Four-fermion weak interactions

$$e^{i S_\Lambda(\phi_L)} = \int_\Lambda \mathcal{D}\phi_H \, e^{i\mathcal{L}_{full}(\phi_L, \phi_H)} = e^{i\int d^4x\, \mathcal{L}_{eff}(\phi_L)}$$

$$\mathcal{L}_{eff}(\phi_L) \leftarrow \sum_i \frac{c_i}{\Lambda^{d-4}} O_i(\phi_L(x))$$

4-F weak interactions

$\Rightarrow$ $\mathcal{L}_{full}$ is known

$$\mathcal{L}_{full} \subset \mathcal{L}_{SM}^{cc} = \frac{g}{2\sqrt{2}} J_\mu W_\mu + h.c.$$

$$J_\mu = V_{ij}\, u_i \gamma_\mu(1-\gamma_5) d_j + \bar{\nu}_i \gamma_\mu (1-\gamma_5) l_i = J_\mu^{had} + J_\mu^{lept}$$

$\rightarrow$ neglecting QCD corrections & h.o. weak interactions
simple gaussian integral

$$S_\Lambda \sim \int d^4x\, d^4y\, D_W^{\mu\nu}(x-y)\, J^\mu(x)\, J^\nu(y)$$

$$D_W^{\mu\nu}(x-y) \to \frac{-i g^{\mu\nu}}{M_W^2}\delta^4(x-y) + O\left(\frac{1}{M_W^4}\right)$$

$$\mathcal{L}_{eff} = -\underbrace{\left(\frac{g^2}{2\sqrt{2}}\right)^2 \frac{1}{M_W^2}}_{= \frac{G_F}{\sqrt{2}}} J_\mu J_\mu^\dagger + O\left(\frac{1}{M_W^4}\right)$$

$$\mathcal{L}_{eff}^{(0)} = -\frac{G_F}{\sqrt{2}} J_\mu J_\mu^\dagger$$

3 basic structures at $O(G_F)$

$J_\mu^H (J_\mu^H)^\dagger$ — non-leptonic

$(J_\mu^H J_\mu^{\ell\dagger} + J_\mu^\ell J_\mu^{H\dagger})$ — semi-leptonic

$J_\mu^\ell (J_\mu^\ell)^\dagger$ — leptonic

Notes:

1. The procedure of "integrating-out" the heavy fields at the functional level is very difficult to be implemented beyond the tree-level (QCD corrections)

$\Downarrow$

matching procedure $\Rightarrow$
- Include all dim.-6 ops allowed by symm.
- Determine the coeff. by appropriate matching cond. on Green functions

2. Interestingly, here everything starts at the level of dim.-6 (irrelevant ops)

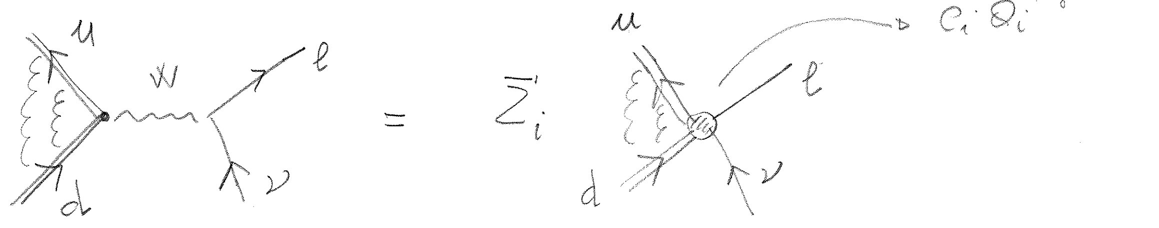
$\left(\frac{E}{M_W}\right)^2 \Rightarrow$ sufficiently small that in general we don't need to include higher-dim. operators

$\hookrightarrow$ but we need to determine these couplings to good accuracy if we want to describe precisely SM effects

why not use the full theory $\Rightarrow$ because it's much more complicated

A) <u>semileptonic - lagrangian</u>

$$\mathcal{L}_{eff}^{(0)} = -\frac{G_F}{\sqrt{2}} V_{ij} \, \bar{u}_i \gamma^\mu (1-\gamma_5) d_j \times \bar{l}_i \gamma^\mu (1-\gamma_5) \nu_i \quad + h.c.$$



⊙ in principle there are several dim-6 operators allowed, but the symmetries of the system restrict a lot their form

possible hadr. currents: $\bar{q}_{L(R)} q_{R(L)}$, $\bar{q}_{L(R)} \sigma^{\mu\nu} q_{R(L)}$, $\bar{q}_{L(R)} \gamma^\mu q_{L(R)}$

QCD preserve chirality (limit $m_q = \phi$) $\Rightarrow$ only $\bar{q}_L \gamma^\mu q_L$ survives

$\downarrow$

Only the leading structure survives.

⊙ QCD corrections are U.V. finite (conserved current) in both theories

$\hookrightarrow$ <u>trivial matching condition</u>:

$$\boxed{\mathcal{L}_{eff} = \mathcal{L}_{eff}^{(0)} \quad \text{to all orders in QCD}}$$

<u>Key observation</u>: The IR of the two theories (full & effective) are the same

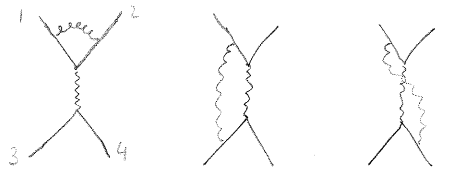
B) Four-quark interactions (e.g. $c \to u d s$)

$$\mathcal{L}_{eff}^{(0)} = -\frac{G_F}{\sqrt{2}} V_{cs}^{*} V_{ud} \underbrace{[\bar{s}\gamma_\mu(1-\gamma_5)c][\bar{u}\gamma_\mu(1-\gamma_5)d]}_{O_2} + h.c.$$

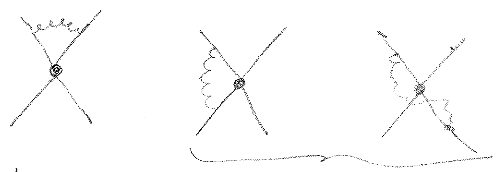
here the situation is more complex :

* extra operator allowed: $Q_1 = \bar{s}^\alpha \gamma_\mu (1-\gamma_5) c^\beta \; \bar{u}^\beta \gamma_\mu (1-\gamma_5) d^\alpha$

* QCD corrections not finite in the eff. theory



$\Longleftrightarrow$



+ self-energy corrections

complicated 1-loop integrals

finite result after quark-field renorm.

+ self.

more divergent than in the full th.

A B

$\Downarrow$

$$i\, A_{full} = \frac{G_F}{\sqrt{2}} V_{cs}^{*} V_{ud} \left\{ \left[1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{-p^2}\right)\right] \langle Q_2 \rangle_T - \frac{3\,\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{-p^2}\right) \langle Q_1 \rangle_T \right\}$$

$m_q^2 \ll -p^2 \ll M_W^2 \Rightarrow -p^2$ act as I.R. cut-off

$$C_1(M_W^2) = 0 + O(\alpha)$$
$$C_2(M_W^2) = 1 + O(\alpha)$$

$\Rightarrow \frac{d}{d\mu^2} C_i(\mu^2) \neq 0$ because of the extra divergences in A & B

$\uparrow$

matching conditions (log vanishes for $\mu^2 \sim M_W^2$)

$$i A_{eff}\Big|_{\overline{MS}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{cd} \Big\{ \left[C_2(\mu)\left(1 + \frac{3\alpha_s}{N_c 4\pi} \ln\left(\frac{\mu^2}{-p^2}\right)\right) + C_1(\mu)\left(-\frac{3\alpha_s}{4\pi} \ln\left(\frac{\mu^2}{-p^2}\right)\right)\right] \langle Q_2 \rangle_T$$

$$M_W^2 \sim \mu^2 \gtrsim -p^2 \gg m_q^2 \qquad \left[C_1(\mu)\left(1 + \frac{3\alpha_s}{N_c 4\pi} \ln\left(\frac{\mu^2}{-p^2}\right)\right) + C_2(\mu)\left(-\frac{3\alpha_s}{4\pi} \ln\left(\frac{\mu^2}{-p^2}\right)\right)\right] \langle Q_1 \rangle_T \Big\}$$

$$\mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{cd} \left[C_1 O_1 + C_2 O_2 \right]$$

$\Rightarrow$ identifying the $\langle Q_2 \rangle$ terms:

$$C_2(\mu^2) = 1 + \frac{3\alpha_s}{N_c 4\pi} \ln\left(\frac{M_W^2}{\mu^2}\right)$$

$\Rightarrow$ identifying the $\langle Q_1 \rangle$ terms:

$$C_1(\mu^2) = 0 - \frac{3\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{\mu^2}\right)$$

<u>In other words:</u>

u.v. sensitivity different

$$A_{full} = \underbrace{1 + \alpha_s \cdot A \ln\left(\frac{M_W^2}{-p^2}\right)}_{\text{full}} = \underbrace{\left[1 + \alpha_s A \ln\left(\frac{M_W^2}{\mu^2}\right)\right]}_{C_i(\mu)} * \underbrace{\left[1 + \alpha_s A \ln\left(\frac{\mu^2}{-p^2}\right)\right]}_{\text{low-energy eff. theory}}$$

IR sensitivity

exactly the same in the two theories

<u>N.B:</u> potential problems with pert. theory if $\mu \ll M_W \Longrightarrow$ need RG impr.

RG-improved perturbation theory

$$\mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{cd} \sum_{i=1}^{2} C_i(\mu)\, Q_i$$

$$iA(i \to f) = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{cd} \sum_i \underbrace{C_i(\mu) \langle Q_i(\mu) \rangle}$$

$\downarrow$

μ-independent

$\Rightarrow$ resummation of large logs by means of RGE

$$\mu \frac{d}{d\mu} \langle Q_i(\mu) \rangle = -\gamma_{ij} \langle Q_j(\mu) \rangle$$

form a complete set

N.B: anomalous dimension of operators

$\downarrow$

|| some irrelevant become more relevant ||

$$\gamma_{ij} = -\frac{\alpha_s}{2\pi} \begin{pmatrix} +3/N_c & -3 \\ -3 & +3/N_c \end{pmatrix}$$

$$\mu \frac{d}{d\mu} \left[\sum_i C_i \langle Q_i(\mu) \rangle \right] = \emptyset = \left[\mu \frac{d}{d\mu} C_j(\mu) - C_i(\mu) \gamma_{ij} \right] \langle Q_j \rangle$$

$$\Rightarrow \mu \frac{d}{d\mu} C_j(\mu) = +(\gamma^T)_{ji} C_i(\mu) \Rightarrow \mu \frac{d}{d\mu} \vec{C} = \gamma(\alpha_s)^T * \vec{C}$$

- Since γ_{ij} depends on μ only via α_s $\Rightarrow$ convenient to change variables

- We need to diagonalise the basis $C_i \to \hat{C}_i$ $\gamma_{ij} \to \hat{\gamma}_{ii}$

$$\mu \frac{d}{d\mu} \ln \hat{C}_i = +\hat{\gamma}_{ii}(\alpha_s) = \beta(\alpha_s) \frac{d}{d\alpha} \ln \hat{C}_i$$

$$\Rightarrow \hat{C}_i(\mu) = \hat{C}_i(M_W)\, e^{\int_{\alpha(M_W)}^{\alpha(\mu)} + \frac{\hat{\gamma}_{ii}(\alpha)}{\beta(\alpha)} d\alpha}$$

$$\hat{C}_\pm = C_1 \pm C_2$$

$$\to \quad \hat{\gamma} = + \frac{3\alpha_s}{2\pi} \begin{pmatrix} 1-\frac{1}{N} & 0 \\ 0 & -1-\frac{1}{N} \end{pmatrix} = + \frac{\alpha_s}{4\pi} \begin{pmatrix} \hat{\gamma}_+ & 0 \\ 0 & \hat{\gamma}_- \end{pmatrix}$$

$$\hat{\gamma}_+ = 6\left(1-\frac{1}{N}\right)$$

$$\hat{\gamma}_- = -6\left(1+\frac{1}{N}\right)$$

$$\beta = \mu \frac{d}{d\mu} \alpha = -\beta_0 \frac{\alpha_s^2}{4\pi} \qquad \text{because of A.F}$$

$$e^{\int_{\alpha(M_W)}^{\alpha(\mu)} \cancel{\frac{\alpha_s}{4\pi}} (\hat{\gamma}_\pm) \frac{1}{(-\beta_0) \cancel{\frac{\alpha_s^2}{4\pi}}} d\alpha_s} = \left[\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right]^{-\frac{\hat{\gamma}_\pm}{\beta_0}} \approx 1 - \hat{\gamma}_\pm \frac{\alpha_s}{4\pi} \overbrace{\ln\left(\frac{M_W^2}{\mu^2}\right)}^{\text{positive}}$$

$$\uparrow$$

$$\frac{\cancel{\alpha_s} - \beta_0 \frac{\alpha_s^{\cancel{2}}}{4\pi} \ln\left(\frac{\mu^2}{M_W^2}\right)}{\cancel{\alpha_s}} = 1 + \beta_0 \frac{\alpha}{4\pi} \ln\left(\frac{M_W^2}{\mu^2}\right)$$

$$\boxed{\hat{C}_\pm(\mu^2) = \hat{C}_\pm(M_W^2) \left(\frac{\alpha_s(\mu^2)}{\alpha_s(M_W^2)}\right)^{-\frac{\hat{\gamma}_\pm}{\beta_0}}}$$

Plugging-in the correct numbers leads to

$$C_\pm(\mu^2) = \underset{\substack{\| \\ 1}}{C_\pm(M_W^2)} \left(\frac{\alpha_s(\mu^2)}{\alpha_s(M_W^2)}\right)^{-\frac{\hat{\gamma}_\pm}{\beta_0}}$$

$$= \begin{cases} C_+ \to \left(\frac{\alpha_s(\mu^2)}{\alpha_s(M_W^2)}\right)^{-\frac{2}{3\beta_0}} < 1 \\[2ex] C_- \to \left(\frac{\alpha_s(\mu^2)}{\alpha_s(M_W^2)}\right)^{+\frac{4}{3\beta_0}} > 1 \end{cases}$$

$$\uparrow$$

Dynamical origin of "$\Delta I = \frac{1}{2}$ Rule"