} \,. \tag{3.31}$$

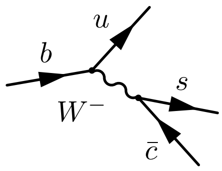
3.4 Operator mixing

Historically, the Fermi theory of weak interactions was devised from a bottom-up perspective to describe the observed low-energy effects of weak interaction. Although today we know about the existence of the weak-scale particles and the SM is an established theory, its low-energy effective description is not less important. E.g., if we are interested in weak decays of B-meson, we are dealing with effects that involve two widely separated mass scales: the low-energy scale given by the mass of the b-quark, and the much higher electroweak scale of the order of the W-boson mass. Hadronic decays of the B-meson involve strong dynamics, i.e., we are interested in QCD corrections. When calculating these effects in the full SM, we will encounter large logarithms multiplying the strong coupling,

$$\alpha_s \log\left(\frac{M_W^2}{m_b^2}\right) = \mathcal{O}(1)\,, \tag{3.32}$$

where the log is about $\log\left(M_W^2/m_b^2\right) \approx 6$ and $\alpha_s = g^2/(4\pi)$. These large logs render higher-order QCD corrections important and instead of a fixed-order calculation one really should use the EFT framework and switch to RG-improved perturbation theory. By using the EFT, we are limiting the calculation to a fixed accuracy in the power counting. This is an approximation that can be systematically improved, with higher-order power corrections suppressed by $m_b^2/M_W^2 \approx 3\times 10^{-3}$—this means that usually the leading order in the power counting is accurate enough. Since in the EFT we are resumming large logarithms, a fixed-order calculation in the full theory (the SM) usually is a worse approximation than a proper EFT calculation, which might appear counter-intuitive at first sight.

In order to see how the general concepts established in Sect. 2 work in a realistic example, we consider the situation of a hadronic B-meson decay $\bar{B}^0 \to D_s^- \pi^+$, which at the quark level is based on a transition $b \to u\bar{c}s$, mediated in the SM at tree level by the exchange of a W^- boson:



In the LEFT, the W boson is integrated out. In complete analogy to the Fermi-theory description of muon decay, the following effective operator is generated at tree-level and dimension six:

$$\mathcal{L}_{\mathrm{LEFT}} = \ldots + \frac{C}{\Lambda^2}(\bar{s}_L\gamma^\mu c_L)(\bar{u}_L\gamma_\mu b_L) + \text{h.c.} + \ldots\,, \tag{3.33}$$

where we use the notation

$$\psi_{L,R} = P_{L,R}\psi\,, \quad \bar{\psi}_{L,R} = (\psi_{L,R})^\dagger\gamma^0 = \bar{\psi}P_{R,L} \tag{3.34}$$

for the chiral fermions and the Wilson coefficient is given by

$$\frac{C}{\Lambda^2} = -\frac{g_2^2}{2M_W^2}V_{cs}^*V_{ub} = -\frac{4G_F}{\sqrt{2}}V_{cs}^*V_{ub}\,. \tag{3.35}$$

The additional factors compared to Eq. (3.8) are due to the fact that the coupling of the W boson to quark currents contains flavor-off-diagonal elements:

$$j_W^\mu = \sum_{\ell=e,\mu,\tau}\bar{\nu}_{\ell L}\gamma^\mu\ell_L + (\bar{u}_L,\bar{c}_L,\bar{t}_L)\gamma^\mu V_{\mathrm{CKM}}\begin{pmatrix} d_L\\ s_L\\ b_L\end{pmatrix}\,, \tag{3.36}$$

where

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{3.37}$$

denotes the unitary CKM quark-mixing matrix relating weak eigenstates to mass eigenstates.

When we explicitly write the color indices of the quarks, we find that there are in fact two different possible contractions. We can write the effective Lagrangian as

$$\mathcal{L}_{\mathrm{LEFT}} = \ldots + L_1' \mathcal{O}_1' + L_2' \mathcal{O}_2' + \mathrm{h.c.} + \ldots \,, \tag{3.38}$$

where

$$\begin{aligned} \mathcal{O}_1' &= (\bar{s}_L^i \gamma^\mu c_L^i)(\bar{u}_L^j \gamma_\mu b_L^j) \,, \\ \mathcal{O}_2' &= (\bar{s}_L^i \gamma^\mu c_L^j)(\bar{u}_L^j \gamma_\mu b_L^i) \,, \end{aligned} \tag{3.39}$$

and only the first operator is generated in the tree-level matching, i.e.,

$$L_1' = -\frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub} \,, \quad L_2' = 0 \,. \tag{3.40}$$

By making use of the first Fierz identity in Eq. (3.29) and the $SU(N_c)$ relation in Eq. (3.31), we can make a basis change to

$$\mathcal{L}_{\mathrm{LEFT}} = \ldots + L_1 \mathcal{O}_1 + L_2 \mathcal{O}_2 + \mathrm{h.c.} + \ldots \,, \tag{3.41}$$

where the operators now contain one bilinear with only down-type and one bilinear with only up-type quarks:

$$\begin{aligned} \mathcal{O}_1 &= (\bar{s}_L \gamma^\mu b_L)(\bar{u}_L \gamma_\mu c_L) \,, \\ \mathcal{O}_2 &= (\bar{s}_L \gamma^\mu T^A b_L)(\bar{u}_L \gamma_\mu T^A c_L) \,. \end{aligned} \tag{3.42}$$

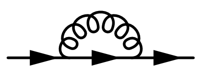
The tree-level matching in this new basis reads

$$L_1 = -\frac{1}{N_c} \frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub} \,, \quad L_2 = -\frac{8G_F}{\sqrt{2}} V_{cs}^* V_{ub} \,. \tag{3.43}$$

If we compute the matching at the weak scale $\mu = M_W$, loop corrections to these relations do not involve any large logarithms. Let us now derive how these Wilson coefficients of the EFT depend on the renormalization scale μ, where the scale dependence arises from QCD one-loop corrections. We use renormalized perturbation theory and relate bare quantities (with a tilde) to renormalized ones (without tilde) via

$$\tilde{q} = Z_q^{1/2} q \,, \quad \tilde{L}_i = Z_{ij} L_j \,, \tag{3.44}$$

where q generically denotes the quark fields. The renormalization of the Wilson coefficients is not only a multiplicative factor, but involves the 2×2 renormalization matrix Z_{ij}, which can contain off-diagonal elements. The $\overline{\mathrm{MS}}$ quark-field renormalization factor is the same for all flavors and obtained by calculating the quark self-energy:


, (3.45)

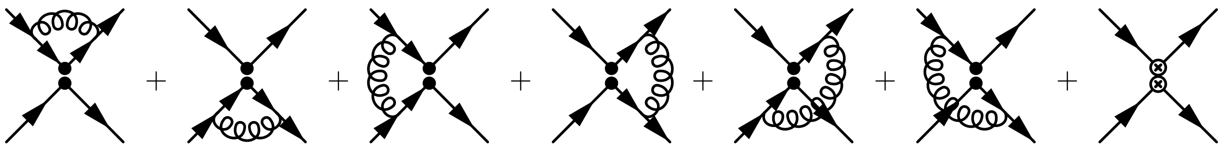
leading to the result

$$Z_q = 1 - \frac{\alpha_s C_F}{4\pi}\frac{1}{\varepsilon}\,, \tag{3.46}$$

where

$$C_F = \frac{N_c^2 - 1}{2N_c} \tag{3.47}$$

is the quadratic Casimir invariant of $SU(N_c)$ in the fundamental representation. The renormalization matrix Z_{ij} is obtained by calculating the four-point 1PI Green's function at one loop, given by:

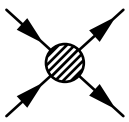

,

where the last diagram denotes the counterterm contribution. The two adjacent dots in the diagrams stand for a single interaction vertex, the four-fermion interaction: the dots represent the two quark bilinears.

Splitting the renormalization factors as

$$Z_q = 1 + \Delta_q\,, \quad Z_{ij} = \delta_{ij} + \Delta_{ij}\,, \tag{3.48}$$

the sum of loop and counterterm diagrams is given by



$$\begin{aligned} &= \left[\frac{\alpha_s}{4\pi\varepsilon}\left(2C_F L_1 - \frac{3C_F}{N_c}L_2\right) + 2\Delta_q L_1 + \Delta_{1k}L_k\right] i(\gamma^\mu P_L)\otimes[\gamma_\mu P_L] \\ &+ \left[\frac{\alpha_s}{4\pi\varepsilon}\left(-6L_1 + \left(2C_F + \frac{6}{N_c}\right)L_2\right) + 2\Delta_q L_2 + \Delta_{2k}L_k\right] i(T^A\gamma^\mu P_L)\otimes[T^A\gamma_\mu P_L] \\ &+ \text{finite}\,. \end{aligned} \tag{3.49}$$

In order to obtain this result, we have made use of relations such as

$$(\gamma^\mu\gamma^\nu\gamma^\lambda P_L)\otimes[\gamma_\mu\gamma_\nu\gamma_\lambda P_L] = 16(\gamma^\mu P_L)\otimes[\gamma_\mu P_L]\,, \tag{3.50}$$

which holds in $D = 4$ dimensions and can be derived using the Fierz relations. Similarly, for the color-space algebra the following $SU(N_c)$ relations are useful:

$$(T^A T^B)\otimes[T^A T^B] = \frac{C_F}{2N_c}(\mathbb{1})\otimes[\mathbb{1}] - \frac{1}{N_c}(T^A)\otimes[T^A]\,, \tag{3.51}$$

which can be derived either by applying Eq. (3.31) multiple times or by using

$$T^A T^B = \frac{1}{2}\left(\frac{1}{N_c}\delta^{AB} + d^{ABC}T^C + if^{ABC}T^C\right)\,. \tag{3.52}$$

If we plug Δ_q into Eq. (3.49), we can read off the values of the $\overline{\text{MS}}$ renormalization constants that render the four-point function finite:

$$\Delta_{ij} = \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} 0 & \frac{3C_F}{N_c} \\ 6 & -\frac{6}{N_c} \end{pmatrix}_{ij}\,. \tag{3.53}$$

Since α_s is the running coupling with

$$\beta(\alpha_s) = \frac{d}{d\log\bar{\mu}}\alpha_s = -2\varepsilon\alpha_s + \mathcal{O}(\alpha_s^2)\,, \tag{3.54}$$

the $\overline{\mathrm{MS}}$ subtraction introduces a renormalization-scale dependence into the Wilson coefficients:

$$0 = \frac{d}{d\log\bar{\mu}}\tilde{L}_i = \frac{d}{d\log\bar{\mu}}L_i + \frac{d\Delta_{ij}}{d\log\bar{\mu}}L_j + \underbrace{\Delta_{ij}\frac{dL_j}{d\log\bar{\mu}}}_{\text{2-loop}}\,, \tag{3.55}$$

hence at one loop accuracy

$$\frac{d}{d\log\bar{\mu}}L_i = \frac{\alpha_s}{4\pi}\begin{pmatrix} 0 & \frac{6C_F}{N_c} \\ 12 & -\frac{12}{N_c}\end{pmatrix}_{ij} \times L_j\,. \tag{3.56}$$

This differential equation is the one-loop QCD RGE for the two Wilson coefficients L_1 and L_2 and describes the running and mixing under variation of the renormalization scale. It is most easily solved by making a basis change that diagonalizes the mixing matrix:

$$L_1 =: \frac{L^+ - L^-}{2} + \frac{L^+ + L^-}{2N_c}\,, \quad L_2 =: L^+ + L^-\,, \tag{3.57}$$

leading to

$$\frac{d}{d\log\bar{\mu}}L^\pm = \frac{\alpha_s}{4\pi}6\left(\pm 1 - \frac{1}{N_c}\right)L^\pm =: \frac{\alpha_s}{4\pi}\gamma^\pm L^\pm\,. \tag{3.58}$$

This diagonal RGE can be solved by separating the variables:

$$L^\pm(\bar{\mu}) = L^\pm(M_W)\exp\left(\int_{\alpha_s(M_W)}^{\alpha_s(\bar{\mu})} d\alpha\frac{\alpha}{4\pi}\frac{\gamma^\pm}{\beta(\alpha)}\right) \tag{3.59}$$

The QCD beta function at one loop (for $\varepsilon \to 0$) is given by

$$\beta(\alpha_s) = -2\alpha_s\left[\frac{\alpha_s}{4\pi}\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \mathcal{O}(\alpha_s^3)\right]\,, \quad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}(n_u + n_d)\,, \tag{3.60}$$

where $n_{u,d}$ denotes the number of up- and down-type quarks. At one loop, we obtain

$$L^\pm(\bar{\mu}) = L^\pm(M_W)\left(\frac{\alpha_s(M_W)}{\alpha_s(\bar{\mu})}\right)^{\frac{\gamma^\pm}{2\beta_0}}\,, \quad \alpha_s(\bar{\mu}) = \frac{\alpha_s(M_W)}{1 + \frac{\alpha_s(M_W)}{4\pi}\beta_0\log\left(\frac{\bar{\mu}^2}{M_W^2}\right)}\,. \tag{3.61}$$

If we evaluate this solution numerically at $\bar{\mu} = m_b$, we obtain with $\alpha_s(M_W) \approx 0.12$ the leading-log (LL), i.e., one-loop RG-improved, result

$$L^+(m_b)\Big|_{\mathrm{LL}} \approx 0.86L^+(M_W)\,, \quad L^-(m_b)\Big|_{\mathrm{LL}} \approx 1.34L^-(M_W)\,. \tag{3.62}$$

The fixed-order one-loop result that follows from re-expanding Eq. (3.61) to linear order in $\alpha_s(M_W)$ gives

$$L^+(m_b)\Big|_{\mathrm{NLO}} \approx 0.89L^+(M_W)\,, \quad L^-(m_b)\Big|_{\mathrm{NLO}} \approx 1.23L^-(M_W)\,. \tag{3.63}$$

Note that the RG-improved result for L^- differs from the fixed-order result by about 10%, even though the difference is a two-loop effect. For $\alpha_s \approx 0.12$, we have

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \approx 9.1 \times 10^{-5} \,, \tag{3.64}$$

which illustrates the numerical effect of the large logarithms.

The running and mixing can also be expressed in terms of our original Wilson coefficients. Numerically,

$$\begin{pmatrix} L_1(m_b) \\ L_2(m_b) \end{pmatrix}\Bigg|_{\text{LL}} \approx \begin{pmatrix} 1.02 & -0.11 \\ -0.48 & 1.18 \end{pmatrix} \begin{pmatrix} L_1(M_W) \\ L_2(M_W) \end{pmatrix} . \tag{3.65}$$

3.5 Scheme dependence and evanescent operators

In Eq. (3.50), we have made use of the Fierz relations, which are valid only in $D = 4$ space-time dimensions. The reason for the Fierz relations (3.28) not being valid away from $D = 4$ lies in the fact that the 16 matrices in Eq. (3.22) do no longer form a basis for the Dirac structures: for non-integer D, the Dirac algebra is in fact infinite dimensional. Since we were only interested in the divergent $1/\varepsilon$ piece in Eq. (3.50), we have put $\varepsilon = 0$ in the numerator and used the relation that is valid for $D = 4$.

Suppose that we changed the RHS of Eq. (3.50) by a term of $\mathcal{O}(\varepsilon)$. Such a term will vanish by itself in $D = 4$ and is called *evanescent*. However, since the term appears in a loop integral that contains a $1/\varepsilon$ divergence, it will lead to a modification of the finite part of the integral. This implies that in a matching calculation at one loop, where we are interested in the finite pieces of the loop integrals, the definition of evanescent terms will affect the matching results. Similarly, evanescent terms affect the $1/\varepsilon$ divergence at two loops, since the loop integral generates $1/\varepsilon^2$ divergences. Therefore, the definition of evanescent terms affects the two-loop RGE.

This might seem disturbing at first sight, but is no reason to be worried: Lagrangian parameters are not physical observables and contain a certain arbitrariness. One particular choice defines a *scheme* for the calculation. In relations between observables, scheme dependences need to cancel up to effects beyond the accuracy of the calculation, but Lagrangian parameters always contain a scheme dependence. We are already used to this from the comparison of different renormalization schemes, which agree on the divergent parts of the counterterms, but differ in the definition of the finite parts of counterterms. In the present context, we see that even within $\overline{\text{MS}}$, there are different scheme choices, which involve evanescent terms. Without going into details, we list a few cases that lead to the appearance of evanescent operators and need to be considered in calculations beyond LL: it is important that in a calculation one defines the scheme choice and consistently applies the scheme throughout the calculation.

- **Evanescent terms in four-fermion structures**

  Evanescent terms arise when reducing four-fermion structures, where the two bilinears contain strings of Dirac matrices. By applying Fierz relations, one can deduce reduction formulas such as Eq. (3.50), which hold in $D = 4$ dimensions. Away from $D = 4$ dimensions, these relations do not hold: the difference of the two sides of the equation becomes an evanescent operator, which has tree-level matrix elements of $\mathcal{O}(\varepsilon)$.

  By using the Dirac algebra, which holds in arbitrary dimensions, one can always decompose a string of Dirac matrices as a linear combination of the basis elements

  $$\Gamma^{(n)} := \frac{1}{n!}\gamma^{[\mu_1} \cdots \gamma^{\mu_n]} \,, \tag{3.66}$$

  where the square brackets denote anti-symmetrization of the Lorentz indices. If $n > 4$, the structure is evanescent, i.e., it vanishes in $D = 4$ dimensions. Finite contributions to matrix elements that arise from a $1/\varepsilon$ loop divergence times an evanescent structure can always be cancelled by finite local counterterms. In such a scheme, evanescents do not contribute to physical matrix elements, but they still need to be taken into account in the renormalization procedure and they affect the two-loop anomalous dimensions. A consistent scheme has been worked out in Ref. [24].

- **Fierz-evanescent terms**

  Fierz-evanescent operators are special cases of four-fermion evanescents and are given by the difference between the LHS and the RHS of the Fierz identity (3.28), e.g.,

$$\mathcal{E} = (\gamma^\mu P_L) \otimes [\gamma_\mu P_L] + (\gamma^\mu P_L] \otimes [\gamma_\mu P_L) \,. \tag{3.67}$$

- **Levi-Civita tensor and Schouten identity**

  The Levi-Civita symbol is an intrinsically four-dimensional object. In DR, this leads to notorious problems related to the definition of $\epsilon^{\mu\nu\lambda\sigma}$ away from $D = 4$ space-time dimensions. In principle, a consistent scheme only allows the indices of the Levi-Civita symbol to run from $\mu = 0 \ldots 3$, but different prescriptions are often used as well. The Schouten identity

$$0 = g_{\mu\nu}\epsilon_{\lambda\alpha\beta\gamma} + g_{\mu\lambda}\epsilon_{\alpha\beta\gamma\nu} + g_{\mu\alpha}\epsilon_{\beta\gamma\nu\lambda} + g_{\mu\beta}\epsilon_{\gamma\nu\lambda\alpha} + g_{\mu\gamma}\epsilon_{\nu\lambda\alpha\beta} \tag{3.68}$$

  is only true in $D = 4$ dimensions and operators involving the RHS of this equation are evanescent.

- **Schemes for γ_5**

  Closely related is the problem with the definition of γ_5 away from $D = 4$ dimensions. The definition $\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\lambda\sigma}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma$ involves again the Levi-Civita symbol. In $D = 4$, we know that γ_5 anticommutes with the Dirac matrices:

$$\{\gamma_5, \gamma^\mu\} = 0 \,. \tag{3.69}$$

  Assuming this relation to be valid for $D \neq 4$ is in conflict with

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] \neq 0 \tag{3.70}$$

  and a cyclic trace. DR with a naively anticommuting γ_5 in D dimension (the NDR scheme) does not reproduce the chiral anomaly. Restricting Eq. (3.69) to $D = 4$ and defining

$$\begin{aligned} \{\gamma_5, \gamma^\mu\} &= 0 \quad \text{for } \mu = 0,1,2,3 \,, \\ [\gamma_5, \gamma^\mu] &= 0 \quad \text{else} \end{aligned} \tag{3.71}$$

  is the self-consistent 't Hooft–Veltman scheme [25, 26]. Although it reproduces the chiral anomaly, it has the disadvantage that in chiral gauge theories it also leads to spurious anomalies that break the Ward identities, because D-dimensional Lorentz covariance and chiral symmetry is broken. The symmetry-breaking terms then need to be cured by finite renormalizations. In this scheme, the anticommutator $\{\gamma_5, \gamma^\mu\}$ is again an evanescent term in D dimensions.

4 Strong interaction at low energies

As a next classic example of effective field theories, we consider the strong interaction at low energies. QCD is the established fundamental theory of the strong interaction of quarks and gluons. Perturbative QCD can be applied to compute corrections to hard scattering processes at high energies: it relies on an expansion in the strong coupling, which leads to the loop expansion. The coupling constant as the expansion parameter depends on the renormalization scale: the running of α_s is dictated by the QCD beta function, which at one loop leads to the solution given in Eq. (3.61):

$$\alpha_s(\bar{\mu}) = \frac{\alpha_s(\bar{\mu}_0)}{1 + \frac{\alpha_s(\bar{\mu}_0)}{4\pi}\beta_0 \log\left(\frac{\bar{\mu}^2}{\bar{\mu}_0^2}\right)} \,. \tag{4.1}$$

We see that for increasing $\bar{\mu}$, the coupling becomes smaller. In order to avoid large logarithms, one should choose the renormalization scale $\bar{\mu}$ to be around the typical energy scale of the process. Therefore, for high energies we can compute with a small expansion parameter. For $\bar{\mu} \to \infty$, QCD becomes a non-interacting theory, an effect known as *asymptotic freedom* [27, 28]. At $\mu \approx M_W$, the value of the strong coupling is $\alpha_s \approx 0.12$. At lower energies, the coupling constant becomes larger and the effects of higher loop orders become more important. At very low energies, the perturbative expansion becomes meaningless since the coupling constant is no longer a small parameter. At the same time, non-perturbative effects that cannot be described by a truncated power series in α_s or not even by a resummation of a power series become more and more important. Therefore, the dynamics of QCD at low energies cannot be described by perturbation theory and one needs to resort to alternative, non-perturbative methods.

Another important aspect of QCD is that it leads to *confinement*: the observed particle states are in fact not quarks and gluons, but composite particles that are not charged under the symmetry group $SU(3)_c$ of QCD, the hadrons. The lightest hadrons are the three pions, $\pi^\pm$ and π^0, which are spin-0 bosons with masses of $M_{\pi^0} \approx 135.0\,\text{MeV}$ and $M_{\pi^\pm} \approx 139.6\,\text{MeV}$. After a mass gap, the next-to-lightest hadrons are the kaons with $M_{K^\pm} \approx 493.7\,\text{MeV}$ and $M_{K^0} \approx 497.6\,\text{MeV}$ and the eta meson with $M_\eta \approx 547.9\,\text{MeV}$. The hadron spectrum also contains a very broad scalar state, the $f_0(500)$ with a T-matrix pole at around $\sqrt{s} \approx (450 - i270)\,\text{MeV}$, decaying into two pions. In the vector (i.e., spin-1) channel, there is the prominent rho resonance with $M_\rho \approx 770\,\text{MeV}$, decaying again into two pions. In the GeV region and above, many hadronic states are present. The lightest spin-1/2 hadronic states, the proton and the neutron, have masses of $m_p \approx 938.3\,\text{MeV}$ and $m_n \approx 939.6\,\text{MeV}$.

In the simple quark model, mesons are understood to be bosonic bound states of a quark and an antiquark, while the baryons are fermionic bound states of three quarks. To a large extent, the structure of the hadronic mass spectrum can be understood based on the symmetries of QCD. In particular, the presence of the lightest mesons is understood as a manifestation of spontaneous symmetry breaking in QCD. Due to their symmetry properties and the mass gap, the interaction of the pions (or even the pions, kaons, and the eta meson) can be described in terms of a low-energy EFT known as chiral perturbation theory (χPT) [29–31].

4.1 QCD and chiral symmetry

QCD describes strong interaction as a non-abelian gauge theory based on the $SU(3)_c$ symmetry group. The Lagrangian is defined in terms of quark and gluon fields as

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \bar{q} i \not{D} q - \bar{q}_R M_q q_L - \bar{q}_L M_q^\dagger q_R \,, \tag{4.2}$$

see also Eq. (3.10). We ignore the CP-violating QCD theta term and consider only the light quark flavors, $q = (u, d, s)$. For the moment, we also ignore QED, i.e., the presence of leptons and photons. The gauge-covariant derivative in the fundamental representation is defined by

$$D_\mu = \partial_\mu + i g T^A G^A_\mu \,. \tag{4.3}$$

In terms of chiral fields, the Lagrangian is given by

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} + \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R - \bar{q}_R M_q q_L - \bar{q}_L M_q^\dagger q_R\,. \tag{4.4}$$

We see that in addition to the gauge symmetry, the classical Lagrangian exhibits a global flavor symmetry

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \end{pmatrix}\,, \quad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \end{pmatrix}\,, \tag{4.5}$$

with $U_{L,R} \in U(2)$ two independent matrices, provided that we neglect the masses of the up and down quarks. Note that the quark masses are very small, $m_u \approx 2.3\,\text{MeV}$, $m_d \approx 4.8\,\text{MeV}$. If we also neglect the somewhat larger strange-quark mass ($m_s \approx 95\,\text{MeV}$), the global symmetry is enlarged to a global $U(3)_L \times U(3)_R$ group:

$$q_L \mapsto U_L q_L\,, \quad q_R \mapsto U_R q_R\,, \quad U_{L,R} \in U(3)\,. \tag{4.6}$$

The limit of vanishing quark masses is called *chiral limit*. The $U(3)$ matrices can be parametrized as

$$U_{L,R} = \exp\left\{-i\theta^a_{L,R}\frac{\lambda^a}{2} - i\theta_{L,R}\right\}\,, \tag{4.7}$$

with λ^a the Gell-Mann matrices in flavor space.[5] The corresponding Noether currents are given by

$$\begin{aligned} L^{\mu,a} &= \bar{q}_L\gamma^\mu\frac{\lambda^a}{2}q_L\,, \quad & R^{\mu,a} &= \bar{q}_R\gamma^\mu\frac{\lambda^a}{2}q_R\,, \\ L^\mu &= \bar{q}_L\gamma^\mu q_L\,, & R^\mu &= \bar{q}_R\gamma^\mu q_R\,. \end{aligned}$$

It is also useful to introduce vector and axial-vector currents:

$$\begin{aligned} V^{\mu,a} &= R^{\mu,a} + L^{\mu,a}\,, \quad & A^{\mu,a} &= R^{\mu,a} - L^{\mu,a}\,, \\ V^\mu &= R^\mu + L^\mu\,, & A^\mu &= R^\mu - L^\mu\,. \end{aligned}$$

At the quantum level, it turns out that the singlet axial current is not conserved due to the triangle anomaly:

$$\partial_\mu A^\mu = \frac{g^2 n_f}{16\pi^2}G^A_{\mu\nu}\widetilde{G}^{A\mu\nu}\,, \tag{4.8}$$

where $n_f = 3$ is the number of quark flavors. The octet vector and axial-vector currents, as well as the singlet vector current are conserved in the chiral limit. The Lagrangian of massless three-flavor QCD therefore has a global $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry. To each current belongs a conserved charge:

$$\begin{aligned} Q^a_V(t) &= \int d^3x V^{0,a}(x) = \int d^3x\, \bar{q}(x)\gamma^0\frac{\lambda^a}{2}q(x) = \int d^3x\, q^\dagger(x)\frac{\lambda^a}{2}q(x)\,, \\ Q^a_A(t) &= \int d^3x A^{0,a}(x) = \int d^3x\, \bar{q}(x)\gamma^0\gamma_5\frac{\lambda^a}{2}q(x) = \int d^3x\, q^\dagger(x)\gamma_5\frac{\lambda^a}{2}q(x)\,, \\ Q_V(t) &= \int d^3x V^0(x) = \int d^3x\, \bar{q}(x)\gamma^0 q(x) = \int d^3x\, q^\dagger(x)q(x)\,. \end{aligned} \tag{4.9}$$

The $2 \times 8 + 1$ charges commute with the Hamiltonian of massless three-flavor QCD.

[5] We are using uppercase letters A, B, … for color $SU(3)_c$ indices and lowercase letters a, b, … for flavor $SU(3)_f$ indices.

However, now the question is if this symmetry is also realized by the states of the theory. Observations indicate a different symmetry pattern for hadrons: the mass spectrum suggests that one can organize the light hadrons as approximately mass-degenerate multiplets of $SU(3)_V \times U(1)_V$, where $SU(3)_V$ is the diagonal subgroup of $SU(3)_L \times SU(3)_R$. The $U(1)_V$ symmetry results in baryon-number conservation and classifies the hadrons into mesons ($B = 0$) and baryons ($B = 1$). If $SU(3)_L \times SU(3)_R$ were realized in nature, one would expect degenerate states with opposite parity. This is not observed, which indicates that this full symmetry is in fact *hidden* and only the diagonal subgroup $SU(3)_V$ is realized in nature. This is also known as *spontaneous symmetry breaking*: although the Lagrangian of the theory is invariant under a symmetry group, the vacuum of the theory and the spectrum break the symmetry.

4.2 Spontaneous symmetry breaking and Goldstone's theorem

Goldstone's theorem states that each spontaneously broken continuous symmetry induces the existence of a massless scalar particle—a Goldstone boson [32]. Assume that a theory is symmetric under the transformations of a Lie group G with dimension n_G. Then, we find n_G Noether charges Q_i, which form a representation of the corresponding Lie algebra in the Hilbert space. These generators of the symmetry commute with the Hamiltonian, since they are conserved quantities:

$$[\mathcal{H}, Q_i] = 0\,, \quad i \in \{1, \dots, n_G\}\,. \tag{4.10}$$

We assume that the vacuum state $|0\rangle$ of the system is invariant only under a subgroup $H \subset G$ of dimension n_H, i.e., the vacuum is annihilated by n_H generators H_i:

$$H_i|0\rangle = 0\,, \quad i \in \{1, \dots, n_H\}\,. \tag{4.11}$$

The generators H_i form the Lie algebra of the subgroup H. Thus, there exist $n_G - n_H$ generators X_i of G that do not annihilate the vacuum: $X_i|0\rangle \neq 0$. This means that the vacuum is degenerate with the $n_G - n_H$ states $X_i|0\rangle$:

$$\mathcal{H}X_i|0\rangle = X_i\mathcal{H}|0\rangle = 0\,. \tag{4.12}$$

We see that there exist $n = n_G - n_H$ massless states, the Goldstone bosons, which emerge from the spontaneous breakdown of the symmetry group G to its subgroup H.

Let us apply Goldstone's theorem to the situation of QCD. For n_f massless flavors, the full global symmetry group is

$$G = SU(n_f)_L \times SU(n_f)_R \times U(1)_V \tag{4.13}$$

with $n_G = (n_f^2 - 1) + (n_f^2 - 1) + 1 = 2n_f^2 - 1$ generators. The vacuum is invariant only under the subgroup

$$H = SU(n_f)_V \times U(1)_V \tag{4.14}$$

with $n_H = (n_f^2 - 1) + 1 = n_f^2$ generators. Therefore, we expect $n = n_f^2 - 1$ massless Goldstone bosons—three for $n_f = 2$, or eight for $n_f = 3$. We identify the broken generators with the axial charges Q_A^a. In our context, the simple argument about the massless states as given above is not quite accurate since the states $Q_A^a|0\rangle$ turn out to have infinite norm. More rigorously, one has to assume the existence of an operator $\mathcal{O}^a$ that satisfies

$$\langle 0|[Q_A^a(t), \mathcal{O}^a(t, \vec{x})]|0\rangle \neq 0 \quad (\text{no sum over } a)\,, \tag{4.15}$$

which of course requires $Q_A^a|0\rangle \neq 0$. In this case, there will be a massless state $|\pi^a\rangle$ that satisfies

$$\langle 0|A^{0,a}(t, \vec{y})|\pi^a\rangle\langle \pi^a|\mathcal{O}^a(t, \vec{x})|0\rangle \neq 0 \quad (\text{no sum over } a) \tag{4.16}$$

and $\mathcal{O}^a$ can be used as an interpolating field for the state $|\pi^a\rangle$, i.e., the corresponding massless Goldstone boson has the quantum numbers of $\mathcal{O}^a$ and $A^{0,a}$. Since the broken generators are the axial charges Q_A^a, the Goldstone bosons are pseudoscalar particles and we can choose the pseudoscalar density as the interpolating field. Then, we have to require that

$$\langle 0|[Q_A^a(t), P^a(t,\vec{x})]|0\rangle \neq 0 \quad (\text{no sum over } a)\,, \quad P^a = i\bar{q}\gamma_5 \frac{\lambda^a}{2} q\,. \tag{4.17}$$

The matrix element can be simplified using the equal-time commutation relations

$$\{q_{\alpha,i}(t,\vec{x}), q^\dagger_{\beta,j}(t,\vec{y})\} = \delta_{\alpha\beta}\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y})\,, \quad \{q_{\alpha,i}, q_{\beta,j}\} = 0\,, \quad \{q^\dagger_{\alpha,i}, q^\dagger_{\beta,j}\} = 0\,, \tag{4.18}$$

leading to[6]

$$i\langle 0|[Q_A^a(t), P^a(t,\vec{x})]|0\rangle = \frac{1}{3}\langle 0|\bar{q}q|0\rangle = \frac{1}{3}\langle 0|\bar{u}u + \bar{d}d + \bar{s}s|0\rangle = \langle 0|\bar{u}u|0\rangle\,. \tag{4.19}$$

We find that the existence of a non-vanishing quark condensate $\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle \neq 0$ implies that chiral symmetry is spontaneously broken, leading to massless pseudoscalar Goldstone bosons.

Both the pseudoscalar density and the axial-charge operator have a non-vanishing matrix element between the vacuum and the massless one-particle state $|\pi^a\rangle$. Due to Lorentz covariance, the matrix element of the axial-vector current between the vacuum and the Goldstone-boson state can be written as

$$\langle 0|A_\mu^a(0)|\pi^b(p)\rangle = ip_\mu F_0 \delta^{ab}\,, \tag{4.20}$$

where F_0 is called the pion decay constant in the chiral limit.

We identify the Goldstone bosons with the lightest mesons: the three Goldstone bosons from the breaking of $SU(2)_L \times SU(2)_R \to SU(2)_V$ are identified with the three pions π^+, π^0, π^-, which form an isospin triplet. The eight Goldstone bosons from the breaking of $SU(3)$ are identified with the meson octet consisting of the three pions, the four kaons, and the η. The fact that these mesons are light but not exactly massless as required by Goldstone's theorem will be explained later by the presence of the mass terms in QCD: the quark-mass terms introduce a small explicit breaking of chiral symmetry in the Lagrangian, which will result in a non-vanishing mass of the Goldstone bosons.

4.3 Nonlinear realization and CCWZ formalism

At low energies, the hadronic degrees of freedom are the pions, kaons, and η mesons. As we discussed, they are identified with the Goldstone bosons of spontaneous breaking of the chiral $SU(3)_L \times SU(3)_R$ down to $SU(3)_V$, i.e., they correspond to the eight generators Q_A^a that belong to the axial-vector currents $A^{\mu,a}$. In order to set up an EFT for the light hadrons, we need to determine how the degrees of freedom transform under the symmetry group. The transformation of the Goldstone bosons will turn out to be nonlinear. This is described by the formalism developed by Callan, Coleman, Wess, and Zumino (CCWZ) [33, 34].

The Goldstone bosons are described by n real scalar fields $\vec{\pi}(x) = (\pi_1(x), \ldots, \pi_n(x))$, which are smooth functions on Minkowski space M^4 and form a vector space $V := \{\vec{\pi} : M^4 \to \mathbb{R}^n\}$. The operation of the symmetry group G as a group action $\vec{\varphi}$ on the vector space V is defined as a mapping

$$\vec{\varphi} : G \times V \to V\,, \quad (g, \vec{\pi}) \mapsto \vec{\varphi}(g, \vec{\pi}) \tag{4.21}$$

[6] Here we are averaging over a, which is possible due to $SU(3)_V$ invariance and can be derived by considering the commutator $[Q_V^a, S^b]$, where S^a is the scalar density.

that fulfills

$$\begin{aligned}&\vec{\varphi}(e,\vec{\pi})=\vec{\pi}\quad\forall\vec{\pi}\in V\ (e\text{ denoting the identity of }G)\,,\\&\vec{\varphi}(g_1,\vec{\varphi}(g_2,\vec{\pi}))=\vec{\varphi}(g_1g_2,\vec{\pi})\quad\forall\vec{\pi}\in V,\ \forall g_1,g_2\in G\,.\end{aligned}\tag{4.22}$$

The ground state $\vec{0}$ is assumed to be invariant under the subgroup $H\subset G$:

$$\vec{\varphi}(h,\vec{0})=\vec{0}\quad\forall h\in H\,.\tag{4.23}$$

It follows

$$\vec{\varphi}(gh,\vec{0})=\vec{\varphi}(g,\vec{\varphi}(h,\vec{0}))=\vec{\varphi}(g,\vec{0})\quad\forall g\in G,\ h\in H\tag{4.24}$$

and $g^{-1}g'\in H$ if $\vec{\varphi}(g,\vec{0})=\vec{\varphi}(g',\vec{0})$:

$$\vec{0}=\vec{\varphi}(g^{-1}g,\vec{0})=\vec{\varphi}(g^{-1},\vec{\varphi}(g,\vec{0}))=\vec{\varphi}(g^{-1},\vec{\varphi}(g',0))=\vec{\varphi}(g^{-1}g',\vec{0})\,.\tag{4.25}$$

Since the Goldstone-boson manifold is obtained by acting with the symmetry group on the vacuum, there exists an isomorphism between the left coset space $G/H=\{gH|g\in G\}$ and the Goldstone boson fields V. The Goldstone bosons transform under $g\in G$ in the following way:

$$\vec{\pi}=\vec{\varphi}(\tilde{g},\vec{0})=\vec{\varphi}(\tilde{g}h,\vec{0})\quad\Rightarrow\quad\vec{\pi}'=\vec{\varphi}(g,\vec{\pi})=\vec{\varphi}(g,\vec{\varphi}(\tilde{g}h,\vec{0}))=\vec{\varphi}(g\tilde{g}h,\vec{0})\,.\tag{4.26}$$

Let us apply this formalism to the case of QCD with $G=SU(3)_L\times SU(3)_R$ and $H=SU(3)_V$. The elements of the coset G/H can be parametrized by

$$\begin{aligned}&\tilde{g}h\in\tilde{g}H\,,\quad\tilde{g}\in G\,,\\&\tilde{g}h=(\tilde{L},\tilde{R})(V,V)=(1,\tilde{R}\tilde{L}^\dagger)(\tilde{L}V,\tilde{L}V)=:(1,U)(V',V')\,,\end{aligned}\tag{4.27}$$

where U transforms as

$$\tilde{g}h\overset{G}{\mapsto}g\tilde{g}h=(L,R)(1,U)(V',V')=(1,RUL^\dagger)(LV',LV')\,,\tag{4.28}$$

i.e., $U\mapsto RUL^\dagger$ and $U^\dagger\mapsto LU^\dagger R^\dagger$. The $SU(3)$ matrix U is then written as

$$U(x)=\exp\left(i\frac{\pi(x)}{F_0}\right)\,,\quad\pi(x)=\sum_{a=1}^{8}\lambda^a\pi^a(x)\,,\quad\mathrm{Tr}(\lambda^a\lambda^b)=2\delta^{ab}\,,\tag{4.29}$$

where $\pi^a(x)$ are identified with the Goldstone boson fields and the parameter F_0 is the pion decay constant in the chiral limit, defined in Eq. (4.20). It is related to the physical decay constant $F_\pi=92.3\,\mathrm{MeV}$, which enters the description of pion decay $\pi^+\to\mu^+\nu_\mu$.

The CCWZ formalism provides us with a convenient parametrization of the degrees of freedom in terms of the matrix $U(x)$ in Eq. (4.29). The transformation under the chiral group $SU(3)_L\times SU(3)_R$ is given by $U\mapsto RUL^\dagger$. The ground state of the system corresponds to $\pi(x)=0$, i.e., $U_0=\mathbb{1}$. Under the diagonal subgroup $H=\{(V,V)|V\in SU(3)\}$, the ground state transforms as

$$U_0\mapsto VU_0V^\dagger=VV^\dagger=\mathbb{1}=U_0\,,\tag{4.30}$$

i.e., it remains invariant. The transformation of the Goldstone-boson fields under the diagonal subgroup is found by expanding Eq. (4.29) in the fields:

$$\begin{aligned}U(x)=\mathbb{1}+i\frac{\pi(x)}{F_0}-\frac{\pi^2(x)}{2F_0}+\ldots\mapsto V\left(\mathbb{1}+i\frac{\pi(x)}{F_0}-\frac{\pi^2(x)}{2F_0}+\ldots\right)V^\dagger\\=\mathbb{1}+i\frac{V\pi(x)V^\dagger}{F_0}-\frac{V\pi(x)V^\dagger V\pi(x)V^\dagger}{2F_0}+\ldots\,,\end{aligned}\tag{4.31}$$

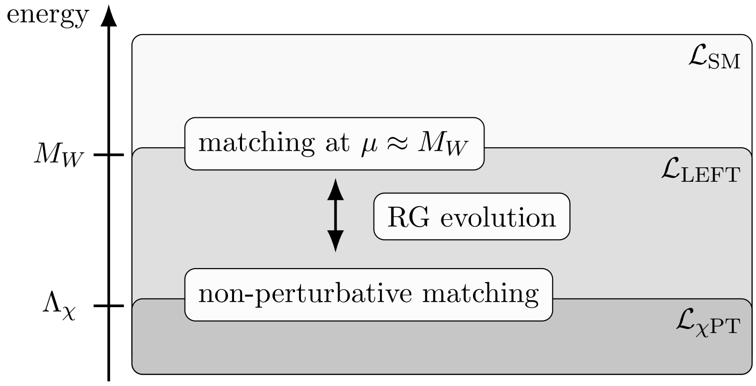


Figure 4: A tower of EFTs: χPT is built on top of the LEFT, which is the low-energy EFT of the SM (and beyond) below the weak scale. The matching between the LEFT and χPT cannot be calculated in perturbation theory.

i.e., the restriction of the group realization to the subgroup H defines a representation on the Hermitian and traceless 3×3 matrices:

$$\pi(x) \mapsto V\pi(x)V^\dagger \,. \tag{4.32}$$

In contrast, the axial transformations of the fields $\pi(x)$ are complicated, non-linear functions. This is known as *nonlinear realization* of chiral symmetry.

In the chiral limit, the building blocks of the EFT Lagrangian are the matrix $U(x)$ and partial derivatives. We will see later that the power counting orders the terms in the chiral Lagrangian by the number of derivatives—the matrix U is dimensionless. A scalar Lagrangian is obtained by taking traces of matrix products. It is easy to show that the leading term is just

$$\mathcal{L}_{p^2} = \frac{F_0^2}{4} \langle (\partial_\mu U)(\partial^\mu U^\dagger) \rangle \,, \tag{4.33}$$

where $\langle \cdot \rangle$ denotes the trace in flavor space (terms without derivatives only give an unphysical constant shift to the Lagrangian). The EFT is known as chiral perturbation theory (χPT). The Wilson coefficients of this EFT usually are called *low-energy constants* (LECs). The coefficient of the leading term is expressed in terms of the same parameter F_0 that appears in the definition of $U(x)$: it is fixed by the requirement that the Goldstone-boson fields have a canonically normalized kinetic term.

4.4 Explicit symmetry breaking, spurions, and external fields

The previous considerations allow us to set up an EFT for massless Goldstone bosons, arising from the spontaneous breaking of chiral symmetry in massless QCD. In reality, the underlying UV theory is not massless QCD, but rather the LEFT, i.e., QCD including symmetry-breaking quark-mass terms plus QED, plus a set of higher-dimension operators that describe the low-energy effects of the weak interaction. Note that here we are really building a tower of EFTs, sketched in Fig. 4: we start from the SM at high energies (or even something more fundamental). Below the weak scale, we are using the LEFT as a low-energy EFT built on top of the SM as the fundamental theory. Now, the LEFT, which is formulated in terms of quark and gluon degrees of freedom, is taking the role of the "fundamental" theory and we are building another EFT on top of it: χPT, which is formulated in terms of hadronic degrees of freedom. Unfortunately, the matching from QCD or the LEFT to χPT is non-perturbative, which means that the free parameters of χPT either have to be obtained with lattice-QCD simulations or from experiment.

We will now extend the treatment to the construction of χPT for the case of QCD including mass terms. We will also include the interaction with static external fields: this allows us to treat, e.g., the interaction of hadrons with photons in a first approximation, i.e., to leading order in $\alpha_{\rm QED} = e^2/(4\pi)$. It also allows us to include the effects of semileptonic four-fermion operators in the LEFT, generated through the weak interaction. The inclusion of dynamical photons and leptons in χPT and the treatment of the effect of four-quark operators goes beyond the scope of this course [35–38].

We start by promoting the global $SU(3)_L \times SU(3)_R$ symmetry to a local one, $L \mapsto L(x)$, $R \mapsto R(x)$. The massless QCD Lagrangian

$$\mathcal{L}^{(0)}_{\rm QCD} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \bar q i \not{D} q = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \bar q_L i \not{D} q_L + \bar q_R i \not{D} q_R \tag{4.34}$$

is not invariant under local $SU(3)_L \times SU(3)_R$ transformations. However, it can be made invariant by adding appropriate source terms, a procedure familiar from the construction of gauge theories: the Lagrangian

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^{(0)}_{\rm QCD} + \mathcal{L}^{\rm ext} \tag{4.35}$$

with

$$\mathcal{L}^{\rm ext} = \bar q_R S q_L + \bar q_L S^\dagger q_R + \bar q_L \gamma^\mu l_\mu q_L + \bar q_R \gamma^\mu r_\mu q_R \,, \tag{4.36}$$

is invariant provided that the sources have the appropriate transformation:

$$\begin{aligned} S(x) &\mapsto R(x) S(x) L^\dagger(x) \,, \\ l_\mu(x) &\mapsto L(x) l_\mu(x) L^\dagger(x) + i L(x) \partial_\mu L^\dagger(x) \,, \\ r_\mu(x) &\mapsto R(x) r_\mu(x) R^\dagger(x) + i R(x) \partial_\mu R^\dagger(x) \,. \end{aligned} \tag{4.37}$$

The sources are 3×3 matrices in flavor space and l_μ and r_μ are assumed to be traceless (otherwise, we could also gauge the $U(1)_V$ part of the symmetry group). The construction of χPT for this case is similar to the previous case, with two differences:

1. the matrix U is not the only degree of freedom, but we also need to explicitly include the external sources;
2. we need to impose invariance not only under the global symmetry group, but under local $SU(3)_L \times SU(3)_R$ transformations.

The external source fields can be regarded as artificial fields, also known as *spurions*: after having constructed the chiral Lagrangian on top of Eq. (4.35) as the fundamental theory, we will fix the spurion fields to the physical values. E.g., we are able to include the effect of symmetry-breaking mass terms and the coupling to electromagnetism by fixing the sources to

$$S \mapsto -M_q \,, \quad l_\mu \mapsto -eQA_\mu \,, \quad r_\mu \mapsto -eQA_\mu \,. \tag{4.38}$$

Similarly, by fixing the external sources to leptonic bilinears, we are able to include the effect of semileptonic four-fermion interactions.

Before fixing the spurions to their physical values, however, we need to construct the chiral Lagrangian that is invariant under the local chiral transformations. We will build the Lagrangian from the matrix U, the external sources S, l_μ, and r_μ, as well as derivatives. Since we impose invariance under the local symmetry, the external currents l_μ and r_μ can only appear in the combination of a covariant derivative. The covariant derivative of $D_\mu U$ is required to have the same transformation properties as U itself, i.e., we require

$$D_\mu U(x) \mapsto R(x) [D_\mu U(x)] L^\dagger(x) \,. \tag{4.39}$$

Therefore, the covariant derivative for U (or any other quantity A transforming as $A \mapsto RAL^\dagger$) is given by

$$D_\mu U = \partial_\mu U + iUl_\mu - ir_\mu U \,, \quad D_\mu U^\dagger = \partial_\mu U^\dagger - il_\mu U^\dagger + iU^\dagger r_\mu \,. \tag{4.40}$$

The commutator of the covariant derivative is

$$\begin{aligned} [D^\mu, D^\nu](\cdot) &= i(\cdot)F_L^{\mu\nu} - iF_R^{\mu\nu}(\cdot) \,, \\ F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu] \,, \\ F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \,. \end{aligned} \tag{4.41}$$

The field-strength tensors of the external fields transform as

$$\begin{aligned} F_L^{\mu\nu}(x) &\mapsto L(x)F_L^{\mu\nu}(x)L^\dagger(x) \,, \\ F_R^{\mu\nu}(x) &\mapsto R(x)F_R^{\mu\nu}(x)R^\dagger(x) \,. \end{aligned} \tag{4.42}$$

The chiral Lagrangian including sources [30, 31] can then be constructed as the most general Lagrangian that is invariant under local $SU(3)_L \times SU(3)_R$ and involves the building blocks

$$U \,, \quad U^\dagger \,, \quad D_\mu \,, \quad F_L^{\mu\nu} \,, \quad F_R^{\mu\nu} \,, \quad S \,, \quad S^\dagger \,. \tag{4.43}$$

Without having established yet the chiral power counting, we state that the leading terms of the Lagrangian will be given by

$$\mathcal{L}_{p^2} = \frac{F_0^2}{4}\langle (D_\mu U)(D^\mu U)^\dagger \rangle - \frac{F_0^2 B_0}{2}\langle SU^\dagger + US^\dagger \rangle \,. \tag{4.44}$$

Here, B_0 is a second LEC that appears in the coefficient of the additional operator. Its meaning can be established by considering the energy density of the ground state $U_0 = \mathbb{1}$. In terms of the effective Hamiltonian with the external source fixed to $S \mapsto -M_q$, one finds

$$\langle \mathcal{H}_{\text{eff}} \rangle_{\text{min}} = -F_0^2 B_0 (m_u + m_d + m_s) \,. \tag{4.45}$$

Now, we compare its derivative with respect to any of the quark masses with the corresponding quantity in QCD:

$$\frac{\partial \langle \mathcal{H}_{\text{eff}} \rangle_{\text{min}}}{\partial m_q} = -F_0^2 B_0 = \left. \frac{\partial \langle 0 | \mathcal{H}_{\text{QCD}} | 0 \rangle}{\partial m_q} \right|_{M_q = 0} = \frac{1}{3} \langle 0 | \bar{q} q | 0 \rangle_0 \,. \tag{4.46}$$

Therefore, B_0 is related to the scalar singlet quark condensate in the chiral limit, which we already encountered in Eq. (4.19):

$$B_0 = -\frac{1}{3F_0^2}\langle 0 | \bar{q} q | 0 \rangle_0 \,. \tag{4.47}$$

Traditionally, this LEC is reabsorbed into $\chi := -2B_0 S$, hence the Lagrangian is written as

$$\mathcal{L}_{p^2} = \frac{F_0^2}{4}\langle (D_\mu U)(D^\mu U)^\dagger \rangle + \frac{F_0^2}{4}\langle \chi U^\dagger + U\chi^\dagger \rangle \,. \tag{4.48}$$

We now fix the spurion χ to the physical value $\chi \mapsto 2B_0 M_q$. If we expand the leading-order Lagrangian in the Goldstone-boson fields, we find that with the inclusion of the symmetry-breaking terms the Goldstone-bosons are no longer massless. Extracting the terms that are quadratic in the fields, we find the following relations for the Goldstone-boson masses:

$$\begin{aligned} M_{\pi^0}^2 = M_{\pi^+}^2 &= B_0 \left(m_u + m_d \right) \,, \\ M_{K^0}^2 &= B_0 \left(m_s + m_d \right) \,, \\ M_{K^+}^2 &= B_0 \left(m_s + m_u \right) \,, \\ M_\eta^2 &= \frac{1}{3} B_0 \left(4m_s + m_u + m_d \right) \,, \end{aligned} \tag{4.49}$$

which are known as Gell-Mann–Oakes–Renner relations [39].

4.5 Power counting and loops

Let us recall the three core principles of EFTs and apply it to χPT, the EFT for strong interaction at low energies.

1. *Degrees of freedom*: we are interested in describing processes at low energies involving only the lightest hadrons, the pseudoscalar mesons $\pi^\pm$, π^0, $K^\pm$, K^0, $\bar{K}^0$, and η. They can be organized as an $SU(3)_V$ flavor octet and we have seen that they can be identified with the Goldstone bosons of the spontaneous breaking of chiral symmetry, $SU(3)_L \times SU(3)_R \to SU(3)_V$. Using the external-field formalism, we are also able to describe the coupling of these mesons to photons and leptons.

2. *Symmetries*: we conveniently write the degrees of freedom in terms of the matrix U, defined in Eq. (4.29). U transforms under the chiral group as $U \mapsto RUL^\dagger$, in contrast to the Goldstone-boson fields, which only transform linearly under the unbroken subgroup $SU(3)_V$, but in a complicated, nonlinear way under axial transformations: chiral symmetry is realized non-linearly on the space of Goldstone bosons, as follows from the CCWZ construction.

3. *Power counting*: finally, we need to establish the power counting of χPT, in order to arrive at a consistent EFT.

Since χPT is a low-energy theory of strong interaction, the small expansion parameter is again the ratio of external small momenta or a small mass divided by the "UV scale," which here is the scale of chiral symmetry breaking Λ_χ. As a first estimate, we take Λ_χ to be of the order of the mass of the lightest resonance, i.e., a hadron that is not included as an explicit degree of freedom. The lightest resonance is located at a mass scale of $M_\rho \approx 770\,\text{MeV}$.

The Lagrangian is written in terms of the $SU(3)$ matrix $U(x)$ and its derivatives. The expansion in momenta corresponds to a counting of derivatives in the Lagrangian. In the absence of symmetry-breaking terms, the terms of the Lagrangian are simply ordered by the number of derivatives. Goldstone bosons are derivatively coupled and the leading term in the Lagrangian contains already two derivatives, denoted as $\mathcal{O}(p^2)$ in the power counting (a chirally invariant scalar function $f(U)$ without derivatives is an irrelevant constant). When including symmetry-breaking terms due to the quark masses, we note that the on-shell relation $p^2 = M^2$ implies that the mass of a Goldstone boson should be counted as $\mathcal{O}(p)$. The spurion χ therefore is of $\mathcal{O}(p^2)$. The low-energy constant B_0 turns out to be large, $B_0 \approx 2.4\,\text{GeV}$ [40], and is counted as $\mathcal{O}(1)$ in the expansion, hence due to the GMOR relation (4.49), a quark mass needs to be counted as $m_q = \mathcal{O}(p^2)$.

Finally, we need to determine how an arbitrary Feynman diagram is to be counted in the chiral expansion. Consider the amplitude $\mathcal{M}$ corresponding to a diagram, which depends on external momenta p_i and meson masses M. When scaling the external momenta as $p_i \mapsto \delta \times p_i$ and the masses as $M^2 \mapsto \delta^2 \times M^2$, the scaling of the diagram

$$\mathcal{M}(\delta \times p_i, \delta^2 \times M^2) = \delta^n \mathcal{M}(p_i, M^2) \tag{4.50}$$

tells us that it is of $\mathcal{O}(p^n)$ in the chiral counting. For a connected graph, there is the topological relation

$$V - I + L = 1\,, \tag{4.51}$$

where V denotes the number of vertices, I the number of internal propagators, and L the number of loops. We assume that the graph contains V_{2k} vertices from the $\mathcal{O}(p^{2k})$ part of the Lagrangian, with $V = \sum_{k=1}^\infty V_{2k}$. Rescaling the external momenta, the masses, and relabelling the loop integration variables, we find in D space-time dimensions *Weinberg's power-counting formula* [29]:

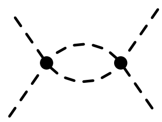
$$n = L \times D - 2I + \sum_{k=1}^\infty 2k \times V_{2k} = 2 + (D-2)L + \sum_{k=1}^\infty 2(k-1)V_{2k}\,. \tag{4.52}$$

We see that $n \geq 2$ for $D = 4$ space-time dimensions. Since there are no interaction terms of $\mathcal{O}(p^0)$, the last term is not negative. This implies that higher loops are of higher order in the chiral power counting—the loop expansion is connected with the EFT expansion. This is a feature of χPT that is in contrast to the linearly realized EFTs that we discussed before, where the loop expansion and the EFT expansion are independent.

Here, the chiral power-counting formula (4.52) tells us that amplitudes at leading order $\mathcal{O}(p^2)$ are obtained from tree-level diagrams with interaction vertices only from $\mathcal{L}_{p^2}$. Contributions to amplitudes at next-to-leading order arise from one-loop diagrams with vertices only from $\mathcal{L}_{p^2}$, as well as tree-level diagrams with the insertion of one vertex from $\mathcal{L}_{p^4}$. At $\mathcal{O}(p^6)$, amplitudes receive contributions from two-loop diagrams with $\mathcal{L}_{p^2}$ vertices, one-loop diagrams with the insertion of one $\mathcal{L}_{p^4}$ vertex, or tree-level diagrams with either two $\mathcal{L}_{p^4}$ vertices or one $\mathcal{L}_{p^6}$ vertex.

As usual in an EFT, χPT contains non-renormalizable interactions in the classical sense, but it is renormalizable order by order in the chiral power counting. UV divergences that arise from one-loop diagrams with $\mathcal{L}_{p^2}$ vertices are local terms that can be cancelled by counterterms split off the $\mathcal{O}(p^4)$ Lagrangian. Inserted into one-loop diagrams, these counterterms also cancel sub-divergences of $\mathcal{O}(p^6)$ two-loop diagrams, while the overall two-loop divergences are cancelled by counterterms split off from $\mathcal{L}_{p^6}$.

In order to get some more insight into the scale of chiral symmetry breaking Λ_χ, we apply some simple power-counting arguments to the case of $\pi\pi$ scattering at one loop. At $\mathcal{O}(p^4)$, there will be a diagram



(4.53)

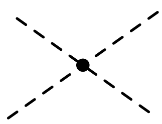
with two vertices from $\mathcal{L}_{p^2}$. From the expansion of $\mathcal{L}_{p^2}$, we see that the vertices have the generic form

$$F_0^2 \frac{\partial^2 \pi^4}{F_0^4} = \frac{\partial^2 \pi^4}{F_0^2}\,, \tag{4.54}$$

which for the loop diagram leads to structures such as

$$\frac{p^4}{F_0^4}\,\mu^{2\varepsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - M^2)^2} \sim \frac{p^4}{F_0^4} \frac{1}{16\pi^2} \left[-\frac{1}{\varepsilon} + 1 + \log\left(\frac{\tilde{\mu}^2}{M^2}\right)\right]\,. \tag{4.55}$$

There will also be a contact interaction from the $\mathcal{L}_{p^4}$ Lagrangian, which provides a counterterm cancelling the UV divergence. The comparison with the tree-level contribution



$$\sim \frac{p^2}{F_0^2} \tag{4.56}$$

shows that the generic suppression of the chiral corrections are of the order of

$$\frac{p^2}{(4\pi F_0)^2}\,, \tag{4.57}$$

i.e., we can identify the scale of chiral symmetry breaking with $\Lambda_\chi \approx 4\pi F_0 \approx 1\,\mathrm{GeV}$ [41], which is of the same order as the mass of the lightest hadronic resonance.

4.6 Chiral Lagrangian at NLO

With Weinberg's chiral power counting, we can apply χPT and calculate low-energy observables in a systematic expansion in the small parameter $p^2/(4\pi F_0)^2$, where p generically stands for an external momentum or a Goldstone-boson mass. The first step is to construct the operator basis of the EFT to the desired order in the expansion. By now, the Lagrangian of the strong sector has been constructed up to $\mathcal{O}(p^8)$. The $\mathcal{O}(p^4)$ three-flavor Lagrangian is given by [31]

$$\begin{aligned}\mathcal{L}_{p^4} = {} & L_1\langle (D_\mu U)(D^\mu U)^\dagger\rangle^2 + L_2\langle (D_\mu U)(D_\nu U)^\dagger\rangle\langle (D^\mu U)(D^\nu U)^\dagger\rangle \\ & + L_3\langle (D_\mu U)(D^\mu U)^\dagger (D_\nu U)(D^\nu U)^\dagger\rangle + L_4\langle (D_\mu U)(D^\mu U)^\dagger\rangle\langle \chi U^\dagger + U\chi^\dagger\rangle \\ & + L_5\langle (D_\mu U)(D^\mu U)^\dagger(\chi U^\dagger + U\chi^\dagger)\rangle + L_6\langle \chi U^\dagger + U\chi^\dagger\rangle^2 \\ & + L_7\langle \chi U^\dagger - U\chi^\dagger\rangle^2 + L_8\langle U\chi^\dagger U\chi^\dagger + \chi U^\dagger \chi U^\dagger\rangle \\ & - iL_9\langle F^R_{\mu\nu}(D^\mu U)(D^\nu U)^\dagger + F^L_{\mu\nu}(D^\mu U)^\dagger(D^\nu U)\rangle + L_{10}\langle U F^L_{\mu\nu} U^\dagger F^{\mu\nu}_R\rangle \\ & + H_1\langle F^R_{\mu\nu}F^{\mu\nu}_R + F^L_{\mu\nu}F^{\mu\nu}_L\rangle + H_2\langle \chi\chi^\dagger\rangle\,, \end{aligned} \tag{4.58}$$

where $F^{L,R}_{\mu\nu}$ are the field-strength tensors for the external fields. Here, the parameters L_i are the new (bare) LECs at next-to-leading order (NLO). The corresponding renormalized values can be determined experimentally or with lattice-QCD simulations. The last two terms in Eq. (4.58) only involve external fields and are called *contact terms*. Their coefficients depend on the type of external fields that are coupled to QCD.

At $\mathcal{O}(p^6)$, the three-flavor Lagrangian contains 90 physical parameters plus 4 contact terms [42]. At $\mathcal{O}(p^8)$, the number explodes to 1233 physical plus 21 contact terms [43].

In an NLO calculation, the divergences of one-loop diagrams are reabsorbed by the counterterms split off the $\mathcal{O}(p^4)$ Lagrangian (4.58). The theory is regularized dimensionally in $D = 4-2\varepsilon$ space-time dimensions. In order for the Lagrangian to have mass dimension D, we replace the decay constant F_0 by a rescaled one,

$$F_0 = \mu^{-\varepsilon} F_0^r\,, \tag{4.59}$$

with mass dimension $[F_0^r] = 1$, introducing the renormalization scale μ of χPT. Since F_0 does not absorb any divergences, we will discard the superscript immediately and relabel the rescaled decay constant by F_0. The bare NLO LECs need to have mass dimension $D-4=-2\varepsilon$. We relate them to renormalized ones by

$$L_i = \mu^{-2\varepsilon}\left[L_i^r(\tilde\mu) - \frac{\Gamma_i}{32\pi^2\varepsilon}\right]\,, \tag{4.60}$$

where the scale dependence of the renormalized LECs can be determined from the counterterms

$$\begin{aligned} &\Gamma_1 = \frac{3}{32}\,,\quad \Gamma_2 = \frac{3}{16}\,,\quad \Gamma_3 = 0\,,\quad \Gamma_4 = \frac{1}{8}\,,\quad \Gamma_5 = \frac{3}{8}\,,\quad \Gamma_6 = \frac{11}{144}\,, \\ &\Gamma_7 = 0\,,\quad \Gamma_8 = \frac{5}{48}\,,\quad \Gamma_9 = \frac{1}{4}\,,\quad \Gamma_{10} = -\frac{1}{4}\,. \end{aligned} \tag{4.61}$$

Traditionally, χPT calculations employ a version of modified minimal subtraction that slightly differs from the $\overline{\text{MS}}$ subtraction scheme used in SM calculations. We call it $\widetilde{\text{MS}}$ here.[7] It corresponds to absorbing into the counterterms some additional finite pieces compared to MS, which is equivalent to redefining the renormalization scale as

$$\mu^2 = \tilde\mu^2 \frac{e^{\gamma_E - 1}}{4\pi} \tag{4.62}$$

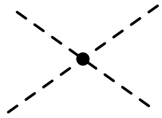
and subtracting the pure $1/\varepsilon$ poles.

The dependence of the renormalized LECs on $\tilde\mu$ is given by the RGE

$$\tilde\mu \frac{d}{d\tilde\mu} L_i^r(\tilde\mu) = -\frac{\Gamma_i}{16\pi^2}\,. \tag{4.63}$$

[7] The χPT literature usually calls the scheme $\overline{\text{MS}}$.

The typical order of magnitude of the renormalized LECs $L_i^r(\tilde{\mu})$ can be estimated using *naive dimensional analysis* (NDA) [41]. Consider again the simple power-counting argument for $\pi\pi$ scattering. The $\mathcal{L}_{p^4}$ contact contribution to $\pi\pi$ scattering is of the generic form



$$\sim L_i \frac{p^4}{F_0^4} \sim \left(L_i^r(\tilde{\mu}) - \frac{\Gamma_i}{32\pi^2\varepsilon} \right) \frac{p^4}{F_0^4} \,. \tag{4.64}$$

This needs to be compared to a typical one-loop contribution (4.55). In the sum, the UV divergences cancel, hence Γ_i are quantities of $\mathcal{O}(1)$. If there are no accidental cancellations, we expect the typical size of the renormalized LECs to be of the order of the variation when changing the renormalization scale. This leads to the estimate of the natural size of the LECs

$$L_i^r(\tilde{\mu}) \sim \mathcal{O}\left(\frac{1}{16\pi^2} \right) \approx \mathcal{O}(6.3 \times 10^{-3}) \,. \tag{4.65}$$

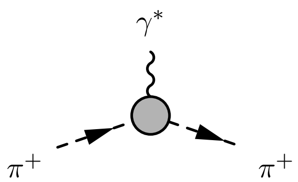
In practice, this estimate turns out to be reasonable: determinations of the NLO LECs from phenomenology or lattice QCD indeed lead to numbers of $\mathcal{O}(10^{-3})$.

4.7 A one-loop calculation in χPT

As an application and illustration, we will calculate the pion vector form factor (VFF) at one loop in three-flavor χPT (without isospin breaking, i.e., we take $m_u = m_d = \hat{m}$). The pion VFF is defined by the matrix element

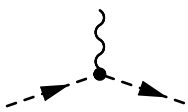
$$\langle \pi^+(p') | j^\mu_\text{em}(0) | \pi^+(p) \rangle = (p'+p)^\mu F_\pi^V(t) \,, \quad t = (p'-p)^2 \,, \tag{4.66}$$

where $j^\mu_\text{em} = \bar{q}\gamma^\mu Q q$ is the electromagnetic current and $Q = \frac{1}{3}\text{diag}(2,-1,-1)$ the quark charge matrix. The pion VFF describes, e.g., the production of a $\pi^+\pi^-$ pair by a virtual photon as a sub-process of $e^+e^- \to \pi^+\pi^-$:



$$= -ie(p'+p)^\mu F_\pi^V(t) \,. \tag{4.67}$$

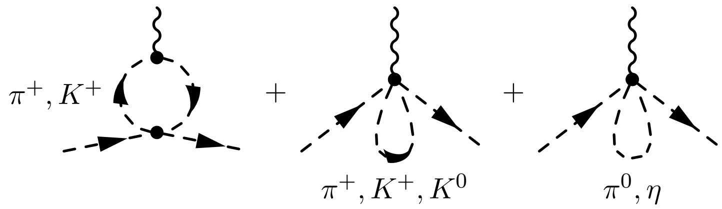
At leading order in χPT, there is a contact interaction from $\mathcal{L}_{p^2}$ contributing to the VFF, which corresponds to scalar QED:



$$= -ie(p'+p)^\mu \quad \Rightarrow \quad F_\pi^V(t) = 1 + \mathcal{O}(p^2) \,. \tag{4.68}$$

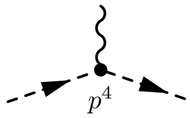
This vertex rule is obtained by expanding in $\mathcal{L}_{p^2}$ the matrix $U(x)$ in powers of the fields and extracting the terms with two Goldstone bosons and one photon, which couples through the covariant derivate upon setting the external fields to $l_\mu = r_\mu = -eQA_\mu$.

The NLO contribution contains several pieces: first, there are the 1PI one-loop diagrams with vertices from $\mathcal{L}_{p^2}$, which are again obtained by expanding the matrix exponential $U(x)$ in powers of the Goldstone-boson fields:



$$= -ie(p'+p)^\mu \frac{4M_K^2 + 8M_\pi^2 + 3t}{192\pi^2 F_0^2}\frac{1}{\varepsilon} + \text{finite}\,. \tag{4.69}$$

At the same order in the power counting, we have a contact interaction with a vertex from $\mathcal{L}_{p^4}$:

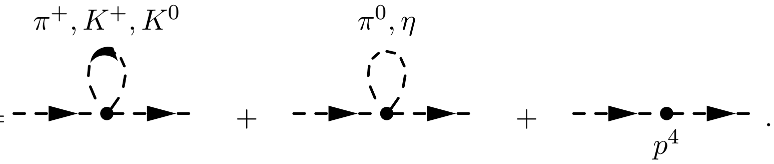


$$= -ie(p'+p)^\mu \frac{2\left(8L_4 M_K^2 + 4(L_4+L_5)M_\pi^2 + L_9 t\right)}{F_0^2}$$

$$= -ie(p'+p)^\mu \left(-\frac{t + 4M_K^2 + 8M_\pi^2}{64\pi^2 F_0^2 \varepsilon} + \frac{2\left(8L_4^r(\tilde\mu)M_K^2 + 4(L_4^r(\tilde\mu) + L_5^r(\tilde\mu))M_\pi^2 + L_9^r(\tilde\mu)t\right)}{F_0^2} \right). \tag{4.70}$$

Another contribution comes from the external-leg corrections for the pions: from the LSZ formula, the wave-function renormalization factor needs to be taken into account, which can be determined by calculating the self-energy:

$$-i\Sigma_\pi(p^2) =$$



(4.71)

The one-loop wave-function renormalization is given by

$$\begin{aligned} Z_\pi = 1 + \delta Z_\pi = 1 + \Sigma_\pi'(M_\pi^2) &= 1 - \frac{M_K^2 + 2M_\pi^2}{48\pi^2 F_0^2}\frac{1}{\varepsilon} - \frac{8\left(2L_4 M_K^2 + (L_4+L_5)M_\pi^2\right)}{F_0^2} \\ &\quad - \frac{M_K^2 \log\left(\frac{\tilde\mu^2}{M_K^2}\right)}{48\pi^2 F_0^2} - \frac{M_\pi^2 \log\left(\frac{\tilde\mu^2}{M_\pi^2}\right)}{24\pi^2 F_0^2} \\ &= 1 + \frac{M_K^2 + 2M_\pi^2}{24\pi^2 F_0^2}\frac{1}{\varepsilon} - \frac{8\left(2L_4^r(\tilde\mu) M_K^2 + (L_4^r(\tilde\mu)+L_5^r(\tilde\mu))M_\pi^2\right)}{F_0^2} \\ &\quad - \frac{M_K^2 \log\left(\frac{\tilde\mu^2}{M_K^2}\right)}{48\pi^2 F_0^2} - \frac{M_\pi^2 \log\left(\frac{\tilde\mu^2}{M_\pi^2}\right)}{24\pi^2 F_0^2}\,. \end{aligned} \tag{4.72}$$

The contribution of the external-leg corrections for the two pion legs is given by $-ie(p'+p)^\mu \delta Z_\pi$. It is easy to very that in the sum of all contributions, the UV divergences cancel. Furthermore, the VFF only depends on the renormalized LEC $L_9^r(\tilde\mu)$, while the dependence on the other NLO LECs drops out. In total, the result reads

$$\begin{aligned} F_\pi^V(t) &= 1 + \frac{1}{F_0^2}\left(2G_\pi(t) + G_K(t)\right), \\ G_P(t) &= \frac{2}{3}tL_9^r(\tilde\mu) + \frac{5t - 24M_P^2}{576\pi^2} + \frac{t\log\left(\frac{\tilde\mu^2}{M_P^2}\right)}{192\pi^2} + \frac{t\sigma_P(t)^3 \log\left(\frac{\sigma_P(t)-1}{\sigma_P(t)+1}\right)}{192\pi^2}, \\ \sigma_P(t) &= \sqrt{1 - \frac{4M_P^2}{t}}\,. \end{aligned} \tag{4.73}$$

Fits to data allow one to determine the LEC $L_9^r(\tilde{\mu})$, resulting in [44]

$$L_9^r(0.77\,\text{GeV}) = (5.93 \pm 0.43) \times 10^{-3}\,, \tag{4.74}$$

in line with the expectation from NDA. A few points should be noted.

- Since we are working with an operator basis without EOM operators, off-shell Green's functions such as the pion two-point function are UV divergent. However, all UV divergences drop out upon suitable field redefinitions and in particular, they cancel in on-shell observables, such as $F_\pi^V(t)$. This can be observed in the LSZ formalism, where UV divergences cancel between the vertex corrections and the wave-function renormalization.
- The scale dependence of the LEC L_9^r cancels the explicit $\tilde{\mu}$-dependence of the chiral logarithm. The observable $F_\pi^V(t)$ is independent of the renormalization scale.
- Inside the NLO correction, we have made use of the GMOR relations (4.49). In particular, we can use the Gell-Mann–Okubo (GMO) relation between the meson masses [45, 46]

  $$4M_K^2 = 3M_\eta^2 + M_\pi^2\,, \tag{4.75}$$

  which holds at LO. Corrections to the GMO relation inside the NLO expression result in an effect at NNLO, which we neglect here.
- Expanding around $t = 0$, one finds $F_\pi^V(t) = 1 + \mathcal{O}(t)$, as required by current conservation. The derivative at zero is related to the pion charge radius:

  $$F_\pi^V(t) = 1 + \frac{1}{6}\langle r_\pi^2\rangle t + \mathcal{O}(t^2)\,. \tag{4.76}$$

  At one loop in χPT, we obtain

  $$\langle r_\pi^2\rangle = \frac{12L_9^r(\tilde{\mu})}{F_0^2} + \frac{1}{32\pi^2F_0^2}\left[2\log\left(\frac{\tilde{\mu}^2}{M_\pi^2}\right) + \log\left(\frac{\tilde{\mu}^2}{M_K^2}\right) - 3\right] \approx 0.4\,\text{fm}^2\,. \tag{4.77}$$

5 EFTs for heavy physics beyond the Standard Model

The SM is a very successful theory, as it correctly describes phenomena in particle physics over a wide range of energies. On the other hand, we know that the SM cannot be the ultimate theory, as it fails to explain certain observations and has some theoretical shortcomings.

- First of all, gravity is not part of the SM, hence it is not a complete theory. However, this might become a problem only at energies around the Planck scale, $M_P \sim 10^{19}$ GeV.
- Astronomical observations show that a large fraction of the matter content of the universe does not consist of ordinary baryonic matter, but some unknown form of *dark matter*. There is no candidate in the SM for a dark-matter particle.
- The *baryon asymmetry* in the universe (i.e., the asymmetry between matter and antimatter) cannot be explained by dynamical processes within the SM alone. In particular, the amount of CP violation does not suffice to explain the baryon asymmetry through SM baryogenesis.
- Unless one adds right-handed neutrinos, it is not possible to have a renormalizable mass term for neutrinos to the SM Lagrangian. However, the observed neutrino oscillations require non-vanishing neutrino masses.

These points provide ample motivation to search for *new physics*, i.e., physics beyond the SM (BSM). The so-called *hierarchy problem* was considered in the past as a strong indication that new physics should consist of yet unknown particles not much heavier than the electroweak scale. Unfortunately, so far the LHC at CERN has not produced any experimental evidence of new particles in direct searches. This means that BSM particles are either extremely weakly coupled to the SM particles, or they are heavier than expected and hence could not yet be produced in high-energy collider experiments. The second scenario naturally lends itself to a description in terms of EFTs and motivates *indirect searches* for new physics. We can regard the SM not only as a successful, but incomplete theory that one day should be replaced by a more fundamental theory, but we can really regard it as a first approximation in a systematic expansion. Provided that BSM particles are heavy, their indirect quantum effects at lower energies can be described in terms of an EFT, with the SM being the leading-order approximation. The most popular variant of such an EFT is known as *Standard Model Effective Field Theory* (SMEFT).

5.1 SMEFT

5.1.1 Symmetries, power counting, and operator basis

The SMEFT is an EFT that describes the indirect low-energy effects of BSM physics consisting of degrees of freedom much heavier than the SM particles at the electroweak scale. To clearly define the EFT, let us discuss the three core principles in this case.

1. *Degrees of freedom*: the degrees of freedom in the SMEFT are the known SM particles. The field content of the SMEFT is identical with the field content of the renormalizable SM.
2. *Symmetries*: the SMEFT is a gauge theory with the same gauge group as the SM, i.e., the field content is organized in terms of irreducible representations of the gauged symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y$. The Higgs scalar particle is part of an $SU(2)_L$ doublet, as in the SM.
3. *Power counting*: the SMEFT follows a dimensional power counting, in analogy to the scalar Toy example that we discussed, or the Fermi theory. The Lagrangian consists of all gauge-invariant local operators, ordered by their mass dimension. The leading-order Lagrangian is just the SM Lagrangian, the higher-dimension terms are suppressed by powers of $1/\Lambda$, where Λ denotes the scale of new physics. With the given field content, this leads to an EFT expansion in terms of the small parameter $\delta \sim p/\Lambda$ or $\delta \sim v/\Lambda$, where p is an external momentum and v denotes the Higgs vev (the electroweak scale).

We write the SM Lagrangian in the form

$$\mathcal{L}_{\rm SM} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} - \frac{1}{4} W^I_{\mu\nu} W^{I\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu H)^\dagger (D^\mu H) + \sum_{\psi=q,u,d,l,e} \overline{\psi}\, i \not{D}\, \psi \\ - \lambda \left(H^\dagger H - \frac{1}{2} v^2 \right)^2 - \left[H^{\dagger i} \overline{d}\, Y_d\, q_i + \widetilde{H}^{\dagger i} \overline{u}\, Y_u\, q_i + H^{\dagger i} \overline{e}\, Y_e\, l_i + \text{h.c.} \right] , \tag{5.1}$$

ignoring possible θ-terms. The covariant derivative is given by

$$D_\mu = \partial_\mu + i g_3 T^A G^A_\mu + i g_2 t^I W^I_\mu + i g_1 Y B_\mu \,, \tag{5.2}$$

with the $SU(3)$ generators T^A, the $SU(2)$ generators $t^I = \tau^I/2$, and the $U(1)$ hypercharge generator Y. The Yukawa matrices $Y_{u,d,e}$ are 3×3 matrices in flavor space. H denotes the Higgs doublet with the leading-order vev $\langle H^\dagger H \rangle = \frac{1}{2} v^2$ and $\widetilde{H}$ is related by $\widetilde{H}_i = \epsilon_{ij} H^{\dagger j}$, where the $SU(2)$-invariant tensor ϵ_{ij} is defined by $\epsilon_{12} = 1$, $\epsilon_{ij} = -\epsilon_{ji}$.

The SMEFT Lagrangian is given by the SM Lagrangian plus a tower of higher-dimension gauge-invariant operators:

$$\mathcal{L}_{\rm SMEFT} = \mathcal{L}_{\rm SM} + \sum_{d \geq 5} \sum_{i=1}^{n_d} C_i^{(d)} Q_i^{(d)} \,, \tag{5.3}$$

where the operator coefficients are of the order $C_i^{(d)} \sim \Lambda^{4-d}$. At dimension five, there is only one operator type plus its Hermitian conjugate [47]:

$$\mathcal{L}^{(5)} = C^{(5)}_{pr}\, Q^{(5)}_{pr} + \text{h.c.} \,, \quad Q^{(5)}_{pr} = \epsilon^{ij} \epsilon^{kl} (l^\top_{ip} C l_{kr}) H_j H_l \,, \tag{5.4}$$

where p and r are flavor indices, l is the left-handed lepton doublet, and C is the charge-conjugation matrix ($C = i\gamma^2\gamma^0$ in the chiral basis). In the broken phase, this dimension-five *Weinberg operator* generates a Majorana-mass term for the neutrinos. It can be generated, e.g., in type-I seesaw models, where the SM is extended by right-handed Majorana neutrinos N_R with a very heavy mass M_N. These models can be matched to the SMEFT by integrating out the heavy Majorana neutrino, leading to a dimension-five operator coefficient of the order $\mathcal{O}(1/M_N)$. In the broken phase, the light neutrinos receive a mass $m_\nu \propto v^2/M_N$. Therefore, very heavy right-handed neutrinos lead to very small masses for the left-handed neutrinos. Note that the Weinberg operator violates lepton number by two units: baryon and lepton number are preserved in the SM due to an accidental symmetry, as discussed earlier. This accidental symmetry is only present in the dimension-four part of the SMEFT Lagrangian.

The SMEFT operator basis at dimension six was classified in [48, 49]. It already contains a rather large number of operators: for three generations of fermions, there are 2499 dimension-six operators that preserve baryon and lepton number, and in addition 273 dimension-six operators with $\Delta B = \Delta L = 1$, plus their Hermitian conjugates. The dimension-six SMEFT operator basis is reproduced in Table 1. The SMEFT operator basis has been constructed explicitly up to dimension nine [50–54].

The higher-dimension operators in the SMEFT lead to new interaction vertices and they can have phenomenological implications in all areas of particle physics. Observations that are in agreement with the SM prediction lead to constraints on the coefficients of higher-dimension SMEFT operators. At present, most of the coefficients of the higher-dimension operators are compatible with zero within the uncertainties. Notable exceptions are related to the very few discrepancies between experimental measurements and SM predictions, which might be interpreted as indirect hints of BSM physics: these are, e.g., four-fermion operators contributing to the flavor anomalies in B physics, or operators contributing to the anomalous magnetic moment of the muon.

1 : X^3	
Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$
$Q_{\widetilde{G}}$	$f^{ABC} \widetilde{G}_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$
Q_W	$\epsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$
$Q_{\widetilde{W}}$	$\epsilon^{IJK} \widetilde{W}_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$

2 : H^6	
Q_H	$(H^\dagger H)^3$

3 : $H^4 D^2$	
$Q_{H\Box}$	$(H^\dagger H)\Box(H^\dagger H)$
Q_{HD}	$\left(H^\dagger D^\mu H\right)^* \left(H^\dagger D_\mu H\right)$

5 : $\psi^2 H^3$ + h.c.	
Q_{eH}	$(H^\dagger H)(\bar{l}_p e_r H)$
Q_{uH}	$(H^\dagger H)(\bar{q}_p u_r \widetilde{H})$
Q_{dH}	$(H^\dagger H)(\bar{q}_p d_r H)$

4 : $X^2 H^2$	
Q_{HG}	$H^\dagger H\, G^A_{\mu\nu} G^{A\mu\nu}$
$Q_{H\widetilde{G}}$	$H^\dagger H\, \widetilde{G}^A_{\mu\nu} G^{A\mu\nu}$
Q_{HW}	$H^\dagger H\, W^I_{\mu\nu} W^{I\mu\nu}$
$Q_{H\widetilde{W}}$	$H^\dagger H\, \widetilde{W}^I_{\mu\nu} W^{I\mu\nu}$
Q_{HB}	$H^\dagger H\, B_{\mu\nu} B^{\mu\nu}$
$Q_{H\widetilde{B}}$	$H^\dagger H\, \widetilde{B}_{\mu\nu} B^{\mu\nu}$
Q_{HWB}	$H^\dagger \tau^I H\, W^I_{\mu\nu} B^{\mu\nu}$
$Q_{H\widetilde{W}B}$	$H^\dagger \tau^I H\, \widetilde{W}^I_{\mu\nu} B^{\mu\nu}$

6 : $\psi^2 X H$ + h.c.	
Q_{eW}	$(\bar{l}_p \sigma^{\mu\nu} \tau^I e_r) H W^I_{\mu\nu}$
Q_{eB}	$(\bar{l}_p \sigma^{\mu\nu} e_r) H B_{\mu\nu}$
Q_{uG}	$(\bar{q}_p \sigma^{\mu\nu} T^A u_r) \widetilde{H}\, G^A_{\mu\nu}$
Q_{uW}	$(\bar{q}_p \sigma^{\mu\nu} \tau^I u_r) \widetilde{H}\, W^I_{\mu\nu}$
Q_{uB}	$(\bar{q}_p \sigma^{\mu\nu} u_r) \widetilde{H}\, B_{\mu\nu}$
Q_{dG}	$(\bar{q}_p \sigma^{\mu\nu} T^A d_r) H\, G^A_{\mu\nu}$
Q_{dW}	$(\bar{q}_p \sigma^{\mu\nu} \tau^I d_r) H\, W^I_{\mu\nu}$
Q_{dB}	$(\bar{q}_p \sigma^{\mu\nu} d_r) H\, B_{\mu\nu}$

7 : $\psi^2 H^2 D$	
$Q_{Hl}^{(1)}$	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{l}_p \gamma^\mu l_r)$
$Q_{Hl}^{(3)}$	$(H^\dagger i \overleftrightarrow{D}^I_\mu H)(\bar{l}_p \tau^I \gamma^\mu l_r)$
Q_{He}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{e}_p \gamma^\mu e_r)$
$Q_{Hq}^{(1)}$	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{q}_p \gamma^\mu q_r)$
$Q_{Hq}^{(3)}$	$(H^\dagger i \overleftrightarrow{D}^I_\mu H)(\bar{q}_p \tau^I \gamma^\mu q_r)$
Q_{Hu}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{u}_p \gamma^\mu u_r)$
Q_{Hd}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{d}_p \gamma^\mu d_r)$
Q_{Hud} + h.c.	$i(\widetilde{H}^\dagger D_\mu H)(\bar{u}_p \gamma^\mu d_r)$

8 : $(\bar{L}L)(\bar{L}L)$	
Q_{ll}	$(\bar{l}_p \gamma^\mu l_r)(\bar{l}_s \gamma_\mu l_t)$
$Q_{qq}^{(1)}$	$(\bar{q}_p \gamma^\mu q_r)(\bar{q}_s \gamma_\mu q_t)$
$Q_{qq}^{(3)}$	$(\bar{q}_p \gamma^\mu \tau^I q_r)(\bar{q}_s \gamma_\mu \tau^I q_t)$
$Q_{lq}^{(1)}$	$(\bar{l}_p \gamma^\mu l_r)(\bar{q}_s \gamma_\mu q_t)$
$Q_{lq}^{(3)}$	$(\bar{l}_p \gamma^\mu \tau^I l_r)(\bar{q}_s \gamma_\mu \tau^I q_t)$

8 : $(\bar{R}R)(\bar{R}R)$	
Q_{ee}	$(\bar{e}_p \gamma^\mu e_r)(\bar{e}_s \gamma_\mu e_t)$
Q_{uu}	$(\bar{u}_p \gamma^\mu u_r)(\bar{u}_s \gamma_\mu u_t)$
Q_{dd}	$(\bar{d}_p \gamma^\mu d_r)(\bar{d}_s \gamma_\mu d_t)$
Q_{eu}	$(\bar{e}_p \gamma^\mu e_r)(\bar{u}_s \gamma_\mu u_t)$
Q_{ed}	$(\bar{e}_p \gamma^\mu e_r)(\bar{d}_s \gamma_\mu d_t)$
$Q_{ud}^{(1)}$	$(\bar{u}_p \gamma^\mu u_r)(\bar{d}_s \gamma_\mu d_t)$
$Q_{ud}^{(8)}$	$(\bar{u}_p \gamma^\mu T^A u_r)(\bar{d}_s \gamma_\mu T^A d_t)$

8 : $(\bar{L}L)(\bar{R}R)$	
Q_{le}	$(\bar{l}_p \gamma^\mu l_r)(\bar{e}_s \gamma_\mu e_t)$
Q_{lu}	$(\bar{l}_p \gamma^\mu l_r)(\bar{u}_s \gamma_\mu u_t)$
Q_{ld}	$(\bar{l}_p \gamma^\mu l_r)(\bar{d}_s \gamma_\mu d_t)$
Q_{qe}	$(\bar{q}_p \gamma^\mu q_r)(\bar{e}_s \gamma_\mu e_t)$
$Q_{qu}^{(1)}$	$(\bar{q}_p \gamma^\mu q_r)(\bar{u}_s \gamma_\mu u_t)$
$Q_{qu}^{(8)}$	$(\bar{q}_p \gamma^\mu T^A q_r)(\bar{u}_s \gamma_\mu T^A u_t)$
$Q_{qd}^{(1)}$	$(\bar{q}_p \gamma^\mu q_r)(\bar{d}_s \gamma_\mu d_t)$
$Q_{qd}^{(8)}$	$(\bar{q}_p \gamma^\mu T^A q_r)(\bar{d}_s \gamma_\mu T^A d_t)$

8 : $(\bar{L}R)(\bar{R}L)$ + h.c.	
Q_{ledq}	$(\bar{l}_p^j e_r)(\bar{d}_s q_{tj})$

8 : $(\bar{L}R)(\bar{L}R)$ + h.c.	
$Q_{quqd}^{(1)}$	$(\bar{q}_p^j u_r)\epsilon_{jk}(\bar{q}_s^k d_t)$
$Q_{quqd}^{(8)}$	$(\bar{q}_p^j T^A u_r)\epsilon_{jk}(\bar{q}_s^k T^A d_t)$
$Q_{lequ}^{(1)}$	$(\bar{l}_p^j e_r)\epsilon_{jk}(\bar{q}_s^k u_t)$
$Q_{lequ}^{(3)}$	$(\bar{l}_p^j \sigma_{\mu\nu} e_r)\epsilon_{jk}(\bar{q}_s^k \sigma^{\mu\nu} u_t)$

$\Delta B = \Delta L = 1$ + h.c.	
Q_{duql}	$\epsilon^{\alpha\beta\gamma}\epsilon^{ij}(d^\top_{\alpha p} C u_{\beta r})(q^\top_{\gamma is} C l_{jt})$
Q_{qque}	$\epsilon^{\alpha\beta\gamma}\epsilon^{ij}(q^\top_{\alpha ip} C q_{\beta jr})(u^\top_{\gamma s} C e_t)$
Q_{qqql}	$\epsilon^{\alpha\beta\gamma}\epsilon^{i\ell}\epsilon^{jk}(q^\top_{\alpha ip} C q_{\beta jr})(q^\top_{\gamma ks} C l_{\ell t})$
Q_{duue}	$\epsilon^{\alpha\beta\gamma}(d^\top_{\alpha p} C u_{\beta r})(u^\top_{\gamma s} C e_t)$

Table 1: The dimension-six operators in SMEFT. The operators that conserve baryon and lepton number are divided into eight classes according to their field content. The class-8 ψ^4 four-fermion operators are further divided into subclasses according to their chiral properties. Operators with + h.c. have Hermitian conjugates. The subscripts p, r, s, t are weak-eigenstate indices.

5.1.2 Broken phase: the scalar sector

It is instructive to consider the modifications of familiar relations in the broken phase of the electroweak sector that arise at tree level in the presence of dimension-six SMEFT operators. Detailed discussions can be found in [20, 55, 56].

The scalar potential and kinetic energy terms in (5.1)

$$\mathcal{L}_H = (D_\mu H)^\dagger (D^\mu H) - \lambda \left(H^\dagger H - \frac{1}{2} v^2 \right)^2 \tag{5.5}$$

get modified by the dimension-six operators $Q_{H\Box}$, Q_{HD}, and Q_H (see Table 1). The minimum of the tree-level potential is given by the modified vacuum-expectation value (vev)

$$\langle H^\dagger H \rangle = \frac{1}{2} v_T^2, \quad v = \left(1 - \frac{3 C_H v_T^2}{8\lambda} \right) v_T \,. \tag{5.6}$$

The Higgs doublet

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_1 \\ [1 + c_{H,\text{kin}}]h + v_T - i\phi_3 \end{pmatrix} \tag{5.7}$$

contains the physical Higgs field h with mass

$$M_H^2 = 2\lambda v_T^2 \left(1 + 2c_{H,\text{kin}} - \frac{3 C_H v_T^2}{2\lambda} \right) , \tag{5.8}$$

where the rescaling factor

$$c_{H,\text{kin}} = \left(C_{H\Box} - \frac{1}{4} C_{HD} \right) v_T^2 \tag{5.9}$$

is required to ensure a canonically normalized kinetic term in the presence of the dimension-six operators.

5.1.3 Gauge sector

The SM Lagrangian (5.1) contains the following Higgs and gauge kinetic terms:

$$\mathcal{L}_{H,\text{kin}} + \mathcal{L}_{\text{gauge}} = (D_\mu H)^\dagger (D^\mu H) - \frac{1}{4} G_{\mu\nu}^A G^{A\mu\nu} - \frac{1}{4} W_{\mu\nu}^I W^{I\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{5.10}$$

In the broken phase, the first term generates the following gauge-boson mass and mixing terms at dimension four:

$$(D_\mu H)^\dagger (D^\mu H) = \frac{v^2}{8} \left(g_2^2 W_\mu^I W_I^\mu + g_1^2 B_\mu B^\mu - 2 g_2 g_1 W_\mu^3 B^\mu \right) + \dots \,. \tag{5.11}$$

At dimension six in the SMEFT, the CP-even operators Q_{HG}, Q_{HW}, and Q_{HB} lead to kinetic terms of the gauge fields in the broken phase that are not canonically normalized. In addition, Q_{HWB} induces an additional kinetic mixing of the weak gauge bosons and the operator Q_{HD} contributes to the gauge-boson mass terms. We define the following rescaled gauge fields and couplings:

$$\begin{aligned} G_\mu^A &= \mathcal{G}_\mu^A (1 + v_T^2 C_{HG}), & W_\mu^I &= \mathcal{W}_\mu^I (1 + v_T^2 C_{HW}), & B_\mu &= \mathcal{B}_\mu (1 + v_T^2 C_{HB}) \,, \\ \bar{g}_3 &= g_3 (1 + v_T^2 C_{HG}), & \bar{g}_2 &= g_2 (1 + v_T^2 C_{HW}), & \bar{g}_1 &= g_1 (1 + v_T^2 C_{HB}) \,, \end{aligned} \tag{5.12}$$

where the fields $\mathcal{G}_\mu^A$, $\mathcal{W}_\mu^I$, and $\mathcal{B}_\mu$ have canonically normalized kinetic terms. Both the kinetic terms and the mass matrix of the neutral gauge bosons can be diagonalized by the transformation

$$\begin{pmatrix} \mathcal{Z}^\mu \\ \mathcal{A}^\mu \end{pmatrix} = \begin{pmatrix} \bar c - \frac{\epsilon}{2}\bar s & -\bar s + \frac{\epsilon}{2}\bar c \\ \bar s + \frac{\epsilon}{2}\,\bar c & \bar c + \frac{\epsilon}{2}\,\bar s \end{pmatrix} \begin{pmatrix} \mathcal{W}_3^\mu \\ \mathcal{B}^\mu \end{pmatrix} , \tag{5.13}$$

where $\epsilon := v_T^2 C_{HWB}$ and

$$\bar c = \cos\bar\theta = \frac{\bar g_2}{\sqrt{\bar g_1^2 + \bar g_2^2}} \left(1 - \frac{\epsilon}{2} \frac{\bar g_1}{\bar g_2} \frac{\bar g_2^2 - \bar g_1^2}{\bar g_1^2 + \bar g_2^2} \right) , \quad \bar s = \sin\bar\theta = \frac{\bar g_1}{\sqrt{\bar g_1^2 + \bar g_2^2}} \left(1 + \frac{\epsilon}{2} \frac{\bar g_2}{\bar g_1} \frac{\bar g_2^2 - \bar g_1^2}{\bar g_1^2 + \bar g_2^2} \right) . \tag{5.14}$$

The charged gauge bosons are defined by $\mathcal{W}_\mu^\pm := \frac{1}{\sqrt{2}}(\mathcal{W}_\mu^1 \mp i\mathcal{W}_\mu^2)$. Overall, one finds the following gauge-boson masses:

$$M_\mathcal{W}^2 = \frac{v_T^2 \bar g_2^2}{4} , \quad M_\mathcal{Z}^2 = \frac{v_T^2}{4} \left(1 + \frac{v_T^2}{2} C_{HD} \right) (\bar g_2^2 + \bar g_1^2) + \frac{v_T^2}{2} \epsilon \bar g_1 \bar g_2 , \tag{5.15}$$

and the photon field $\mathcal{A}_\mu$ remains massless. The electroweak ρ-parameter is modified by the coefficient of the operator Q_{HD}.

5.1.4 Fermions

The fermion mass matrices and Yukawa couplings of the SM

$$\mathcal{L} = -\left[H^{\dagger i} \bar d Y_d q_i + \widetilde H^{\dagger i} \bar u Y_u q_i + H^{\dagger i} \bar e Y_e l_i + \text{h.c.} \right] \tag{5.16}$$

are modified in the SMEFT by the dimension-six operators of the class $\psi^2 H^3$. In the spontaneously broken SMEFT, the fermion mass terms are given by

$$\mathcal{L}_M = -[M_\psi]_{rs} \bar\psi_{Rr} \psi_{Ls} + \text{h.c.} , \quad [M_\psi]_{rs} = \frac{v_T}{\sqrt{2}} \left([Y_\psi]_{rs} - \frac{v_T^2}{2} C^*_{\underset{sr}{\psi H}} \right) , \quad \psi = u, d, e . \tag{5.17}$$

The Yukawa interaction in the broken phase is modified to

$$\begin{aligned} \mathcal{L}_Y &= -[\mathcal{Y}_\psi]_{rs} h \bar\psi_{Rr} \psi_{Ls} + \text{h.c.} , \quad \psi = u, d, e , \\ [\mathcal{Y}_\psi]_{rs} &= \frac{1 + c_{H,\text{kin}}}{v_T} [M_\psi]_{rs} - \frac{v_T^2}{\sqrt{2}} C^*_{\underset{sr}{\psi H}} . \end{aligned} \tag{5.18}$$

We see that at dimension six, the Yukawa interaction is no longer proportional to the mass matrix.

The interactions of the fermions with the weak gauge bosons get modified by the dimension-six operators of the class $\psi^2 H^2 D$. For example, the operator Q_{Hud} induces an interaction of the $\mathcal{W}$ bosons with *right-handed* quark currents:

$$C_{\underset{rs}{Hud}} (\widetilde H^\dagger i D_\mu H)(\bar u_{Rr} \gamma^\mu d_{Rs}) = -\bar g_2 \frac{v_T^2}{2\sqrt{2}} C_{\underset{rs}{Hud}} \mathcal{W}_\mu^+ (\bar u_{Rr} \gamma^\mu d_{Rs}) + \ldots , \tag{5.19}$$

where the ellipsis denotes terms involving scalar fields.

5.2 Effects beyond the SM in the LEFT

In Sect. 3.2, we already discussed the EFT below the electroweak scale, which only contains the light SM particles. This EFT can be regarded not only as an effective low-energy description of SM weak interaction, but in its most general form as an EFT for any heavy physics at or above the electroweak scale. This includes scenarios with heavy particles beyond the SM. Several differences arise compared to the Fermi theory of SM weak interactions.

1. Within the SM, baryon and lepton number are accidental symmetries. The low-energy EFT matched to the SM inherits these symmetries. In a more general BSM scenario, the accidental SM symmetries do not need to be respected and the symmetry group of the LEFT is just the gauged $SU(3)_c \times U(1)_{\mathrm{em}}$. E.g., the LEFT contains a new dimension-3 operator that violates lepton-number by two units, the Majorana-mass term for the light neutrinos:

$$\mathcal{L}_{\mathrm{LEFT}} = \ldots - \frac{1}{2}\left(\nu_L^\top C M_\nu \nu_L + \mathrm{h.c.}\right) + \ldots . \tag{5.20}$$

At dimension five, we can add another $\Delta L = 2$ operator and its Hermitian conjugate:

$$\mathcal{L}_{\mathrm{LEFT}} = \ldots + \underset{pr}{L_{\nu\gamma}} \underset{pr}{\mathcal{O}_{\nu\gamma}} + \mathrm{h.c.} + \ldots , \quad \underset{pr}{\mathcal{O}_{\nu\gamma}} = \nu_{Lp}^\top C \sigma^{\mu\nu} \nu_{Lr} F_{\mu\nu} , \tag{5.21}$$

where p and r are flavor indices. This is a neutrino dipole operator, describing an effective interaction of neutrinos with photons. At dimension six, one can write down many four-fermion operators, which violate lepton number or lepton and baryon number. There is scalar operator with $\Delta L = 4$ and its Hermitian conjugate,

$$\underset{prst}{\mathcal{O}_{\nu\nu}^{S,LL}} = (\nu_{Lp}^\top C \nu_{Lr})(\nu_{Ls}^\top C \nu_{Lt}) , \tag{5.22}$$

and several $\Delta L = \pm 2$ operators can be constructed with a scalar or tensor neutrino bilinear (as in the Majorana-mass term or the dipole operator), supplemented by another leptonic or quark bilinear. Furthermore, there are operators with $\Delta B = \Delta L = \pm 1$, similarly to the SMEFT operators in Table 1, and even operators with $\Delta B = -\Delta L = \pm 1$ exist. The complete basis up to dimension six can be found in [20].

2. The SM with massless left-handed neutrinos conserves not only overall lepton number, but it is invariant under a global $U(1)_e \times U(1)_\mu \times U(1)_\tau$ symmetry group that implies lepton-family-number (or lepton-flavor) conservation. Lepton-flavor violation is observed in nature in neutrino oscillations, which can be explained by the presence of very small neutrino masses. In a generic BSM scenario, the LEFT operator basis is constructed without the restriction of the global lepton-flavor symmetry group and therefore it contains lepton-flavor-violating effective operators, such as off-diagonal dipoles

$$\underset{pr}{\mathcal{O}_{e\gamma}} = \bar{e}_{Lp} \sigma^{\mu\nu} e_{Rr} F_{\mu\nu} . \tag{5.23}$$

The non-observation of the charged lepton-flavor-violating process $\mu \to e\gamma$ puts very strong constraints on the coefficients of the operators $\underset{e\mu}{\mathcal{O}_{e\gamma}}$ and $\underset{\mu e}{\mathcal{O}_{e\gamma}^\dagger}$: the experimental bound from the PSI experiment MEG is $\mathrm{BR}(\mu \to e\gamma) < 4.2 \times 10^{-13}$ [57].

3. Apart from the missing accidental global symmetries, the difference between the SM and a generic BSM scenario is not reflected in the actual structure of the LEFT, but rather in the value of its Wilson coefficients. When performing the matching from the SM at the weak scale, many Wilson coefficients only receive contributions that are highly suppressed either by loop factors, small couplings, powers of the LEFT expansion parameter m/v, or combinations thereof. An example are flavor-changing neutral-current interactions or CP-violating processes. Within a generic BSM scenario, a priori there could be contributions to the corresponding effective operators that dominate over the suppressed SM contribution. Accordingly, if the experimental measurements are compatible with the SM prediction, strong constraints on any such BSM scenario are obtained.

The EFT framework provides an efficient tool to compile indirect constraints as bounds on operator coefficients. The RGEs allow one to consistently combine constraints arising at different energy scales

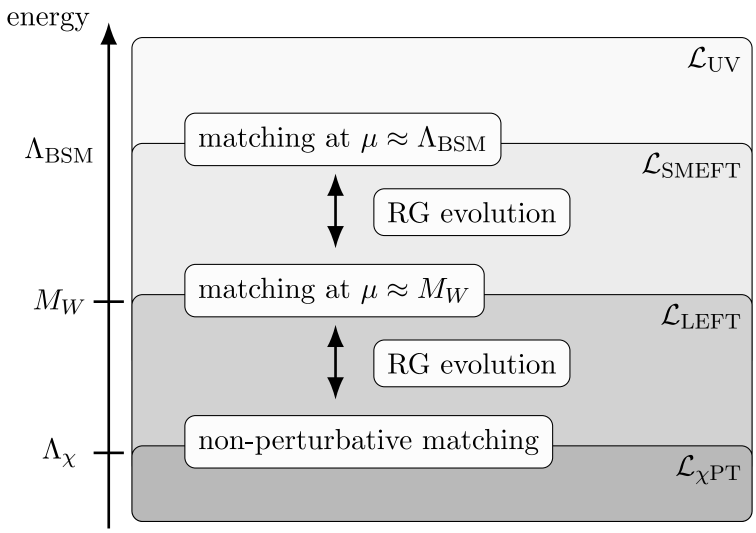


Figure 5: A tower of EFTs: χPT is built on top of the LEFT, which at the weak scale can be matched to the SMEFT. We expect BSM physics to appear at some high-energy scale Λ_{BSM}, where some concrete model for new physics can be matched to the SMEFT.

and to resum large logarithms. At the appropriate scale of new physics, a model for BSM physics can be matched to the EFT and the existing bounds on the operator coefficients directly translate into constraints on the model parameters. If new physics is much heavier than the electroweak scale, the general strategy relies on a tower of EFTs, as illustrated in Fig. 5: by integrating out the heavy BSM particles at a scale Λ_{BSM}, one performs the matching of a specific model for new physics to the SMEFT. At the electroweak scale, the SMEFT can be matched to the LEFT. If one is interested in processes at very low energies at the hadronic scale, one needs to take into account the non-perturbative nature of the strong interaction. The matching to χPT at the hadronic scale cannot be performed in perturbation theory.

5.3 Nonlinear realization of electroweak symmetry

In Sect. 5.1, we discussed the SMEFT, which is an EFT for heavy physics beyond the SM and relies on the assumption that the physical Higgs boson can be described by a field that is collected together with the Goldstone bosons of electroweak symmetry breaking in a fundamental $SU(2)_L$ doublet. This resulted in a linear realization of electroweak symmetry and an EFT with a power counting based on canonical mass dimensions of the operators. Although well motivated, this is not the most general EFT for heavy BSM physics. Experimentally, the symmetry breaking pattern $SU(2)_L \times U(1)_Y \to U(1)_{\mathrm{em}}$ is well established, which implies the presence of three Goldstone boson modes (due to the fact that the symmetries are gauged, the Goldstone bosons are "eaten" and become the longitudinal degrees of freedom of the gauge bosons). Furthermore, the presence of a scalar particle is established—the Higgs boson. Its couplings are compatible with the prediction of the SM, but so far, the uncertainties in the scalar sector are much larger than in the other sectors of the SM. In particular, the self-couplings of the Higgs boson are at present barely constrained experimentally.

A more general framework than SMEFT does not rely on the linear realization of electroweak symmetry, but is rather based on the most general parametrization of the degrees of freedom in the scalar sector: three Goldstones from spontaneous symmetry breaking, together with a (physical) scalar boson. The most general formulation leads to a nonlinear realization of electroweak symmetry, in analogy to χPT as a low-energy description of QCD. In the case of heavy new physics, the underlying UV theory is not known. Scenarios that require a nonlinear realization instead of SMEFT include strongly coupled new sectors, but also simple extensions by scalar degrees of freedom mixing with the

Higgs field in the non-decoupling region of parameter space.

Let us start with by considering again the scalar sector of the SM Lagrangian:

$$\mathcal{L} = (\partial_\mu H)^\dagger (\partial^\mu H) - \lambda \left(H^\dagger H - \frac{1}{2} v^2 \right)^2 . \tag{5.24}$$

In the usual linear representation of the SM, we write the Higgs doublet as

$$H(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2(x) + i\phi_1(x) \\ h(x) + v - i\phi_3(x) \end{pmatrix} . \tag{5.25}$$

Since

$$H^\dagger H = \frac{1}{2} \left((h+v)^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 \right) , \tag{5.26}$$

we see that the scalar sector has an enlarged symmetry group $SO(4)$, which spontaneously breaks down to $SO(3)$. We perform a field redefinition and introduce the 2×2 matrix

$$\Sigma(x) := \left(\widetilde{H}(x), H(x) \right)^\dagger = \frac{1}{\sqrt{2}} \left((h(x) + v)\mathbb{1} - i\vec{\tau} \cdot \vec{\phi}(x) \right) . \tag{5.27}$$

In terms of this matrix, we can write the scalar Lagrangian as

$$\mathcal{L} = \frac{1}{2} \langle (\partial_\mu \Sigma)(\partial^\mu \Sigma)^\dagger \rangle - \frac{\lambda}{4} \left(\langle \Sigma^\dagger \Sigma \rangle - v^2 \right)^2 , \tag{5.28}$$

where $\langle \cdot \rangle$ denotes the trace. The Lagrangian in terms of the redefined fields now exhibits an $SU(2)_L \times SU(2)_R$ symmetry, which has a Lie algebra that is isomorphic to the one of $SO(4)$. The vev spontaneously breaks the symmetry to the diagonal subgroup $SU(2)_V$. We perform another field redefinition

$$\Sigma(x) =: \frac{\hat{h}(x) + v}{\sqrt{2}} U(x) , \tag{5.29}$$

where

$$U(x) = \exp \left(i \frac{\vec{\tau} \cdot \vec{\pi}(x)}{v} \right) . \tag{5.30}$$

We have now changed from the fields $\vec{\phi}$ and h, which parametrize the Goldstone bosons and the Higgs scalar, to the fields $\vec{\pi}$ and $\hat{h}$, which are related by a nonlinear field redefinition. As we know, physics is unchanged by field redefinitions and we can equally well interpret $\hat{h}$ as the Higgs field and $\vec{\pi}$ as the Goldstone bosons. ("The Higgs field" actually is not a unique notion at all.)

In terms of the new fields, the scalar sector of the Lagrangian reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \hat{h} \partial^\mu \hat{h} - \frac{1}{2} M_H^2 \hat{h}^2 + \frac{v^2}{4} \mathcal{F}(\hat{h}/v) \langle (\partial_\mu U)(\partial^\mu U)^\dagger \rangle - V(\hat{h}/v) , \tag{5.31}$$

where $M_H^2 = 2\lambda v^2$ and

$$\mathcal{F}(\hat{h}/v) = \left(1 + \frac{\hat{h}}{v} \right)^2 , \quad V(\hat{h}/v) = v^4 \left(\frac{M_H^2}{2v^2} \left(\frac{\hat{h}}{v} \right)^3 + \frac{M_H^2}{8v^2} \left(\frac{\hat{h}}{v} \right)^4 \right) . \tag{5.32}$$

We have arrived at a description of the scalar sector that corresponds to the nonlinear realization of the spontaneous breakdown of $SU(2)_L \times SU(2)_R \to SU(2)_V$, in analogy to χPT. The matrix U transforms as

$$U(x) \mapsto R U(x) L^\dagger , \quad L, R \in SU(2) . \tag{5.33}$$

We can promote the symmetry to a local one by introducing spurion fields and defining the covariant derivative as

$$D_\mu U = \partial_\mu U + iU\hat{W}_\mu - i\hat{B}_\mu U \,, \tag{5.34}$$

where $\hat{W}_\mu$ and $\hat{B}_\mu$ are 2×2 matrix spurion fields, formally transforming as

$$\begin{aligned} \hat{W}_\mu(x) &\mapsto L(x)\hat{W}_\mu(x)L^\dagger(x) + iL(x)\partial_\mu L^\dagger(x)\,, \\ \hat{B}_\mu(x) &\mapsto R(x)\hat{B}_\mu(x)R^\dagger(x) + iR(x)\partial_\mu R^\dagger(x)\,. \end{aligned} \tag{5.35}$$

The SM scalar sector including the interaction with the weak gauge bosons is then recovered from the Lagrangian

$$\mathcal{L}_{p^2}^{\text{scalar}} = \frac{1}{2}\partial_\mu \hat{h}\partial^\mu \hat{h} - \frac{1}{2}M_H^2 \hat{h}^2 + \frac{v^2}{4}\mathcal{F}(\hat{h}/v)\langle (D_\mu U)(D^\mu U)^\dagger \rangle - V(\hat{h}/v) \tag{5.36}$$

upon fixing the spurion fields to

$$\hat{W}_\mu(x) \mapsto -g_2 \frac{\tau^I}{2} W_\mu^I(x)\,, \quad \hat{B}_\mu(x) \mapsto -g_1 \frac{\tau^3}{2} B_\mu(x)\,, \tag{5.37}$$

which introduces an explicit symmetry breaking of the chiral group but leaves $SU(2)_L \times U(1)_Y$ intact.

After adding the gauge sector and the fermion sector to the Lagrangian for the scalar sector, we can regard the SM as a special case of the leading term in an effective chiral Lagrangian: the Higgs field $\hat{h}$ is a singlet under the SM gauge group and the symmetries of the theory do not fix the functions $\mathcal{F}(\hat{h}/v)$ and $V(\hat{h}/v)$, which in general can be written as a power series

$$\mathcal{F}(\hat{h}/v) = 1 + \sum_{n=1}^{\infty} a_n \left(\frac{\hat{h}}{v}\right)^n, \quad V(\hat{h}/v) = v^4 \sum_{n=3}^{\infty} b_n \left(\frac{\hat{h}}{v}\right)^n. \tag{5.38}$$

The NLO Lagrangian is constructed in close analogy to the NLO χPT Lagrangian, but again with operators multiplied by arbitrary functions of $\hat{h}/v$.

This EFT is known under several names: HEFT (Higgs effective field theory), EWχL (electroweak chiral Lagrangian), or EWET (electroweak effective theory) [58–60]. It is a generalization of SMEFT that can be characterized as follows:

1. *Degrees of freedom*: The physical degrees of freedom agree with the ones in the SM: the EFT contains the SM gauge bosons, fermions, and a scalar boson that corresponds to the Higgs particle discovered at the LHC. However, the scalar particle is not treated as part of an electroweak doublet.

2. *Symmetries*: The EFT is based on the same gauge symmetries as the SM, including the spontaneous breaking pattern $SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_{\text{em}}$. In contrast to the SMEFT, the electroweak symmetry is nonlinearly realized. The Higgs boson is described by a singlet field, the three Goldstone bosons of spontaneous symmetry breaking are collected in the matrix exponential U.

3. *Power counting*: Due to the nonlinear realization, the EFT power counting is not in canonical mass dimensions, but rather follows a chiral loop expansion, similar to χPT extended by photons and leptons. It can be organized either in terms of chiral dimensions or by an generalization of NDA rules [61, 62].

While a linear representation can be transformed into a nonlinear realization by a field redefinition, the reverse is not always possible. That the nonlinearly realized electroweak EFT is a generalization of SMEFT can be understood in geometric terms, as explained in Ref. [63].

6 Dispersion relations

Dispersion relations are a technique to relate different observables in a (largely) model-independent way by making use on the fundamental principles of unitarity and analyticity, which are satisfied in a local QFT. They were very popular in the context of S-matrix theory during the 1950s and 1960s, when attempts were made to arrive at a theory of strong interaction through a bootstrapping procedure. On the one hand, this research triggered the development of string theory, on the other hand it was largely abandoned in the context of strong interaction with the success of perturbative QCD. In the context of low-energy QCD, where perturbative methods do not work, dispersion relations were resurrected in the 1990s [64] in a "marriage with χPT" [65]. The combination of dispersion relations with EFTs is useful because of the strength of the two approaches is somewhat complementary.

- Dispersion relations often involve free parameters, which a priori are specific to the process in question. A matching to an EFT allows one to express these parameters in terms of the free parameters of the EFT (Wilson coefficients, or LECs in the case of χPT), which do not depend on the process. This matching can be performed in the very low-energy region, where the EFT is most reliable. On the other hand, dispersion relations often allow one to extend the validity of the description beyond the range of a pure EFT calculation.
- Certain assumptions on the asymptotic behavior of the amplitudes result in sum rules and/or bounds that follow from dispersion relations and that can be formulated as bounds on the EFT parameters. Such kind of bounds have been derived for chiral LECs, and they are becoming popular again in the context of EFTs for new physics. They hold under the assumption that the underlying UV theory fulfills the fundamental principles used in the derivation of the dispersion relation (usually analyticity, unitarity, and crossing symmetry), and up to corrections of higher order in the EFT expansion.

In this last chapter, we will introduce the basic concepts of dispersion relations and give a flavor of the connection with EFTs and resulting applications, without going into much detail. Sects. 6.1 and 6.2 introduce the notions of unitarity and analyticity. In Sect. 6.3, we discuss the matching between EFTs and dispersion relations and in Sect. 6.4, we will see how one can obtain bounds on EFT parameters from dispersion relations.

6.1 Unitarity of the S-matrix

The S-operator, or S-matrix relates the asymptotic incoming states to the asymptotic outgoing states of a scattering process by

$$|p\rangle_{\text{in}} = S|p\rangle_{\text{out}} \,, \quad |p\rangle_{\text{out}} = S^{-1}|p\rangle_{\text{in}} \,, \tag{6.1}$$

and hence

$${}_{\text{in}}\langle p| = {}_{\text{out}}\langle p|S^\dagger \,, \quad {}_{\text{out}}\langle p| = {}_{\text{in}}\langle p|(S^{-1})^\dagger \,. \tag{6.2}$$

In order to conserve probability, S needs to be a unitary operator:

$${}_{\text{out}}\langle p|p\rangle_{\text{out}} = {}_{\text{in}}\langle p|p\rangle_{\text{in}} = {}_{\text{in}}\langle p|S^\dagger S|p\rangle_{\text{in}} \,, \tag{6.3}$$

hence $S^\dagger S = \mathbb{1}$ and therefore

$${}_{\text{out}}\langle p| = {}_{\text{in}}\langle p|S \,. \tag{6.4}$$

We split the S-matrix into a trivial part without any scattering, plus the nontrivial T-matrix:

$$S = \mathbb{1} + iT \,. \tag{6.5}$$

Unitarity of S then implies for the T-matrix

$$i(T^\dagger - T) = T^\dagger T\,. \tag{6.6}$$

Taking the matrix element of this equation between initial and final states $|i\rangle$ and $|f\rangle$ leads to

$$i(\mathcal{T}_{if}^* - \mathcal{T}_{fi}) = \sum_n (2\pi)^4 \delta^{(4)}(p_f - p_n) \mathcal{T}_{nf}^* \mathcal{T}_{ni}\,, \tag{6.7}$$

where we inserted a complete set of intermediate states $|n\rangle$. Time-reversal invariance leads to the *optical theorem*:

$$\mathrm{Im}(\mathcal{T}_{fi}) = \frac{1}{2}\sum_n (2\pi)^4 \delta^{(4)}(p_f - p_n) \mathcal{T}_{nf}^* \mathcal{T}_{ni}\,. \tag{6.8}$$

Let's consider the example of a $2 \to 2$ scattering of scalar particles. If we are in an energy region, where the only intermediate state is given by the final state, $|n\rangle = |f\rangle$, we are dealing with elastic final-state rescattering. In this case, the phase-space integral in the optical theorem consists of six integrals. Four of them can be performed trivially by making use of the delta function, resulting in

$$\mathrm{Im}\big(\mathcal{T}_{fi}(s,\vartheta)\big) = \frac{1}{8(2\pi)^2 S}\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2s}\int d\Omega\, \mathcal{T}_{ff}^*(s,\vartheta'')\mathcal{T}_{fi}(s,\vartheta')\,, \tag{6.9}$$

where the symmetry factor is $S = 2$ for indistinguishable particles and $S = 1$ otherwise, $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)$ is the Källén triangle function, s is the Mandelstam variable, and ϑ, ϑ', and ϑ'' are the s-channel scattering angles of the processes $i \to f$, $i \to n$, and $n \to f$, respectively. $m_{1,2}$ denote the masses of the final-state particles. We expand the scattering amplitudes into s-channel partial waves:

$$\begin{aligned}\mathcal{T}_{fi}(s,\vartheta) &= \sum_{l=0}^{\infty} P_l(\cos\vartheta) f_l(s)\,,\\ \mathcal{T}_{ff}(s,\vartheta) &= \sum_{l=0}^{\infty} P_l(\cos\vartheta) t_l(s)\,,\end{aligned} \tag{6.10}$$

where P_l are the Legendre polynomials.[8] The addition theorem for Legendre polynomials allows us to perform the angular integrals:

$$\mathrm{Im}\big(f_l(s)\big) = \frac{1}{8(2\pi)^2(2l+1)S}\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2s} t_l^*(s) f_l(s)\,\theta(s-(m_1+m_2)^2)\,, \tag{6.11}$$

with the step function θ. We make a couple of observations.

- The imaginary part of the partial-wave scattering amplitude starts at the two-particle threshold $s = (m_1 + m_2)^2$.
- The right-hand side of the equation needs to be a real quantity. Therefore, the phase of f_l agrees with the phase of the elastic scattering partial wave t_l (modulo π). This is known as *Watson's final-state theorem* [67].
- Applied to elastic scattering, $|i\rangle = |f\rangle$, it allows us to write the partial wave in terms of the elastic phase shift $\delta_l(s)$, which is a real function:

$$t_l(s) = 8\pi(2l+1)S\,\frac{2s\sin\delta_l(s)e^{i\delta_l(s)}}{\lambda^{1/2}(s,m_1^2,m_2^2)} \quad \text{if } s > (m_1+m_2)^2\,. \tag{6.12}$$

- For the generic process $i \to f$, this implies

$$\mathrm{Im}\big(f_l(s)\big) = f_l(s)e^{-i\delta_l(s)}\sin\delta_l(s)\,\theta(s-(m_1+m_2)^2)\,. \tag{6.13}$$

[8] For particles with non-zero spin, the partial-wave expansion of helicity amplitudes involves the Wigner d-functions [66].

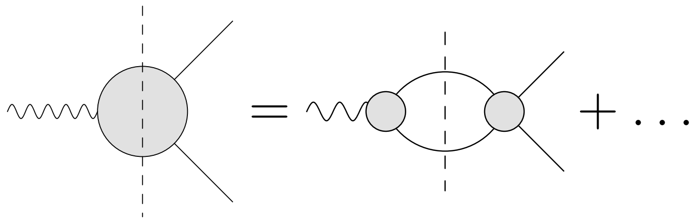


Figure 6: Taking only elastic unitarity into account, the imaginary part of the pion VFF gets related to the elastic $\pi\pi$-scattering phase shift.

6.2 Analyticity of scattering amplitudes

In the context of the harmonic oscillator or the Kramers–Kronig relations in optics,[9] one already encounters the connection between causality and analyticity, which is formulated in Titchmarsh's theorem. In relativistic QFTs, a similar connection with micro-causality holds, resulting in the principle of maximal analyticity: scattering amplitudes are analytic or holomorphic functions of the scalar invariants, promoted to *complex variables*, apart from singularities dictated by unitarity.

We will illustrate this with an example that we already discussed in Sect. 4.7: the pion VFF, defined in Eq. (4.66). Denoting the argument of the VFF by s, which is regarded to be a complex variable, elastic unitarity implies

$$\mathrm{Im}\big(F_\pi^V(s)\big) = F_\pi^V(s) e^{-i\delta_1^1(s)} \sin\delta_1^1(s)\, \theta(s-4M_\pi^2)\,, \tag{6.14}$$

where $\delta_1^1(s)$ is the elastic $\pi\pi$-scattering phase shift in the P-wave and for isospin $I=1$. If this phase shift is known (e.g., from experiment), unitarity determines the imaginary part of the VFF above the two-pion threshold, as illustrated in Fig. 6. As we will see shortly, analyticity also allows us to obtain the real part of the VFF.

We first discuss the general procedure. Consider a function of one complex variable $f(s)$ that is analytic on the cut complex plane $\mathbb{C}\setminus\Gamma$, $\Gamma := [s_0,\infty)\subset\mathbb{R}$ and real below the cut: $f(s)\in\mathbb{R}\ \forall s<s_0$. The Schwarz reflection principle implies

$$f(s^*) = f^*(s) \quad \forall\, s\in\mathbb{C}\setminus\Gamma\,. \tag{6.15}$$

We can apply Cauchy's integral formula

$$f(s) = \frac{1}{2\pi i}\oint_\gamma ds' \frac{f(s')}{s'-s} \tag{6.16}$$

to the integration path γ shown in figure 7. If we assume that the function $f(s)$ tends to zero for $|s|\to\infty$, the integral over the arc vanishes for $R\to\infty$ and the integral over γ_c remains.

$$f(s) = \frac{1}{2\pi i}\int_{\gamma_c} ds' \frac{f(s')}{s'-s} = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{s_0}^\infty ds' \frac{f(s'+i\epsilon)-f(s'-i\epsilon)}{s'-s} = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{s_0}^\infty ds' \frac{\mathrm{Im} f(s'+i\epsilon)}{s'-s}\,. \tag{6.17}$$

By evaluating this equation just above the cut, we arrive at:

$$f(s) = \frac{1}{\pi}\int_{s_0}^\infty ds' \frac{\mathrm{Im} f(s')}{s'-s-i\epsilon}\,, \tag{6.18}$$

[9] This is the origin of the term *dispersion relations*.

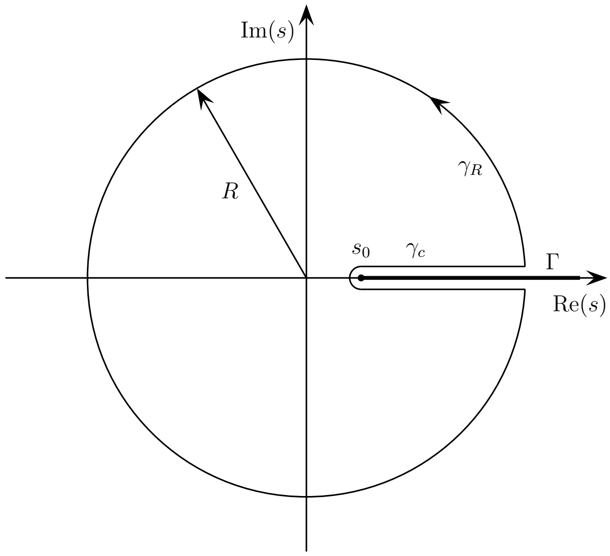


Figure 7: The integration path γ consists of a part γ_c circumnavigating the branch cut and an arc γ_R with radius R.

where we suppress the limit $\epsilon \to 0$ for simplicity and $f(s)$ and $f(s')$ are understood to denote the analytic continuation to the real axis from above the cut. Equations like (6.18) are called *dispersion relations*.

If the function $f(s)$ does not fall off for $|s| \to \infty$, we can define a new function

$$g(s) := \frac{f(s) - f(\bar{s})}{s - \bar{s}}, \tag{6.19}$$

where $\bar{s} < s_0$ is called the subtraction point. g has the same analytic properties as f but falls off faster. If it tends to zero for $|s| \to \infty$, we can write a dispersion relation for $g(s)$:

$$\frac{f(s) - f(\bar{s})}{s - \bar{s}} = \frac{1}{\pi} \int_{s_0}^{\infty} ds' \frac{1}{s' - s - i\epsilon} \mathrm{Im} \left(\frac{f(s') - f(\bar{s})}{s' - \bar{s}} \right) . \tag{6.20}$$

Since $\mathrm{Im} f(\bar{s}) = 0$, this is equivalent to the *subtracted dispersion relation*

$$f(s) = f(\bar{s}) + \frac{s - \bar{s}}{\pi} \int_{s_0}^{\infty} ds' \frac{\mathrm{Im} f(s')}{(s' - s - i\epsilon)(s' - \bar{s})} . \tag{6.21}$$

In this case, $f(s)$ is not determined by its imaginary part alone, but one needs to know the *subtraction constant* $f(\bar{s})$.

In the example of the pion VFF, let us assume that $F_\pi^V(s)$ does not have any zeros in the complex s-plane and that the phase shift $\delta_1^1(s)$ is bounded. In this case, $f(s) := \log F_\pi^V(s)$ is an analytic function of s, apart from the unitarity branch cut $\Gamma = [4M_\pi^2, \infty)$. Elastic unitarity implies that

$$\mathrm{Im} f(s) = \delta_1^1(s) \tag{6.22}$$

for s below the first inelastic thresholds. Gauge invariance fixes the value of the pion VFF at zero, $F_\pi^V(0) = 1$, hence $f(0) = 0$. Neglecting inelasticities, we can write a once-subtracted dispersion relation for $f(s)$ (with subtraction point $\bar{s} = 0$), since $\delta_1^1(s)/s \to 0$ for $s \to \infty$. The pion VFF has the following form:

$$F_\pi^V(s) = \exp \left(\frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{(s' - s - i\epsilon)s'} \right) , \tag{6.23}$$

which is called *Omnès function* [68]. This form even holds if inelastic effects are included, as long as we replace the $\pi\pi$ phase shift $\delta_1^1(s)$ above the first inelastic threshold by the phase of the VFF itself.

6.3 Matching dispersion relations with EFTs

Dispersion relations allow us to obtain relations between observables that are based on the fundamental principles of unitarity and analyticity (and often crossing symmetry). The analytic structure of scattering amplitudes is determined through these principles as follows:

- single-particle intermediate states lead to poles in the scattering amplitudes;
- multi-particle intermediate states lead to branch cuts.

Unitarity of the S-matrix determines the residue of single-particle poles and the discontinuity along multi-particle branch cuts in terms of the respective sub-processes. By making use of the Cauchy integral formula, one then derives dispersion integrals, which reproduce the entire amplitude from its pole residues and discontinuities, potentially up to the constants in a subtraction polynomial. This is the point where EFTs enter the game: the subtraction constants in a dispersion relation correspond to the values of the amplitude and its derivatives at the subtraction point, which is chosen below threshold, i.e., in a region of very low energies. This is the region, where an EFT description of the amplitude works best. Depending on the form of the dispersion relation, the identification of the subtraction constants with linear combinations of the free parameters of an EFT is more or less direct and follows by matching the value of the amplitude and its derivatives at the subtraction point between the EFT description and the dispersive representation.

As we have seen in Sect. 6.2, dispersion relations require subtractions in the case that the amplitude does not fall off fast enough at infinity. In this case, the subtraction constants are free parameters in the dispersion relation, i.e., unitarity and analyticity do not fix them alone: e.g., in such a case the relative energy dependence of an amplitude might be determined through unitarity and analyticity, whereas the overall normalization of the amplitude is not fixed. While the subtraction constant is specific to the process, the matching to an EFT description expresses it in terms of the free parameters of the EFT, which are universal: the same EFT parameters enter in the description of many different processes.

Consider for simplicity a subtraction point $\bar{s} = 0$. The identity

$$\frac{1}{s'-s} = \frac{1}{s'} + \frac{s}{(s'-s)s'} \tag{6.24}$$

allows us to rewrite an unsubtracted dispersion relation (6.18) as

$$f(s) = \underbrace{\frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds'}{s'}\mathrm{Im}f(s')}_{f(0)} + \frac{s}{\pi}\int_{s_0}^{\infty} ds' \frac{\mathrm{Im}f(s')}{(s'-s-i\epsilon)s'} \,. \tag{6.25}$$

The expression for $f(0)$ in terms of an integral over the imaginary part is called *sum rule*. If the imaginary part is known well enough, one can use the relation to determine $f(0)$ and, by matching the amplitude, to determine a free parameter of the EFT. On the other hand, if the imaginary part is poorly known at high energies, it might be better to use input from an EFT instead of the sum rule to determine $f(0)$, even if the sum-rule integral converges in principle. In this case, we speak of an *over-subtracted dispersion relations*: input at low energies reduces the dependence on the imaginary part at high energies. The additional power of s' in the denominator in the second integral of Eq. (6.25) suppresses the high-energy tail of the integrand compared to the unsubtracted relation in Eq. (6.18).

If the amplitude describes a measured process, we can use the (over-)subtracted dispersion relation for a fit to data and determine the subtraction constant experimentally. The matching to the EFT will then allow us to determine an EFT parameter from data. Using the dispersive representation (together with input on the imaginary part from the unitarity relation) as an intermediate step usually leads to more reliable determinations compared to a direct fit of the EFT amplitude to data: note that by using the dispersion relation, we are relying on the EFT expansion only around the point $s = 0$, i.e., below the physical threshold, instead of using an EFT description in the physical region $s > s_0$.

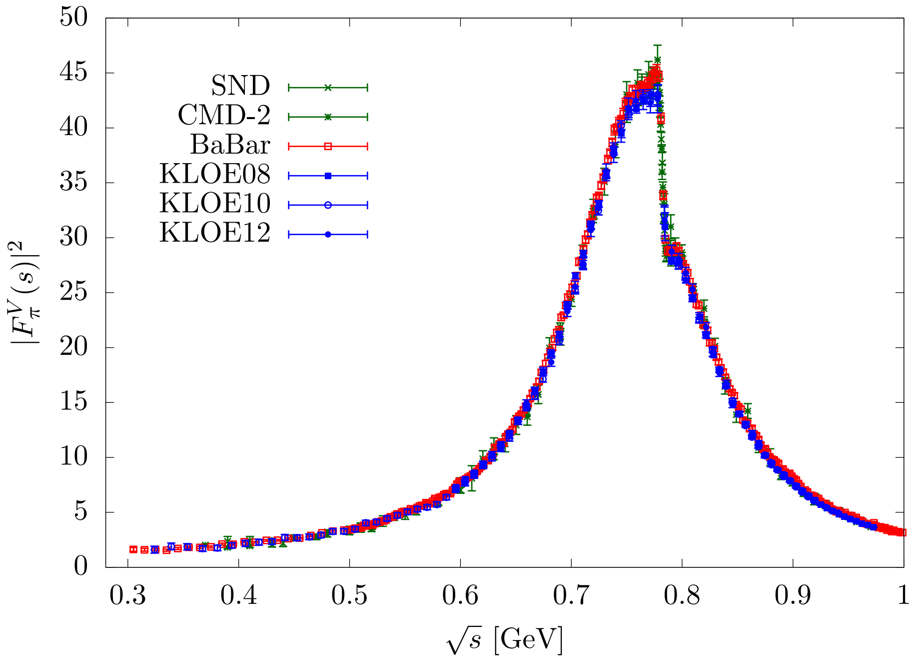


Figure 8: Experimental measurement of the pion VFF by the SND [69, 70], CMD-2 [71–74], BaBar [75, 76], and KLOE experiments [77–80].

Take as an example the pion VFF. In the *time-like physical region* $s > 4M_\pi^2$, it is accessible through measurements of the process $e^+e^- \to \pi^+\pi^-$, see Fig. 8. It is well-known that the VFF is approximately described by a vector-meson dominance (VMD) model with a ρ-meson propagator:

$$F_\pi^V(s) \approx \frac{M_\rho^2}{M_\rho^2 - s}\,, \tag{6.26}$$

which reproduces the prominent resonance around $M_\rho \approx 770\,\text{MeV}$, visible in Fig. 8. The region of the resonance is where the EFT description of the VFF breaks down—χPT only describes the VFF at low energies. The expansion around $s = 0$

$$F_\pi^V(s) = 1 + \frac{1}{6}\langle r_\pi^2\rangle s + \mathcal{O}(s^2) \tag{6.27}$$

leads to the VMD estimate of the pion charge radius

$$\langle r_\pi^2\rangle \approx \frac{6}{M_\rho^2} \approx 0.4\,\text{fm}^2\,. \tag{6.28}$$

Through the matching to χPT around $s = 0$ given in Eq. (4.77), one can extract the NLO LEC L_9^r.

The VMD estimate can be refined by using a dispersion relation for the VFF. The Omnès representation (6.23) can be generalized to

$$F_\pi^V(s) = P(s)\Omega_1^1(s)\,, \quad \Omega_1^1(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{(s'-s-i\epsilon)s'}\right)\,, \tag{6.29}$$

where gauge invariance requires $P(0) = 1$. Elastic unitarity tells us that $P(s) = F_\pi^V(s)/\Omega_1^1(s)$ has a branch cut starting only at the inelastic three-pion threshold $s_{3\pi} = (3M_\pi)^2$. However, the three-pion intermediate state violates isospin symmetry and is only relevant in the region of the very narrow ω-resonance, which can be described well by a Breit–Wigner factor. The next threshold is due to four pions, $s_{4\pi} = (4M_\pi)^2$. Phenomenologically, the four-pion intermediate state only becomes relevant above the $\pi\omega$ threshold, $s_{\pi\omega} = (M_\pi + M_\omega)^2 \approx (0.9\,\text{GeV})^2$. Therefore, if we stay at low energies away from the inelastic branch cuts, the function $P(s)$ can be expanded into a Taylor series and approximated by a polynomial, $P(s) = 1 + as + bs^2 + \ldots$. With input on the elastic $\pi\pi$ phase shift $\delta_1^1(s)$ (which is precisely known from another dispersion relation and data input [81]), the parameters a, b, … can be determined in a fit to $e^+e^- \to \pi^+\pi^-$ data. If the narrow ω resonance is included, a representation of the pion VFF can be obtained that is accurate up to energies of almost 1 GeV, see, e.g., Ref. [82]. The expansion around $s = 0$ again determines the pion charge radius and, by matching to χPT, provides access to the LEC L_9^r. In this procedure, the chiral expansion is used only at $s = 0$, but not in the physical time-like region $s > 4M_\pi^2$. Alternatively, χPT provides a good description of the pion VFF in the very low-energy *space-like region* $s < 0$, which is accessible in the crossed process $e^-\pi^- \to e^-\pi^-$ [83].

6.4 Analyticity and unitarity constraints on EFT parameters

In the previous section, we have seen that dispersion relations can be used to extend the range of validity of the description of an amplitude compared to a pure EFT treatment, provided that the input for the imaginary part is available: the unitarity relation determines the imaginary part in terms of the sub-processes with the intermediate states, which are again observables. EFTs can then be used to describe the subtraction polynomial in a dispersion relation, i.e., the EFT expansion is used only at very low energies.

When performing the matching to the EFT in the low-energy region, unitarity and analyticity sometimes provide constraints on the EFT parameters even in the case where the imaginary part in the dispersion integrals is not known. We briefly discuss another example from χPT [84], but the methods are generic and can be also applied, e.g., to EFTs for physics beyond the SM.

Consider the $2 \to 2$ scattering process $\pi^0\pi^0 \to \pi^0\pi^0$, which is totally crossing symmetric and described in terms of an amplitude $\mathcal{T}(s,t,u)$, where the Mandelstam variables fulfill

$$s + t + u = 4M_\pi^2\,. \tag{6.30}$$

The optical theorem implies that the amplitude develops an imaginary part as soon as one of the Mandelstam variables is larger than the two-pion threshold, i.e., $s > 4M_\pi^2$, $t > 4M_\pi^2$, or $u > 4M_\pi^2$. The linear constraint (6.30) implies that there are only two independent dynamic variables. This can be illustrated in terms of a *Mandelstam diagram*: since the sum $s + t + u$ is constant, the Mandelstam variables can be represented in one plane, using the fact that the sum of distances of a point to the sides of an equilateral triangle is constant. The Mandelstam diagram for $\pi^0\pi^0 \to \pi^0\pi^0$ scattering is shown in Fig. 9. The physical scattering regions of the three channels are highlighted in blue. The dark-gray and light-gray regions satisfy $s, t, u < 4M_\pi^2$, i.e., all Mandelstam variables are below threshold and the amplitude is real in these regions.

We will now focus on the dark-gray region A, defined by $s, t, u < 4M_\pi^2$ and $t > 0$. We see that along a line of fixed t, the amplitude develops an imaginary part for $s > 4M_\pi^2$ and for $u > 4M_\pi^2$, i.e., for $s < -t$. Compared to the simple form-factor kinematics discussed previously, a dispersion relation for a $2 \to 2$ scattering process is slightly more complicated and involves two dispersion integrals: one along the right-hand s-channel unitarity cut, another one along the left-hand crossed-channel (u-channel) unitarity cut. From basic principles, it can be derived that the amplitude does not grow faster than $\mathcal{T} \asymp s\log^2(s)$ or $|\mathcal{T}| < \text{const}\, s^2$ for fixed t [85, 86], leading to a twice-subtracted dispersion relation:

$$\begin{aligned}\mathcal{T}(s,t,u(s,t)) = a(t) + b(t)\,s + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty ds'\,\frac{\text{Im}\mathcal{T}(s',t,u(s',t))}{(s'-s)s'^2} \\ + \frac{u(s,t)^2}{\pi}\int_{4M_\pi^2}^\infty du'\,\frac{\text{Im}\mathcal{T}(s(t,u'),t,u')}{(u'-u(s,t))u'^2}\,,\end{aligned} \tag{6.31}$$

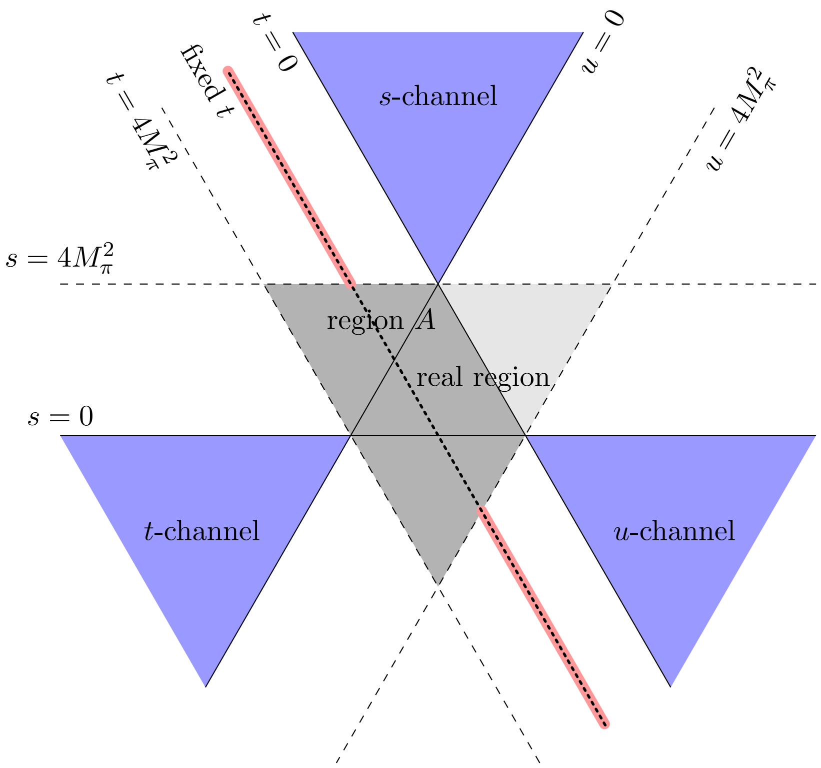


Figure 9: Mandelstam diagram for $\pi^0\pi^0 \to \pi^0\pi^0$ scattering. The light-blue areas are the physical scattering regions (s-, t-, and u-channels). The gray regions are the unphysical regions below all thresholds, where the amplitude is real. The dotted line at fixed t contains two parts, highlighted in red, where the amplitude develops an imaginary part due to s-channel or u-channel intermediate states. The dark-gray region is the region A where the bounds of Ref. [84] are derived.

where $u(s,t) = 4M_\pi^2 - s - t$ and $s(t,u) = 4M_\pi^2 - t - u$. Taking the second derivative and using crossing symmetry, one obtains

$$\frac{d^2}{ds^2}\mathcal{T}(s,t,u(s,t)) = \frac{2}{\pi}\int_{4M_\pi^2}^{\infty} dx \left[\frac{1}{(x-s)^3} + \frac{1}{(x-u(s,t))^3}\right] \mathrm{Im}\mathcal{T}(x,t)\,, \tag{6.32}$$

where we denote $\mathcal{T}(x,t) = \mathcal{T}(x,t,u(x,t))$. Next, we use a partial-wave expansion to write

$$\mathcal{T}(s,t) = \sum_{l=0}^{\infty} P_l(\cos\vartheta)t_l(s) = \sum_{l=0}^{\infty} P_l\left(1 + \frac{2t}{s-4M_\pi^2}\right)t_l(s)\,. \tag{6.33}$$

The optical theorem (6.8) relates the imaginary part of the partial waves to the partial-wave cross section, hence

$$\mathrm{Im}t_l(s) > 0\,. \tag{6.34}$$

Since the Legendre polynomials $P_l(z)$ are positive for $z > 1$ for any l, we conclude that

$$\mathrm{Im}\mathcal{T}(x,t) > 0 \quad \text{for } x > 4M_\pi^2 \text{ and } t > 0\,. \tag{6.35}$$

For (s,t) in the region A, also the square bracket in Eq. (6.32) is positive and hence the integral is positive as well. The second derivative of the amplitude in the region A can be computed in χPT and at NLO depends on chiral LECs. The chiral expansion is only performed in the very low-energy sub-threshold region A, where the EFT description is expected to work well. The bound

$$\frac{d^2}{ds^2}\mathcal{T}(s,t,u(s,t)) > 0 \quad \text{for } (s,t) \in A \tag{6.36}$$

directly translates into a bound on a linear combination of NLO LECs. Similar bounds can also be obtained for $\pi\pi$-scattering including charged pions. The bounds obtained in Ref. [84] are fulfilled by phenomenological determinations of the LECs. We note that χPT itself does not predict the values of the LECs: analyticity, unitarity, crossing, and an asymptotic behavior following from axiomatic principles provide additional constraints, which need to be fulfilled by any sensible UV theory underlying the EFT. These kind of constraints are of particular interest in the case where the underlying UV theory is not known, e.g., in the context of the SMEFT. Even if the bounds are derived by using the EFT expansion only in a region of very low energies, one should not forget that usually they are subject to corrections of higher order in the power counting.

A Short questions

The following questions were asked as a warm-up at the start of each lecture. They do not represent the entire content of the lecture and they do not always address the most important aspects—they were simply meant to stimulate discussions and to help focusing the attention to the current subject.

- Why does the presence of one operator with dimension $[\mathcal{O}] > 4$ induce an infinite tower of operators?
- Why is the presence of an infinite tower of operators in an EFT not a problem?
- Why is it important to have a complete set of operators?
- Why should we work with an operator basis and remove redundant operators?
- What is the difference between on-shell matching and off-shell matching? What are the advantages/disadvantages of the two methods?
- Why does the expansion of loop integrals in the soft scales in general introduce $1/\varepsilon_{\text{IR}}$ divergences?
- How do the UV and IR divergences cancel in the matching equation?
- What are the fundamental building blocks for the LEFT operators?
- Which classes of operators appear at $d = 5$ and $d = 6$ in the LEFT?
- What needs to be calculated to determine the mixing of the coefficients of

  $$\mathcal{O}_1 = (\bar{s}_L \gamma^\mu b_L)(\bar{u}_L \gamma_\mu c_L)\,,$$
  $$\mathcal{O}_2 = (\bar{s}_L \gamma^\mu T^A b_L)(\bar{u}_L \gamma_\mu T^A c_L)$$

  under QCD renormalization?
- Do these operators mix under QED renormalization as well?
- QCD at low energies: why does it make sense to study symmetries in the chiral limit $m_{u,d,s} \to 0$?
- Away from the chiral limit: why is isospin symmetry $m_u = m_d$ a good approximation? What is the symmetry group and the relation to chiral symmetry?
- What is a characteristic feature of Goldstone-boson interactions?
- What is the relation between quark masses and Goldstone-boson masses? Why are the kaons and the eta meson heavier than the pions?
- In the LEFT, we have encountered RG mixing effects:

  $$\mu \frac{d}{d\mu} C_i = \gamma_{ij} C_j$$

  What kind of RG mixing is possible in χPT?
- What is the range of validity of χPT? What is the expansion parameter?
- (*Catch question*) In which representation of chiral symmetry does the pion transform?
- In which sense can we understand the mesons to be bound states of a quark and an antiquark?
- What are the different types of chiral symmetry breaking?
- Why are there no dimension-5 dipole operators in the SMEFT?
- Which class of dimension-6 SMEFT operators leads to a modification of the coupling of the weak gauge bosons to fermion currents?
- How would you determine if a term in an effective Lagrangian contributes to CP violation? What follows in the case that an operator transforms as $\mathcal{O} \overset{CP}{\mapsto} \mathcal{O}^\dagger$?
- What is the main difference between the (bosonic sector of) HEFT / EWχL and χPT?

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