



**Universität  
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# MASTER THESIS

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## ON THE COMBINATIONS OF TRIANGLES, MEMORIES, AND GYROSCOPES

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# 1 Introduction I

The idea of gravitational waves was first proposed by Heaviside [34] long before the theory of general relativity was completed. It stemmed from the similarities between the electromagnetic and gravitational inverse square laws. However, due to the absence of negative gravitational charges, the analogy is not complete. Einstein attempted to develop the correct concept of gravitational waves and published his results in 1918 [25], where he pointed out that gravitational waves should propagate at the speed of light and that there must be three types of them. By 1922, Arthur Eddington showed that two of these types were artefacts of the coordinate system and, by an appropriate choice of coordinates, could be made to propagate at any speed, leading him to sceptically remark that they travel “at the speed of thought” [39].

Although Eddington showed that the third type propagates at the speed of light regardless of the coordinate system, its physicality was so doubtful that in 1936, Einstein wrote a letter to Max Born, stating, “Together with a young collaborator [N. Rosen], I arrive at the interesting result that *gravitational waves do not exist*” [20]. This uncertainty in the contemporary physics community was mainly due to the impossibility of solving the full general relativity equations. The only way to develop a working gravitational wave formalism was to linearise the theory. Up until the 1960s, gravitational waves were treated as perturbations.

Discussions of gravitational wave physics continued. Advancements in differential geometry and topology, coupled with a lack of technical tools for solving field equations, pushed the development of a new framework. Previously, coordinate systems used in calculations were chosen for mathematical simplicity rather than physical convenience. In 1960, Bondi proposed a new metric specifically suitable for the description of outgoing radiation [9]. The construction of the Bondi coordinates and further explanations of the framework were elaborated in 1962 [10]. Later, Rainer K. Sachs generalised the formalism for non-axisymmetric spacetimes [56].

## 2 Bondi-Sachs Formalism

### 2.1 Bondi Coordinates

Bondi's insight was to use a family of outgoing null rays to construct coordinates suitable for describing outgoing gravitational waves. We will depict these coordinates in a spacetime diagram 1. The waves propagate along null rays with  $c = 1$ . The source is depicted as a star at the origin of the coordinate system.

A natural choice for the temporal coordinate is the null retarded time  $u = t - r$ , as it remains constant along the outgoing wave trajectory. The wave travels into the future null infinity  $\mathcal{I}^+$  (pronounced as 'scri plus' for 'script I plus'), which corresponds to  $r \rightarrow \infty, t \rightarrow +\infty$ . Note that the spacetime is asymptotically flat at  $\mathcal{I}^+$ , allowing us to avoid contribution from the near-field regime.

Similarly, if we are to study incoming gravitational waves from past null infinity  $\mathcal{I}^-$  (i.e.  $r \rightarrow \infty, t \rightarrow -\infty$ ), we would choose the advanced time  $v = t + r$ .

Finally, the two remaining coordinates,  $\theta$  and  $\phi$ , are the usual spherical coordinates, describing the direction of propagation of the waves on a 2-sphere at a given  $r$ . In the context of Bondi coordinates, they are typically denoted as  $x^A$  with  $A, B, \dots = 2, 3$ . This choice provides a natural way to decompose the angular dependence of the gravitational waves into spherical harmonics:

$$h(t, r, \theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^l h^{lm}(t, r) {}_{-2}Y_{lm}(\theta, \phi). \quad (1)$$

### 2.2 Bondi-Sachs Metric

Following the approach of [56], we now construct the metric, which focuses on the outgoing null hypersurfaces  $u = \text{const}$ . The normal covector to this hypersurface is given by the gradient of  $u$ :  $k_a = \partial_a u = (1, 0, 0_A)$ . This covector is collinear with the wave trajectory, implying that it is also null:

$$0 = \partial^a u \partial_a u = g^{ab} \partial_a u \partial_b u = g^{00} \partial_0 u \partial_0 u = g^{00} = 0.$$

The corresponding vector  $\partial^a u = g^{ab} \partial_b u = g^{a0} \partial_0 u$  has a vanishing  $u$ -component, indicating that it is tangent to null hypersurfaces ( $u = 0 = \text{const}$ ). On the other hand, we can choose  $r$  as an arbitrary parameter along the null geodesics. The velocity  $\frac{\partial x^a}{\partial r}$  must be collinear with the tangent vector. Thus, up to a rescale factor  $\mathcal{R}$ :

$$k^a = \mathcal{R} \frac{\partial x^a}{\partial r},$$

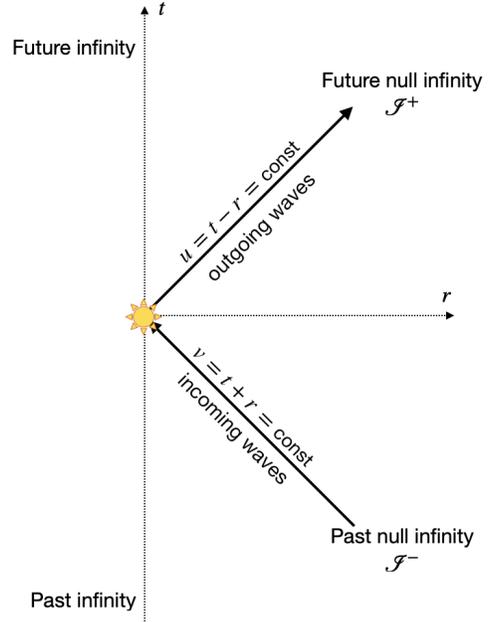


Figure 1: A schematic picture of Bondi coordinates.

yielding  $k^a = \mathcal{R} \frac{\partial x^a}{\partial r} = (0, \mathcal{R}, 0^A)$ . For radially travelling waves with constant velocity, a regular radial coordinate  $r^2 = x^\mu x_\mu$  is a suitable parameter choice. The rescale factor  $\mathcal{R}$  is traditionally<sup>1</sup> written as  $-e^{-2\beta}$ , where  $\beta$  is an arbitrary coordinate function. In this way,  $r$  remains non-singular as long as  $\beta$  is finite. With that, we recalculate:

$$k^a = g^{ab} k_b = g^{a0} k_0 + g^{a1} k_1 + g^{aA} k_A = g^{a0} = (g^{00}, g^{10}, g^{A0}) = (0, -e^{-2\beta}, 0^A).$$

Alternatively,  $g^{A0} = 0$  can also be deduced given that angular coordinates are constant along the null rays, i.e., their directional derivative  $k^a \partial_a x^A = 0$ . Since covariant and contravariant components are related by  $g^{ab} g_{bc} = \delta_c^a$ , we obtain:

$$0 = \delta_1^0 = g^{0b} g_{b1} = g^{00} g_{01} + g^{01} g_{11} + g^{0A} g_{A1} = g^{01} g_{11},$$

$$0 = \delta_A^0 = g^{0b} g_{bA} = g^{00} g_{0A} + g^{01} g_{1A} + g^{0B} g_{BA} = g^{01} g_{1A},$$

leading to  $g_{11} = g_{1A} = 0$ . Furthermore:

$$1 = \delta_1^1 = g^{1b} g_{b1} = g^{10} g_{01} + g^{11} g_{11} + g^{1B} g_{B1} = g^{10} g_{10} = (-e^{2\beta}) \cdot (-e^{-2\beta}).$$

Other components may be derived similarly. In particular, the 1-components should all contain the  $e^{-2\beta}$  factor to ensure conformal freedom in choosing the radial coordinate.

The inverse metric generally takes the form:

$$g^{-1} = \begin{bmatrix} 0 & -e^{2\beta} & 0^A \\ -e^{2\beta} & U e^{-2\beta} & -U^A e^{2\beta} \\ 0^B & -U^B e^{2\beta} & g^{AB} \end{bmatrix}.$$

It is the only appearance of the metric that satisfies all the constraints. For further details on non-degeneracy conditions, refer to [56].

We can additionally specify the metric components by considering its asymptotic flatness at future null infinity, as discussed in [10]. Taking  $r \rightarrow \infty$  at finite  $u$ , we have:

$$U = 1 + \frac{M(u, x^A)}{r} + \mathcal{O}(r^{-1}),$$

where  $M$  is the Bondi mass aspect. It is a measure of the system's mass-energy content and is a generalisation of mass that incorporates gravitational radiation. It is traditional to denote the  $g^{11}$  component of the inverse metric as  $\frac{V}{r} e^{-2\beta}$ , where  $V$  is the quantity related to  $M$ .

Therefore, the line element is:

$$ds^2 = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} dudr + g_{AB} (dx^A - U^A du)(dx^B - U^B du). \quad (2)$$

Here, as the mass aspect lies in the 00-component, one may see its relation to energy. This will be discussed further in section 2.4. Other components at future null infinity satisfy  $\lim_{r \rightarrow \infty} \beta = \lim_{r \rightarrow \infty} U^A = 0$  [10].

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<sup>1</sup>Usually, the normal vector is defined pointing outwards, i.e.,  $k^a = -\partial^a u$ , and the rescale factor is then  $e^{-2\beta}$ .

Finally, we shall discuss the angular part of the metric  $g_{AB}$ . The only coordinate that varies along the null rays is  $r$ , so it is chosen to be an areal coordinate. At future null infinity,  $r^{-2}g_{AB}$  approaches a regular 2-sphere metric, namely  $q_{AB} = \text{diag}(1, \sin^2 \theta)$ . Hence, in this limit, we write  $g_{AB} = r^2 q_{AB}$ .

However, in the presence of gravitational radiation, the metric must also account for distortions and rotations it causes. Hence, in the limit of  $r \rightarrow \infty$ , we state

$$g_{AB} = q_{AB} + \frac{C_{AB}(u, x^A)}{r} + \mathcal{O}(r^{-2}).$$

$C_{AB}$  is the leading-order perturbation due to gravitational waves. It must, therefore, be symmetric and trace-free. We will discuss it in more detail in section 2.4.

Outside the future null infinity, the angular metric is denoted as  $g_{AB} = r^2 h_{AB}$ . For the sake of volume preservation,  $\det h_{AB} = \det q_{AB} = \sin^2 \theta$ . Considering these two conditions, the metric  $h_{AB}$  has two surviving degrees of freedom. We will denote them as  $\gamma$  and  $\delta$ .

In the original paper by H. Bondi [10], only the axisymmetric spacetime case was described. Thus, because of the rotational symmetry in the  $\phi$  direction, only one degree of freedom was left. Later, R. Sachs [56] generalised this result to an arbitrary spacetime. Finally, M. van der Burg in [16] rewrote the metric to provide a more natural description of gravitational waves. It is related to Sachs' version with  $\gamma \rightarrow \frac{\gamma+\delta}{2}$  and  $\delta \rightarrow \frac{\gamma-\delta}{2}$ , whereas Bondi's initial result is obtained by setting  $\delta = 0$  (as well as  $U^3 = 0$ ):

$$h_{AB} dx^A dx^B = (e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2) \cosh 2\delta + 2 \sin \theta \sinh 2\delta d\theta d\phi.$$

Note also that Bondi's version is invariant under  $\phi \rightarrow -\phi$ , so his metric is not suitable for describing a rotating black hole despite the system being axisymmetric.

### 2.3 The Electromagnetic Field

As in the case of gravity, electromagnetic waves also possess two degrees of freedom and propagate along null hypersurfaces. They thus provide a straightforward demonstration of the Bondi formalism application. We describe the electromagnetic field in Minkowski spacetime following the approach introduced in [63] and refined by [45].

Let us introduce the 3-dimensional timelike worldtube  $\Sigma$  of radius  $R$ , encompassing all charge and current distributions of the system (see Fig. 2). The null cones  $N_u$  intersect  $\Sigma$  at a retarded time  $u$  in spacelike spheres of radius  $R$  coordinated by  $x^A$ .

Consider the outgoing null spherical coordinates  $(u, r, x^A)$ . The flat version of the Bondi-Sachs metric may be achieved by taking the  $r \rightarrow \infty$  limit in the absence of gravity, therefore excluding the first-order corrections. Hence, we immediately set:

$$\beta = U^A = 0, \quad U = 1, \quad g_{AB} = r^2 q_{AB}, \tag{3}$$

and the metric (2) takes the form:

$$ds^2 = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B.$$

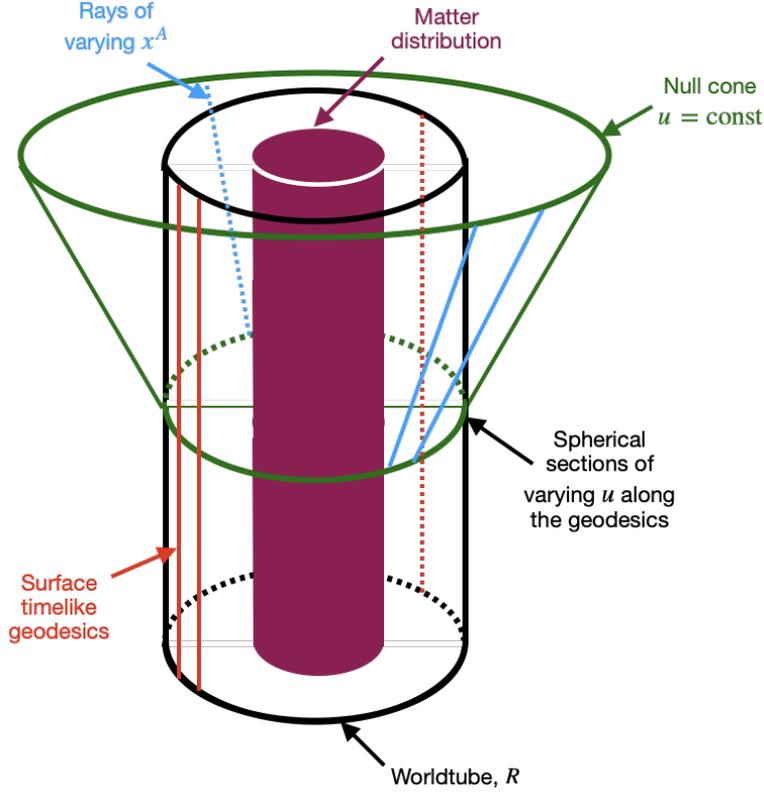


Figure 2: Illustration of the worldtube.

To develop a working formalism, we introduce the vacuum field equations  $\nabla_a F^{ab} = 0$ . The field strength tensor  $F_{ab}$  is defined through the vector potential as  $F_{ab} = 2A_{[a;b]}$ . The potential has a gauge freedom:

$$A_a \rightarrow A_a + \nabla_a \chi,$$

where  $\chi$  is an arbitrary function. In the next section, we will show that the metric components are the potential of the gravitational field and are thus analogues of  $A_a$ . We shall adopt the gauge conditions:

$$A_0 = 0, \text{ for } r = R,$$

$$A_1 = 0, \text{ for } r \geq R,$$

analogous to the condition  $g^{00} = g^{0A} = 0$ , obtained in Sec. 2.2. There remains a gauge freedom

$$A_A \rightarrow A_A + \nabla_A \chi(x^B).$$

We will now hold onto the following strategy. The 0- and  $A$ -components of Maxwell's equations:

$$\nabla_a F^{a0} = \nabla_a F^{aA} = 0,$$

will be considered 'main' equations. The corresponding 1-component will be deemed supplemental. From Maxwell's equations, we derive another identity:

$$0 = \nabla_b \nabla_a F^{ab} = \partial_0 \nabla_a F^{a0} + \frac{1}{r^2} \partial_1 (r^2 \nabla_a F^{a1}) + \frac{1}{\sin \theta} \partial_A (\sin \theta \nabla_a F^{aA}). \quad (4)$$

If the main equations are satisfied, it reduces to

$$\partial_1(r^2\nabla_a F^{a1}) = 0, \quad (5)$$

so that the supplementary condition is satisfied everywhere if it is satisfied at some specified value of  $r$ , for instance,  $r = R$  or  $r \rightarrow \infty$ .

We start working with the main equations. The 0-component will be dubbed the hypersurface equation:

$$0 = \nabla_a F^{a0} \Rightarrow \partial_1(r^2\partial_1 A_0) = \partial_1(\eth_A A^A),$$

where  $\eth_A$  henceforth denotes the covariant derivative with respect to  $q_{AB}$  with  $\eth^A = q^{AB}\eth_B$ . It is an ordinary differential equation along the null rays, which determines  $A_0$ . A formal integration of the hypersurface equation yields:

$$\partial_1 A_0 = \frac{Q(u, x^A) + \eth_B A^B}{r^2} + \mathcal{O}(r^{-3}).$$

Here,  $Q(u, x^A)$  enters as a function of integration. We can now express the radial component of the electric field. In our gauge,  $A_1 = 0$ , therefore,  $E_1 = F_{10} = \partial_1 A_0$ . Using Gauss's theorem, we eliminate  $\eth_B A^B$  and obtain the total charge encompassed by a large sphere at a certain moment of a retarded time:

$$q(u) := \lim_{r \rightarrow \infty} \frac{1}{4\pi} \oint_{S^2} E_1 r^2 \sin \theta \, d\theta \, d\phi = \frac{1}{4\pi} \oint_{S^2} Q(u, x^A) \sin \theta \, d\theta \, d\phi. \quad (6)$$

Thus, we call  $Q(u, x^A)$  the *charge aspect*.

The second main equation (the  $A$ -component of Maxwell's equation), which we will hereafter call the evolution equation, in turn, gives us an ordinary differential equation that allows for determining  $\partial_0 A_B$ :

$$0 = \nabla_a F^{aA} \Rightarrow \partial_1 \partial_0 A_B = \frac{1}{2} \partial_1 (\partial_1 A_B + \partial_B A_0) - r^2 \eth^C \eth_{[B} A_{C]}.$$

The supplementary condition in an explicit form:

$$0 = \nabla_a F^{a1} \Rightarrow \partial_0(r^2\partial_1 A_0) = \eth^B(\partial_1 A_B - \partial_0 A_B + \eth_B A_0).$$

Its integral over a large sphere gives the conservation law:

$$\frac{dq(u)}{du} = 0.$$

Hence, given  $A_B$  on the initial null cone  $N_{u_0}$ ,  $\partial_1 A_0$  on its spherical cross-section ('slice')  $S_{u_0}$ , and  $\partial_0 A_B$  on the pre-defined  $\Sigma$ , we develop a hierarchical integration scheme, summarised as an algorithm:

1. Define a gauge such that  $A_0 = 0$  at  $\Sigma$  (i.e. at  $r = R$ ). This is analogous to Bondi's 'news' function, which we will state in Section 2.4.

2. With the initial data, integrate the main equations:

- (a) Given the initial condition  $\partial_0 A_B$  on the spherical cross-section  $S_{u_0}$ , integrate the evolution equation radially to determine  $\partial_0 A_B$  on the entire cone  $N_{u_0}$ . Specifically, we are interested in the evolution of the derivative along the cone to  $u_0 + \Delta u$ . In the finite difference approximation,  $A_B$  at  $u_0 + \Delta u$  can be calculated using:

$$\partial_0 A_B = \lim_{\Delta u \rightarrow 0} \frac{A_B(u_0 + \Delta u) - A_B(u_0)}{\Delta u} \Rightarrow A_B(u_0 + \Delta u) = A_B(u_0) + \partial_0 A_B \cdot \Delta u.$$

- (b) With the other two initial values,  $A_B$  on the initial null cone  $N_{u_0}$  and  $\partial_1 A_0$  on its cross-section  $S_{u_0}$ , integrate the hypersurface equation along the null rays of the cone to determine  $A_0$  on that cone. Afterwards, use the supplementary condition to specify  $\partial_0 \partial_1 A_0$  on the cross-section. Finally, in a finite difference approximation,  $\partial_1 A_0$  at  $u_0 + \Delta u$  can be obtained.

3. Iterate the procedure to determine  $A_B$  and  $A_0$  in the finite difference approximation on the null cone  $u_0 + n\Delta u$ .

There is, however, a peculiarity arising from this treatment. Because of the timelike character of  $\Sigma$ , defining the ‘news’  $A_0$  on a small region of the worldtube right away determines  $A_0$  on a large part of  $\Sigma$ . Thus, we cannot freely assign the ‘news’ on the worldtube. Consequently, the initial-value formulation is incorrectly set for a non-analytical case [37]. Although no differential constraints emerge, some functional constraints must be imposed [63]. These are automatically satisfied in the analytical case.

Therefore, this approach is an overreaction for the treatment of linear systems, especially considering its analytic weaknesses. However, it is invaluable for non-linear equations, such as those of General Relativity, as they can be analysed in the same way as linear systems.

## 2.4 General Relativity Equations

We shall now demonstrate the advantages of Bondi’s formalism for describing gravity by following the approach introduced in Sec. 2.3. First, we will state the main and conservation conditions, obtain supplementary equations, and then work separately with the hypersurface and evolution equations. After establishing the hierarchical integration scheme, we will delve deeper into the crucial results of the formalism.

### 2.4.1 Main Equations and Conservation Condition

In geometric units ( $G = c = 1$ ), Einstein’s field equations read:

$$E_{ab} := G_{ab} - 8\pi T_{ab} = R_{ab} - \frac{1}{2}g_{ab}R - 8\pi T_{ab} = 0,$$

where we introduced the symmetric quantity  $E_{ab}$  for simplicity of all the following computations. Other quantities are  $R_{ab}$ , which stands for the Ricci tensor,  $R = R^a_a$ , the Ricci scalar,  $G_{ab}$  is the Einstein tensor, and  $T_{ab}$  is the symmetric matter energy-momentum tensor.

In analogy to the electromagnetic case, we establish the ‘main’ equations, namely the hypersurface ones:

$$E_a^u = 0, \tag{7}$$

and the evolution equations:

$$E_{AB} - \frac{1}{2}g_{AB}g^{CD}E_{CD} = 0. \quad (8)$$

We further add a conservation condition similar to (4):

$$0 = E_{b;a}^b = \frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}E_a^b) + \frac{1}{2}E_{bc}\partial_a g^{bc}, \quad (9)$$

where we assumed that the matter stress-energy tensor is divergence-free ( $T_{a;b}^b = 0$ ). This condition is a consequence of the contracted Bianchi identities.

### 2.4.2 Supplemental Conditions

If the main equations are satisfied for the metric (2), we get

$$E_1^b = -e^{-2\beta}E^{0b} = -e^{-2\beta}g^{ba}E_a^0 = 0,$$

and the  $a = 1$  component of the conservation condition (9) yields  $E_{AB}\partial_1 g^{AB} = -2r^{-1}g^{AB}E_{AB} = 0$ , so that, trivially,  $g^{AB}E_{AB} = 0$ . With this, the 0- and  $A$ -components of the conservation condition (9) reduce to:

$$\partial_1(r^2 e^{2\beta} E_0^1) = 0, \quad (10a)$$

$$\partial_1(r^2 e^{2\beta} E_A^1) = 0. \quad (10b)$$

These relations are satisfied everywhere if they hold at least for some value of  $r$  (e.g.,  $r \rightarrow \infty$  or  $r = R$ ). Called supplementary conditions by Bondi and Sachs, they are analogous to the supplementary condition (5) in the electromagnetic case. If evaluated at  $r \rightarrow \infty$ , the last two equations are related to the asymptotic flux conservation laws for total energy and angular momentum. In particular, the first equation leads to the famous Bondi mass loss formula (33).

### 2.4.3 Hypersurface Equations

Let us now examine the main equations in more detail. The hypersurface equations (7) help us evaluate the metric components. For instance, a first-order radial differential equation determines  $\beta$  along the null rays:

$$E_1^0 = 0 \Rightarrow \partial_1 \beta = \frac{r}{16}h^{AB}h^{CD}(\partial_1 h_{AC})(\partial_1 h_{BD}) + 2\pi r T_{11}. \quad (11)$$

Furthermore, it contains two second-order radial differential equations, which allow for determining  $U^A$ :

$$E_A^0 = 0 \Rightarrow \partial_1 [r^4 e^{-2\beta} h_{AB} \partial_1 U^B] = 2r^4 \partial_1 \left[ \frac{1}{r^2} D_A \beta \right] - r^2 h^{CD} D_C \partial_1 h_{AD} + 16\pi r^2 T_{1A}, \quad (12)$$

and an equation to determine  $V$ :

$$E_0^0 = 0 \Rightarrow 2e^{-2\beta} \partial_1 V = \mathfrak{R} - 2h^{AB}(D_A D_B \beta - D_A \beta \cdot D_B \beta) + \frac{e^{-2\beta}}{r^2} D_A \partial_1 (r^4 U^A) - \frac{1}{2} r^2 e^{-4\beta} h_{AB} \partial_1 U^A \cdot \partial_1 U^B + 8\pi [h^{AB} T_{AB} - r^2 T_a^a]. \quad (13)$$

Here,  $\mathfrak{R}$  is the Ricci scalar with respect to the angular metric  $h_{AB}$ , and  $D_A$  denotes the covariant derivative of  $h_{AB}$  with  $D^A = h^{AB}D_B$ . We notice that metric components here are analogues of  $A^a$ .

#### 2.4.4 Evolution Equations

We may significantly simplify the evolution equation by introducing a complex polarisation dyad  $m^a$  tangent to null hypersurfaces so that  $m^a \nabla_a u = 0$ . Furthermore, we want it to point in the angular direction, i.e.,  $m^a = (0, 0, m^A)$ . We now impose the normalisation so that:

$$h^{AB} = \frac{1}{\chi\bar{\chi}}(m^A\bar{m}^B + \bar{m}^A m^B), \quad (14)$$

with  $\chi \in \mathbb{C}$ , and thus  $m_A\bar{m}^A = \chi\bar{\chi}$ ,  $m^A = h^{AB}m_B$ ,  $m^A m_A = 0$ .

This way,  $m^A$  is determined up to the phase freedom (a factor of  $e^{in}$ ), up to a convention. For instance, numerical applications mostly use  $\chi\bar{\chi} = 2$  to avoid the  $\sqrt{2}$  factors [71], but Newman and Penrose in [49] prefer  $\chi\bar{\chi} = 1$ . Note also that the null vector of their formalism is related to the complex dyad as  $m^a_{(NP)} = r^{-1}m^a$  because they define the null vector with respect to  $g_{AB}$  rather than  $h_{AB} = r^{-2}g_{AB}$ .

Having defined that, the  $E_{AB}$  tensor is expanded as

$$E_{AB} = \frac{1}{(\chi\bar{\chi})^2}E_{CD}(m^C m^D \bar{m}_A \bar{m}_B + \bar{m}^C \bar{m}^D m_A m_B) + \frac{1}{2}h_{AB}h^{CD}E_{CD},$$

and we have shown in Sec. 2.4.2 that  $h^{CD}E_{CD}$  vanishes identically. Thus, the evolution equations take the form  $m^A m^B E_{AB} = 0$ , which in full form writes [71]:

$$\begin{aligned} m^A m^B \left[ r \partial_1 (r \partial_0 h_{AB}) - \frac{1}{2} \partial_1 (r V \partial_1 h_{AB}) - 2e^\beta D_A D_B e^\beta + h_{CA} D_B \partial_1 (r^2 U^C) - \right. \\ \left. - \frac{1}{2} r^4 e^{-2\beta} h_{AC} h_{BD} \partial_1 U^C \cdot \partial_1 U^D + \right. \\ \left. + r^2 \left( \frac{1}{2} D_C U^C + U^C D_C \right) \partial_1 h_{AB} - r^2 \partial_1 h_{AC} \cdot h_{BE} [D^C U^E - D^E U^C] - 8\pi e^{2\beta} T_{AB} \right] = 0. \quad (15) \end{aligned}$$

#### 2.4.5 Initial Data

Before we construct the sequential integration scheme, we need to define the initial conditions for some retarded time  $u_0$ . There will be four of them [45]:

1. On an initial null cone  $N_{u_0}$ , the asymptotic expression for the metric in the asymptotic inertial frame, usually called the Bondi frame:

$$h_{AB}(u_0, r, x^C) = q_{AB} + \frac{C_{AB}(u_0, x^C)}{r} + \frac{d_{AB}(u_0, x^C)}{r^2} + \dots \quad (16)$$

Thus, it approaches the Minkowski metric at null infinity with (3). Correspondingly,

$$h^{AB} = q^{AB} - \frac{C^{AB}}{r} - \frac{d^{AB} - q^{AC} C^{BD} C_{CD}}{r^2} + \dots \quad (17)$$

and the indices of both  $C_{AB}$  and  $d_{AB}$  are raised and lowered with  $q_{AB}$ . Additionally, the condition  $\det h_{AB} = \det q_{AB}$  requires:

$$q^{AB}C_{AB} = 0, \quad q^{AB}d_{AB} = \frac{1}{2}C^{AB}C_{AB}, \quad q^{AB}\partial_0 C_{AB} = 0, \quad q^{AB}\partial_0 d_{AB} = C^{AB}\partial_0 C_{AB}. \quad (18)$$

2. The  $\frac{1}{r}$  coefficient of  $h_{AB}$ , which describes the temporal dependence of gravitational waves:

$$C_{AB}(u_0, x^C) := \lim_{r \rightarrow \infty} r(h_{AB} - q_{AB}).$$

We can now utilise the dyad introduced in (14). Considering  $q^A = \lim_{r \rightarrow \infty} m^A$  on the unit sphere, so that  $q^{AB} = \frac{1}{\chi\bar{\chi}}(q^A\bar{q}^B + \bar{q}^A q^B)$  and  $q^A = \frac{\chi}{\sqrt{2}}(1, \frac{i}{\sin\theta})$ . Then, we introduce the radiative strain:

$$\sigma_0 = \frac{1}{2\chi\bar{\chi}}q^A q^B C_{AB} = \frac{1}{2} \left( C_{\theta\theta} - \frac{C_{\phi\phi}}{\sin^2\theta} \right) + \frac{iC_{\theta\phi}}{\sin\theta}. \quad (19)$$

The real and imaginary parts of  $\sigma_0$  are the + and  $\times$  polarisation modes measured by a gravitational wave detector at large distances from the source [65]. Note also that it corresponds to the leading order of the spin coefficient  $\sigma$  of the Newman-Penrose formalism (see, e.g., [49]).

Moreover, we can now establish a link between the covariant derivative  $D_A$  of  $h_{AB}$  and the  $\bar{\partial}_A$  of  $q_{AB}$  in the  $\frac{1}{r}$  expansion. Considering the total metric expansion with a constant leading term  $q_{AB}$  and the coordinate-dependent correction  $C_{AB}$ , we introduce the asymptotic affine connection:

$$\Gamma_{BC}^A = \frac{1}{2r}q^{AD}(\bar{\partial}_B C_{CD} + \bar{\partial}_C C_{BD} - \bar{\partial}_D C_{BC}) + \mathcal{O}(r^{-2}),$$

and the covariant derivatives relate as a flat  $\bar{\partial}^A$  plus the curvature correction, namely:

$$D_A V^B = \bar{\partial}_A V^B + \Gamma_{AD}^B V^D.$$

3. A function  $M(u, x^A)$ , which we call the *mass aspect*:

$$M(u_0, x^A) := \frac{1}{2} \lim_{r \rightarrow \infty} (V(u_0, r, x^B) - r).$$

The reasoning for its name may be illustrated as follows. In a static spherically symmetric case with  $h_{AB} = q_{AB}$ ,  $\beta = U^A = 0$  and  $M(u, x^A) = \mathcal{M}$ , the Bondi metric (2) takes the form of the Eddington-Finkelstein metric for a Schwarzschild mass  $\mathcal{M}$ .

4. A co-vector field  $L_A(u_0, x^B)$ , which is the *angular momentum aspect*:

$$L_A(u_0, x^C) := -\frac{1}{6} \lim_{r \rightarrow \infty} (r^4 e^{-2\beta} h_{AB} \cdot \partial_1 U^B - r \bar{\partial}^B C_{AB}). \quad (20)$$

Another useful quantity can be defined as the following retarded time derivative:

$$N_{AB} = \frac{1}{2} \partial_0 C_{AB}(u, x^C), \quad (21)$$

where the factor  $\frac{1}{2}$  is introduced by Bondi for his axisymmetric case.  $N_{AB}$  is referred to as the *news tensor*, and it determines the energy flux of gravitational radiation. It is a geometrically

defined tensor field on  $\mathcal{I}^+$  and is independent of the conformal areal coordinate choice and, as a consequence, of  $u$ -foliation (see [30] or Sec. 2.5 for further discussion).

Sometimes, the *Bondi news* function is used, defined as:

$$N = \frac{1}{\chi\bar{\chi}} q^A q^B N_{AB}, \quad (22)$$

which is a retarded time derivative of the strain  $N = \partial_0 \sigma_0$ .

#### 2.4.6 Hierarchical Integration Scheme

We can now radially integrate the main equations and construct the hierarchical integration scheme, as outlined in Sec. 2.3. We will follow the original approach by Bondi [10] and Sachs [56]. They considered compactified matter sources and expanded all the solutions in terms of  $\frac{1}{r}$  in an asymptotic inertial frame, often called the Bondi frame. For a more general treatment with logarithmic terms in the far field, refer to [70].

Given the asymptotic gauge conditions (3) and the initial data on  $N_{u_0}$  described in Sec. 2.4.5, we introduce the algorithm for the formal integration of the main equations [45]:

1. Integration of the hypersurface equation for  $\beta$  (11) yields:

$$\beta(u_0, r, x^A) = -\frac{1}{32} \frac{C^{AB} C_{AB}}{r^2} + \mathcal{O}(r^{-3}). \quad (23)$$

2. Insert the initial data for the metric (16) along with the solution for beta (23) obtained in the previous step into the hypersurface equation for  $U^A$  (12). It induces:

$$\partial_1(r^4 e^{-2\beta} h_{AB} \partial_1 U^B) = \bar{\partial}^C C_{AC} + \frac{S_A(u_0, x^D)}{r} + \mathcal{O}(r^{-2}). \quad (24)$$

Thus, unless  $S_A(u_0, x^D) = \bar{\partial}^B(2d_{AB} - q^{CE} C_{BE} C_{AC}) = 0$ , the integration of this equation brings in a logarithmic  $r^{-4} \ln r$  term in  $\partial_1 U^B$ , which is ruled out by the assumption of an asymptotic  $1/r$  expansion.

The determinant conditions (18) allow us to write:

$$q^A q^B q^{CD} C_{AC} C_{BD} = \frac{1}{2} q^A q^B (q^C \bar{q}^D + \bar{q}^C q^D) C_{AC} C_{BD} = 0,$$

and hence,

$$q^{CD} C_{AC} C_{BD} = \frac{1}{2} q_{AB} C^{CD} C_{CD}.$$

As a result:

$$S_A = \bar{\partial}^B(2d_{AB} - \frac{1}{2} q_{AB} C^{CD} C_{CD}).$$

Using the determinant conditions (18) once again:

$$S_A = 2\bar{\partial}^B b_{AB} = 0, \quad (25)$$

where we defined a symmetric trace-free tensor  $b_{AB} = d_{AB} - \frac{1}{2} q_{AB} q^{CD} d_{CD}$ . We will now

show that from the condition  $S_A = 0$ , it immediately follows that  $b_{AB} = 0$ . We first utilise  $q^{AB}b_{AB} = 0$  to arrive at:

$$S_A = 2\tilde{\partial}^B b_{AB} = 2q^{BC}\tilde{\partial}_C b_{AB} = 2q^{BC}(\tilde{\partial}_C b_{AB} - \tilde{\partial}_A b_{CB}).$$

So from (25), we deduce

$$\epsilon^{CA}\tilde{\partial}_C b_{AB} = 0,$$

where  $\epsilon_{AB} = \frac{i}{\chi\bar{\chi}}(q_A\bar{q}_B - \bar{q}_A q_B)$  is an antisymmetric surface area tensor on a unit sphere. Let us introduce a Killing vector  $\xi^A$  on that sphere. Then, the component  $\xi^A\epsilon^{BC}\tilde{\partial}_B b_{AC}$  vanishes, and we may expand:

$$0 = \xi^A\epsilon^{BC}\tilde{\partial}_B b_{AC} = \epsilon^{BC}\tilde{\partial}_B(b_{CA}\xi^A) - \epsilon^{BC}b_{CA}q_{BD}\tilde{\partial}^D\xi^A. \quad (26)$$

However, Killing's equation  $\tilde{\partial}^{(A}\xi^{B)} = 0$  and the tracelessness of  $b_{AB}$  dictate for the last term:

$$\begin{aligned} \epsilon^{BC}b_{CA}q_{BD}\tilde{\partial}^D\xi^A &= \epsilon^{BC}b_{CA}q_{BD}(\tilde{\partial}^{(D}\xi^{A)} + \tilde{\partial}^{[D}\xi^{A]}) = \epsilon^{BC}b_{CA}q_{BD}\epsilon^{DA}\epsilon_{EF}\tilde{\partial}^E\xi^F = \\ &= \epsilon^{BC}b_{CA}\delta_B^A\epsilon_{EF}\tilde{\partial}^E\xi^F = \epsilon^{AC}b_{CA}\epsilon_{EF}\tilde{\partial}^E\xi^F = 0, \end{aligned} \quad (27)$$

where we used that for an arbitrary 2-dimensional antisymmetric tensor  $T_{AB}$ , relation  $T_{AB} = \frac{1}{2}\epsilon_{AB}\epsilon^{CD}T_{CD}$  holds.

Therefore, inserting (27) in (26), we obtain

$$\epsilon^{BC}\tilde{\partial}_B(b_{CA}\xi^A) = 0,$$

so that

$$b_{CA}\xi^A = \tilde{\partial}_C b$$

for some scalar  $b$ . Plugging this relation into (25) results in  $S_A\xi^A = 2\tilde{\partial}_A\tilde{\partial}^A b = 0$ . The only solution to this equation is  $b = \text{const}$  and thus  $b_{AB}\xi^B = 0$ . It is sufficient to demonstrate that both independent components of  $b_{AB}$  vanish. It follows more straightforwardly from the Newman and Penrose [49]  $\tilde{\partial}$ -calculus.

This condition means that  $d_{AB}$  is purely a trace term, defined by the determinant conditions (18). Imposing this constraint and integrating (24), we finally get:

$$r^4 e^{-2\beta} h_{AB} \partial_1 U^B = -6L^A(u_0, x^C) + r\tilde{\partial}_B C^{AB} + \mathcal{O}(r^{-1}). \quad (28)$$

3. For the radial integration of  $\partial_1 U^A$ , we use the initial data for the contravariant metric (17) and the angular momentum aspect (20) along with the expression for  $\beta$  (23), while rearranging (28):

$$U^A(u_0, r, x^B) = -\frac{\tilde{\partial}_C C^{AC}}{2r^2} + \frac{1}{r^3}(2L^A + \frac{1}{3}C^{AC}\tilde{\partial}^D C_{CD}) + \mathcal{O}(r^{-4}). \quad (29)$$

This corrects the non-linear coefficients of  $\mathcal{O}(r^{-3})$  of the original works of Bondi and Sachs and matches them with the corresponding coefficient of [2] up to a rescale  $L^A \rightarrow -3L^A$ .

4. We now integrate the last hypersurface equation to determine  $V$ . To achieve this, we use the initial values of  $\beta$  (23),  $U^A$  (29), and the metric (16):

$$V(u_0, r, x^A) = r - 2M(u_0, x^A) + \mathcal{O}(r^{-1}). \quad (30)$$

5. Inserting all the solutions of hypersurface equations ( $\beta$  from (23),  $U^A$  from (29), and  $V$  from (30)) into the evolution equation (15) confirms that to leading order of  $q^A q^B \partial_0 d_{AB} = 0$  is consistent with the determinant conditions (18).
6. Having the asymptotic solutions of the metric components, the leading-order coefficient of the supplemental condition for  $E_0^1$  (10a) brings us to:

$$2\partial_0 M = \eth_A \eth_B N^{AB} - N_{AB} N^{AB}. \quad (31)$$

Assuming  $N_{AB}$  is known after some  $u_0$ , we can determine  $M$  in terms of its initial value  $M(u_0, x^A)$  through integration.

7. The second supplemental condition for  $E_A^1$ , in the same way, determines the evolution of the angular momentum aspect  $L^A$ :

$$\begin{aligned} -3\partial_0 L^A = \eth_A M - \frac{1}{4}\eth^B(\eth_B \eth^C C_{AC} - \eth_A \eth^C C_{BC}) + \\ + \frac{1}{8}\eth_A C_{BC} N^{BC} - \eth_B C^{BC} N_{CA} + \frac{1}{2}C^{BC} \eth_A N_{BC}. \end{aligned} \quad (32)$$

Just as in the case of the mass aspect (31), the time evolution of the angular momentum aspect after  $u_0$  is fully determined by  $N_{AB}$  and the initial values of  $L_A$ ,  $M$ , and  $C_{AB}$  at  $u_0$ .

We can now also motivate the name of  $L^A(u, x^B)$ . In the non-vacuum case, the  $E_A^1$  equation (10b), governing the  $L^A$  evolution, is coupled to the matter angular momentum flux  $r^2 T_A^1$  at null infinity.

The equation (32) corrects the original Bondi and Sachs expressions. For the metric in which  $\gamma(u, r, \theta) = \frac{c(u, \theta)}{r} + \mathcal{O}(r^{-3})$ , it takes the form:

$$-3\partial_1 L_3 = \partial_3 M + \frac{1}{2}c \cdot \partial_3 N - \frac{3}{2}N \cdot \partial_3 c,$$

where we used the definition for the axisymmetric Bondi news function  $N = \partial_0 c$ .

This integration algorithm demonstrates that a unique formal solution of the field equations in the asymptotic  $\frac{1}{r}$  expansion is entirely governed by the determinant conditions (18) and the initial data set in Sec. 2.4.5. Specifically, the supplemental conditions from Sec. 2.4.2 define the temporal derivatives of the mass aspect  $M$  and the angular momentum aspect  $L^A$ , and the hypersurface equations of Sec. 2.4.3 determine the higher-order expansion coefficients.

The formal solution cannot be deemed a well-formulated evolution problem for the metric at  $u_0$  because the needed data (e.g.  $C_{AB}(u > u_0, x^C)$ ) lives in the future of the initial hypersurface. Nevertheless, it brought Bondi to the first clear understanding of mass loss due to gravitational wave emission and gave rise to the interpretation of the supplemental condition as energy, momentum, and angular momentum flux conservation laws [63].

### 2.4.7 Bondi Mass Loss

For an isolated system, the time-dependent *Bondi mass*  $m(u)$  (cf. charge integral (6)) is defined as:

$$m(u) := \frac{1}{4\pi} \oint_{S^2} M(u, x^A) \sin \theta \, d\theta \, d\phi.$$

After the integration of (31) over a sphere, the first term is eliminated due to the divergence theorem. Using the definition of the Bondi news function  $N$  (22), we arrive at the famous Bondi mass loss formula:

$$\frac{d}{du} m(u) = -\frac{1}{4\pi} \oint |N|^2 \sin \theta \, d\theta \, d\phi. \quad (33)$$

The minus sign before the integral indicates that the emission of gravitational radiation (i.e., if there is news) causes the Bondi mass to decrease. If there is no news (i.e.,  $N = 0$ ), the Bondi mass remains constant<sup>2</sup>.

## 2.5 Bondi–Metzner–Sachs Group

### 2.5.1 Introduction

After establishing the general relativity equations in the Bondi formalism and finding convincing confirmation of gravitational wave existence, the energy transmission by gravitational radiation was to be investigated. According to Noether’s theorem, all conservation laws arise from some underlying symmetries. Therefore, the symmetries of general relativity had to be studied.

Naively, one might expect that for asymptotically flat spacetimes, the symmetries must be the same as those of special relativity. That is to say, an observer distant from all sources of gravity should observe the ten symmetries of the Lorentz group, which include four translations in space and time, three boosts, and three rotations. Bondi, van der Burg, Metzner, and Sachs [10] not only refrained from making assumptions about the nature of their asymptotic symmetry group, but they also did not assume the existence of such a group at all. They began by imposing some physically sensible boundary conditions for the gravitational field at  $\mathcal{I}^+$  to characterise the asymptotic flatness.

Note that the null infinities remained obscure until the Bondi coordinates were introduced. In Cartesian coordinates, only two temporal and one spatial infinity are obvious. However, all five infinities are well-revealed in the conformal approach by Penrose [55], which we will describe later in this section.

After the boundary conditions were imposed, Bondi and his collaborators considered the transformations that preserve the form of their boundary conditions. It turned out that those symmetries indeed form an infinite-dimensional extension of the Lorentz group, now referred to as the *BMS group*. This group extends the spacetime translations to the infinite-dimensional Abelian group of supertranslations. Interestingly, the structure of that group did not depend on the particular gravitational field present. Thus, at least at spatial infinity, one may separate the dynamics of spacetime from that of gravity.

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<sup>2</sup>Bondi mass and Bondi mass loss formulae were generalised for arbitrary-dimensional spacetimes in, e.g., [64]

### 2.5.2 Conformal Bondi Coordinates

In this section, we justify the asymptotic conditions and symmetries of the metric at future null infinity in terms of Penrose compactification and show that the assumption of asymptotic series expansion in  $\frac{1}{r}$  is, in fact, a smoothness condition. We follow [55], [63], and [45].

Penrose highlighted that the metric structure of spacetime offers no adequate basis for points at infinity. Nevertheless, employing a coordinate transformation, we may assign some finite coordinates to those points, introducing an ‘unphysical spacetime’, described by a  $(3 + \varepsilon)$ -smooth metric  $\hat{g}_{ab}$ . It is an extension of the physical spacetime with metric  $g_{ab}$ , containing the endpoints of null geodesics. The two metrics are conformally related with  $\hat{g}_{ab} = \Omega^2 g_{ab}$ , and  $\Omega = 0$  at  $\mathcal{I}^+$ .

Evidently, the unphysical metric must be singular at the null infinity points so that the points in their neighbourhood are infinitely distant. In this section, we will show that one may eliminate those singularities by introducing a conformal metric, regular at null infinity. For a detailed treatment of the endpoints of null geodesics, refer to [63].

Furthermore, for the spacetime to be asymptotically flat, the topology of  $\mathcal{I}^+$  must be that of a time and spatial spherical cross-section, i.e.  $\mathbb{R} \times \mathbb{S}^2$ , and the covariant derivative of  $\Omega$  with respect to  $\hat{g}_{ab}$ ,  $\hat{\nabla}_a \Omega$ , must be non-vanishing at the null infinity, ensuring it is a non-degenerate boundary.

The unphysical and physical Ricci tensors are then related by:

$$\Omega^2 R_{ab} = \Omega^2 \hat{R}_{ab} + 2\Omega \hat{\nabla}_a \hat{\nabla}_b \Omega + \hat{g}_{ab} [\Omega \hat{\nabla}^c \hat{\nabla}_c \Omega - 3 \hat{\nabla}^c \Omega \cdot \hat{\nabla}_c \Omega]. \quad (34)$$

Segregating the trace of this equation, the physical space vacuum ( $R_{ab} = 0$ ) field equations at  $\mathcal{I}^+$  yield:

$$0 = \hat{\nabla}^c \Omega \cdot \hat{\nabla}_c \Omega \Big|_{\mathcal{I}^+}, \quad (35a)$$

$$0 = \hat{\nabla}_a \hat{\nabla}_b \Omega - \frac{1}{4} \hat{g}_{ab} \hat{\nabla}^c \hat{\nabla}_c \Omega \Big|_{\mathcal{I}^+}. \quad (35b)$$

The first condition confirms that  $\mathcal{I}^+$  is null, and the second one guarantees the existence of a further conformal transformation  $\hat{\Omega}^{-2} \hat{g}_{ab} = \tilde{\Omega}^{-2} \tilde{g}_{ab}$ , where  $\tilde{\nabla}_a \tilde{\nabla}_b \tilde{\Omega} \Big|_{\mathcal{I}^+} = 0$ . Thus, there is a preferred conformal factor  $\tilde{\Omega}$ , for which the null infinity is shear-free ( $\tilde{\nabla}_a \tilde{\nabla}_b \tilde{\Omega} \Big|_{\mathcal{I}^+} = 0$ ) and divergence-free ( $\tilde{\nabla}^c \tilde{\nabla}_c \tilde{\Omega} \Big|_{\mathcal{I}^+} = 0$ ).

We introduce the inverse areal coordinate  $\rho = \frac{1}{r}$ , which will also be a convenient choice of a conformal factor  $\Omega$ . We now compactify the space by a transformation of physical Bondi coordinates introduced in Sec. 2.1:  $\hat{x}^a = (u, \rho, x^A) = (u, \frac{1}{r}, x^A)$ . The conformal metric thus reads:

$$d\hat{s}^2 = -\rho^3 V e^{2\beta} du^2 + 2e^{2\beta} dud\rho + h_{AB}(dx^A - U^A du)(dx^B - U^B du). \quad (36)$$

The leading-order coefficients of the metric are governed by (34):

$$\begin{aligned} \beta &= H(u, x^C) + \mathcal{O}(\rho^2), \\ U^A &= H^A(u, x^B) + 2\rho e^{2H} H^{AB} D_B H + \mathcal{O}(\rho^2), \end{aligned}$$

$$\rho^2 V = D_A H^A(u, x^B) + \rho \left[ \frac{1}{2} \mathfrak{R} + D^A D_A e^{2H} \right] + \mathcal{O}(\rho^2),$$

$$h_{AB} = H_{AB}(u, x^C) + \rho C_{AB}(u, x^C) + \mathcal{O}(\rho^2),$$

where  $H$ ,  $H^A$ , and  $H_{AB}$  have some general form and do not correspond to an asymptotic inertial frame. We represented the  $H^{AB}$ -related quantities as  $\mathfrak{R}$  for the Ricci scalar and  $D_A$  for the covariant derivative. The contravariant metric components at future null infinity are found by raising indices of (36) and taking the  $\mathcal{I}^+$  limit:

$$\hat{g}^{ab} \Big|_{\mathcal{I}^+} = \begin{bmatrix} 0 & e^{-2H} & 0 \\ e^{-2H} & 0 & -H^A e^{-2H} \\ 0 & -H^A e^{-2H} & H^{AB} \end{bmatrix}$$

To introduce the inertial coordinates properly, we first consider the null vector  $\hat{n}^a = \hat{g}^{ab} \hat{\nabla}_b \rho$ . By definition, it is tangent to the null geodesics, analogously to what was shown in Sec. 2.1. Its components at the future null infinity are:

$$\hat{n}^a \Big|_{\mathcal{I}^+} = (e^{-2H}, 0, -e^{-2H} H^A).$$

For the introduction of the inertial version of angular coordinates, we demand:

$$\hat{n}^a \partial_a x^A \Big|_{\mathcal{I}^+} = 0,$$

resulting in  $H^A = 0$ .

For the inertial version of retarded time, we require that  $u$  is an affine parameter along the  $\mathcal{I}^+$ -generating null geodesics and:

$$\hat{n}^a \partial_a u \Big|_{\mathcal{I}^+} = 1,$$

giving  $H = 0$ . As a result, we also find that  $\rho$  is a preferred conformal factor for divergence- and shear-free infinity. From  $\hat{\nabla}_a \hat{\nabla}_b \rho \Big|_{\mathcal{I}^+} = 0$  we extract  $\partial_0 H_{AB} = 0$ . We can thus adjust the conformal factor by a static conformal transformation  $\rho \rightarrow \omega(x^A) \rho$  so that  $H_{AB} \rightarrow q_{AB}$ , and the cross-section of  $\mathcal{I}^+$  has a unit sphere geometry. Simultaneously adjusting  $u$ , we keep  $H = 0$ .

Therefore, we can establish an inertial coordinate system at the future null infinity to justify the Bondi-Sachs boundary conditions. After eliminating  $H$ ,  $H^A$ , and  $H_{AB}$ , the asymptotic behaviour of the metric in these coordinates is:

$$\beta = \mathcal{O}(\rho^2), \tag{37a}$$

$$U^A = -\frac{1}{2} \check{\partial}_B C^{AB} \rho^2 + 2L^A \rho^3 + \mathcal{O}(\rho^4), \tag{37b}$$

$$\rho^3 V = \rho^2 - 2M \rho^3 + \mathcal{O}(\rho^4), \tag{37c}$$

$$h_{AB} = q_{AB}(u, x^C) + \rho C_{AB}(u, x^C) + \mathcal{O}(\rho^2). \tag{37d}$$

It also shows that  $C_{AB}$ ,  $M$ , and  $L_A$ , introduced in Sec. 2.4.5, are indeed the leading-order coefficients of the series expansion at null infinity with respect to the preferred conformal factor  $\rho$ .

From (35b), we also deduce that  $\rho^{-1}\hat{\nabla}_a\hat{\nabla}_b\rho$  is finite at the future null infinity. In inertial coordinates, the news tensor is defined as:

$$N_{ab} = f^* \left( \lim_{\rho \rightarrow 0} \rho^{-1} \hat{\nabla}_a \hat{\nabla}_b \rho \right),$$

where  $f^*$  denotes the pull-back to  $\mathcal{I}^+$  [30]. It is thus immediately seen that if we change the conformal factor by some<sup>3</sup>  $\omega$ , the structure of the news tensor ensures that it remains independent of  $\omega$ . Hence, as mentioned in Sec. 2.4.5, the news tensor is geometrically defined on  $\mathcal{I}^+$  and is independent of the  $u$ -slicing choice.

### 2.5.3 The Asymptotic Symmetries

The Bondi–Metzner–Sachs (BMS) group is the asymptotic isometry group of the Bondi metric (2). The infinitesimal generators  $\xi^a$  of the BMS group thus satisfy *asymptotic* Killing's equation:

$$\Omega^2 \mathcal{L}_\xi g_{ab} \Big|_{\mathcal{I}^+} = -2\Omega^2 \nabla^{(a} \xi^{b)} \Big|_{\mathcal{I}^+} = 0,$$

where  $\mathcal{L}_\xi$  is the Lie derivative along  $\xi^a$ . In terms of the metric (37), this yields:

$$\left[ \hat{\nabla}^{(a} \xi^{b)} - \rho^{-1} \hat{g}^{ab} \xi^c \partial_c \rho \right]_{\rho=0} = 0. \quad (38)$$

Consequently, we require that the generator is tangent to  $\mathcal{I}^+$ , i.e.,  $\xi^a \partial_a \rho = 0$  and  $\rho^{-1} \xi^a \partial_a \rho \Big|_{\mathcal{I}^+} = \partial_1 \xi^1 \Big|_{\mathcal{I}^+}$ . Explicitly writing the Lie derivative and considering these conditions, (38) takes the form:

$$\left[ \hat{g}^{ac} \partial_c \xi^b + \hat{g}^{bc} \partial_c \xi^a - \xi^c \partial_c \hat{g}^{ac} - \hat{g}^{ac} \partial_1 \xi^1 \right] \Big|_{\mathcal{I}^+} = 0. \quad (39)$$

The metric reduces to:

$$\hat{g}^{ab} \Big|_{\mathcal{I}^+} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q^{AB} \end{bmatrix}$$

Since it is the only component of (39), we can analyse that equation straightforwardly. Its general solution is:

$$\xi^a \partial_a \Big|_{\rho=0} = [\alpha(x^C) + \frac{u}{2} \partial_B \zeta^B(x^C)] \partial_0 + \zeta^A(x^C) \partial_A, \quad (40)$$

where we denoted the conformal Killing vector of a unit sphere as  $\zeta^A(x^C)$ , with

$$\partial^{(A} \zeta^{B)} - \frac{1}{2} q^{AB} \partial_C \zeta^C = 0.$$

These constitute the generators of the BMS group. The transformations with  $\alpha = 0$  describe the 6-parameter subgroup of conformal transformations of the unit sphere, isomorphic to the orthochronous Lorentz transformations, i.e., those preserving the time direction [56]. The transformations with  $\zeta^A = 0$  form the normal subgroup of *supertranslations*. According to [10], two spacetimes related by a supertranslation must be deemed physically equivalent.

As  $\alpha$ 's only depend on angular coordinates, they may be expanded as an infinite series of spherical harmonics. Of special importance are those consisting of  $l = 0$  and  $l = 1$  harmonics,

<sup>3</sup>We assume  $\omega > 0$  from the definition of the (inverse) areal coordinate.

e.g.,  $\alpha = a + a_x \sin \theta \cos \phi + a_y \sin \theta \sin \phi + a_z \cos \theta$ , as they construct an invariant 4-dimensional translation group, formed by temporal ( $a$ ) and spatial ( $a_x, a_y, a_z$ ) translations. These allow for a unique definition of energy and momentum. However, as the Lorentz group is not an invariant subgroup of BMS, the angular momentum definition is generally ambiguous. Nevertheless, in special cases, such as stationary spacetimes, we can identify a preferred Lorentz group and fix a unique angular momentum definition, see [63].

In general, however, the defining equations of the asymptotic symmetry group are covariant under the differential transformations of  $\mathcal{I}^+$ . Thus, the group structure is an invariant property of the future null infinity. Had we chosen a different conformal coordinate frame, the resulting asymptotic symmetry group would be isomorphic to the one we found.

Let us consider a finite supertranslation  $\tilde{u} = u + \alpha(x^A) + O(\rho)$  with  $\tilde{x}^A = x^A$ . Note that the  $O(\rho)$  term is added to maintain  $u$  as a null coordinate. Under this supertranslation, the radiation strain  $\sigma$  transforms as follows:

$$\begin{aligned} \sigma(u, x^C) &= \frac{1}{\rho^2} \frac{1}{\chi\bar{\chi}} q^A q^B \nabla_A \nabla_B u \Big|_{\mathcal{I}^+} \Rightarrow \\ &\Rightarrow \tilde{\sigma}(u, x^C) = \frac{1}{\rho^2} \frac{1}{\chi\bar{\chi}} q^A q^B \nabla_A \nabla_B \tilde{u} \Big|_{\mathcal{I}^+} = \sigma(u, x^C) + \frac{1}{\chi\bar{\chi}} q^A q^B \bar{\partial}_A \bar{\partial}_B \alpha(x^C). \end{aligned}$$

This transformation reveals the gauge freedom of the strain under supertranslations. This gauge freedom only affects the ‘electric’ component [45], since  $\alpha$  is real.

### 3 Introduction II

The existence of gravitational waves, though rigorously supported by theoretical work, still was not fully accepted within the physics community, mainly due to the experimental challenges in their detection. According to general relativity, gravitational waves are generated by accelerating masses, similar to how accelerated charges produce electromagnetic radiation. These waves propagate as transverse distortions of spacetime, causing masses to oscillate in a plane perpendicular to their direction of travel. Notably, the amplitude of these oscillations increases linearly with the transverse distance between them [40]. By measuring the restoring force, the signal may be reconstructed.

The primary hindrance in detecting gravitational waves is the small value of  $G$ , the gravitational constant. The strains generated by laboratory sources of achievable mass and acceleration are undetectable. Extraterrestrial events, unless very nearby, must be extremely violent to produce measurable radiation. Even then, thermal oscillations of the detector materials often overshadow the potential signal [31]. This led to a sense of despair in the field of gravitational wave physics.

Interest in the field was revived after J. Weber claimed to have detected gravitational waves [67]. He introduced a detector, now known as Weber bars [68], which operates like a massive harmonic oscillator. Weber's approach was to measure the single vibration mode of a large, solid aluminium bar. When a gravitational wave passes through the bar, it induces oscillations that persist long after the wave has passed, allowing for the signal detection. This method avoided the need for assembling and isolating multiple masses, as well as the need for large dimensions of the bar.

Weber continuously used seismometers and other detectors to filter out irrelevant events, as gravitational radiation does not disturb seismometers [40]. Even so, in 1970, he sometimes registered as many as *seven gravitational wave events per day* [40]. Despite numerous reported detections, Weber's results were never independently confirmed [7, 13, 23, 24, 35]. It is now believed that what he detected was most likely noise, processed with poor data analysis techniques.

Nevertheless, to prove him wrong, several detectors in different countries were built, restoring faith in the possibility of their detection and leading to a widespread discussion on the topic and a peak in enthusiasm for gravitational waves. Y. Zeldovich and A. Polnarev [73] then discovered a new effect, later dubbed *gravitational memory* by V. Braginskii and L. Grishchuk [14], thus conceiving a new field.

## 4 Gravitational Memory Effect

### 4.1 Linear Memory

In 1973, Y. Zeldovich and A. Polnarev published a paper focusing on the instrumental sensitivity and frequency characteristics of resonance detectors, like those proposed by Weber [68], required to have a reasonable chance of detecting pulses from binary star formations in superdense clusters within our galaxy [73]. More importantly, they proposed a different type of detector: a non-resonance detector. The authors suggested that the distance between two non-interacting bodies, such as satellites, would change after encountering a gravitational wave. After the wave passes, the velocities of the particles would vanish, leaving a permanent displacement. This effect was obtained in linearised general relativity and, therefore, was later called the *linear* memory effect.

However, in full general relativity, H. Bondi and F. Pirani argued that the residual motion persists and the bodies will inevitably fall together within a finite amount of time [11, 12]. Thus, the memory effect, as introduced by Y. Zeldovich and A. Polnarev, does not exist<sup>4</sup>. Instead, the so-called *velocity memory* arises.

We provide a simple exemplary calculation of the linear memory, following [27]. When we speak of a gravitational source with memory, we mean that at least one of the emitted gravitational wave polarisations possesses a property:

$$\Delta h_{+, \times} = \lim_{t \rightarrow +\infty} h_{+, \times} - \lim_{t \rightarrow -\infty} h_{+, \times} \neq 0. \quad (41)$$

That is to say, if not for memory, a truly free-falling detector will return to the initial state after the wave passes.

The linear memory effect mostly arises in unbound systems after a scattering event or when a burst of particles abandons the system. For our sample calculations, let us consider a hyperbolic binary. According to (1), the gravitational wave polarisations are conveniently decomposed into a sum over  $(l, m)$ -modes:

$$h(t, r, \theta, \phi) = h_+ - ih_\times = \sum_{l=2}^{\infty} \sum_{m=-l}^l h^{lm}(t, r) {}_{-2}Y_{lm}(\theta, \phi), \quad (42)$$

with  $\theta$  and  $\phi$  indicating the source position relative to the observer and  ${}_{-2}Y_{lm}$  being a spin-weighted spherical harmonic. These are defined in terms of Wigner  $d$ -functions:

$${}_{-s}Y^{lm}(\theta, \phi) = (-1)^s \sqrt{\frac{2l+1}{4\pi}} d_{ms}^l(\theta) e^{im\phi},$$

where

$$d_{ms}^l(\theta) = \sqrt{(l-m)!(l+m)!(l-s)!(l+s)!} \sum_{k=\max(0, m-s)}^{\min(l+m, l-s)} \frac{(-1)^k (\sin \frac{\theta}{2})^{2k+s-m} (\cos \frac{\theta}{2})^{2l-2k-s+m}}{k!(l+m-k)!(l-s-k)!(s-m+k)!}.$$

---

<sup>4</sup>For an independent proof, see [74].

The  $h_{lm}$  modes can be further decomposed into mass ( $U_{lm}$ ) and current ( $V_{lm}$ ) multipoles as follows<sup>5</sup> [26]:

$$h^{lm} = \frac{G}{\sqrt{2}rc^{l+2}} \left[ U^{lm}(u) - \frac{i}{c} V^{lm}(u) \right]. \quad (43)$$

Here,  $r$  is the distance to the source, and  $u$  is the retarded time as defined in Sec. 2.1. As before, we assume  $G = c = 1$ .

Thus, at the leading order, the polarisations take the form:

$$h_+ - ih_\times \approx \sum_{m=-2}^2 \frac{\ddot{I}_{2m}}{r\sqrt{2}} {}_{-2}Y^{lm}(\theta, \phi). \quad (44)$$

Introducing the total mass  $M = m_1 + m_2$ , reduced mass  $\mu = \frac{m_1 m_2}{M}$ , relative orbital separation  $R(t)$ , and phase angle  $\varphi(t)$  for a Keplerian binary in the  $x$ - $y$ -plane, the mass quadrupole formula is:

$$I_{2m} = \frac{16\pi}{5\sqrt{3}} \mu R^2(t) Y_{2m}^* \left( \frac{\pi}{2}, \varphi(t) \right).$$

We further specify the eccentricity  $e_0$  and true anomaly  $\nu = \varphi - \omega_p$  (with  $\omega_p = 0$  being the periastron direction on the  $x$ -axis) for the Keplerian orbit. Then, the orbit is described by:

$$r = \frac{p}{1 + e_0 \cos \nu} \quad \text{and} \quad \dot{\nu} = \dot{\varphi} = \frac{\sqrt{pM}}{r^2}.$$

Thus, the quadrupole moment derivatives are:

$$\begin{aligned} \ddot{I}_{20} &= -8\sqrt{\frac{\pi}{15}} \frac{M\mu}{p} e_0 (e_0 + \cos \nu), \\ \ddot{I}_{2\pm 2} &= -4\sqrt{\frac{2\pi}{5}} \frac{M\mu}{p} e^{\mp 2i\varphi(t)} [1 - e_0^2 + (1 + e_0 \cos \nu)(1 + 2e_0 e^{\pm i\nu})], \end{aligned}$$

and give the waveform together with (44). We observe that for bound orbits  $0 \leq e_0 < 1$ , the waveforms are oscillatory, but for unbound ones ( $e_0 \geq 1$  with  $\omega_p = 0$ ), the phase angle approaches:

$$\lim_{t \rightarrow -\infty} \varphi = \varphi_- = \nu_- = -\arccos(-e_0^{-1}),$$

and

$$\lim_{t \rightarrow +\infty} \varphi = \varphi_+ = \nu_+ = +\arccos(-e_0^{-1}).$$

Note that for a parabolic orbit, there is no memory since  $\varphi_- = \varphi_+$ . The difference in the derivatives of mass multipoles (which in turn generates the difference in the waveforms):

$$\begin{aligned} \Delta \ddot{I}_{20} &= 0, \\ \Delta \ddot{I}_{2\pm 2} &= \pm 16\sqrt{\frac{2\pi}{5}} \frac{M\mu}{p} \frac{(e_0^2 - 1)^{3/2}}{e_0^2}. \end{aligned}$$

---

<sup>5</sup>We move from the natural units for the moment to give the reader an idea of the dimensions of radiative moments.

Hence, the leading-order linear memory ‘resides’ in the imaginary part of the  $(l, m) = (2, 2)$  mode. As in the example we provided, memory always arises from a change in at least the quadrupole mass or current moment.

As mentioned, however, memory can also be derived from the linearised field equations for a system of  $N$  particles with masses  $m_i$ :

$$\square \bar{h}_{ij} = -16\pi T_{ij},$$

where  $T_{ij}$  is the energy-momentum tensor of the system, and  $\bar{h}_{ij}$  is the trace-reversed space-space piece of the metric perturbation.  $\square$  is the flat-space D’Alembert operator. Considering constant velocities of the particles  $v_i$ , we may solve the equation using Liénard-Wiechert potentials and using the transverse-traceless (TT) gauge [15]:

$$\Delta h_{ij}^{TT} = \Delta \sum_{k=1}^N \frac{4m_k}{r \sqrt{1-v_k^2}} \left[ \frac{v_k^i v_k^j}{1-v_k^l \hat{n}_l} \right]^{TT}, \quad (45)$$

where  $\hat{n}_l$  points from the source to the observer and  $\Delta$  denotes the difference as before. Physically, the particles we mean are usually [27]:

- (i) pieces of a disrupted binary;
- (ii) a gamma-ray burst jet;
- (iii) individual radiated neutrinos or pieces of ejected material in a supernova explosion.

## 4.2 Non-Linear Memory

Although it was generally recognised that the nonlinearity of Einstein’s equations must be taken into account to study the generation of gravitational waves from strong sources, as all of them are far, far away from Earth and their amplitude is so tiny, it was widely believed that linearised theory sufficed. In 1991, D. Christodoulou in [21] showed that this assumption was incorrect. Using the mathematically elegant Bondi formalism described in Sec. 2, he discovered the so-called non-linear memory. Unlike other non-linear effects of general relativity, the memory is non-oscillatory (see Fig. 3), meaning that it builds up over time as the system loses its energy. This led K. Thorne [66] to propose that what happens is similar to what we described in Sec. 4.1; however, instead of losing a body from the system, what goes away instead are gravitons<sup>6</sup>. The lost energy contributes to the change of multipole moments of the source, meaning that memory is sensitive to the entire radiative history of the system and influences the gravitational waves emitted afterwards.

We will provide the argument by K. Thorne [66] refined by M. Favata [27] that shows that the linear memory expression (45) actually includes the non-linear contribution. As the latter is the graviton burst, each graviton must be included in the final  $\sum_k$  (but, of course, excluded from the initial sum). The quantity  $\frac{m_k}{\sqrt{1-v_k^2}}$  is, in fact, the graviton’s energy as measured in the

---

<sup>6</sup>Note that non-linear memory was also independently discovered by L. Blanchet and T. Damour in 1992, who had already used the term ‘bundle of gravitons’ [8].

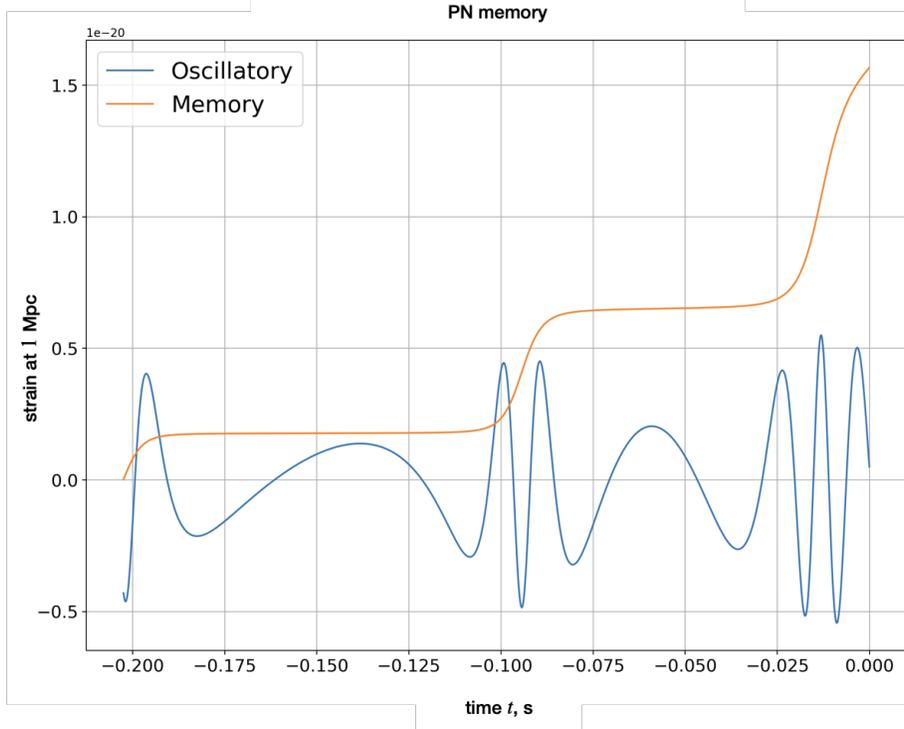


Figure 3: PN waveform along with its memory part for an equal-mass binary with  $M_{\text{tot}} = 60$  and  $e_0 = 0.4$ .

detector's rest frame. The speed of a graviton is  $c$  (see, e.g., [41]). Thus, we write:

$$\Delta h_{ij}^{TT} = \frac{4}{r} \int_{-\infty}^u dt \int \frac{dE}{d\Omega dt} \left[ \frac{c'_j c'_k}{1 - c^j \hat{n}_l} \right]^{TT} d\Omega. \quad (46)$$

This can also be deduced by considering second-order corrections to the field equations and incorporating second-order terms in metric perturbations into the energy-momentum tensor. The time integral is what gives memory its hereditary nature, constantly accounting for all the changes that happened before. Equations (45) and (46) are analogous to the expressions of Christodoulou [21] for linear and non-linear contributions, respectively.

If we decompose the gravitational wave field into modes as in Sec. 4.1, one can show that it enters 2.5PN and higher-order corrections of the mass moments [8]:

$$U_{lm}^{\text{nonlinmem}} = 32\pi \sqrt{\frac{(l-2)!}{2(l+2)!}} \int_{-\infty}^u dt \int \frac{dE}{d\Omega dt} Y_{lm}^*(\Omega) d\Omega,$$

whereas the current moments  $V_{lm}$  do not have non-linear memory contributions.

Note also that memory is not related to the mass loss of the source or the Coulomb part change; these contribute only indirectly through the change of multipole moments. For instance, spherically symmetric mass loss yields  $\frac{dE}{d\Omega} = 0$  and thus produces no gravitational waves.

### 4.3 Null and Ordinary Memory

In the previous sections, we demonstrated that memory arises from particles escaping to infinity. For linear memory, this phenomenon primarily involves massive bodies. Non-linear memory effects, however, result from gravitons flowing into  $\mathcal{I}^+$ . Further research has shown that energy transmission to future null infinity by other massless particles also induces non-linear-type memory, even within the linearised theory, including electromagnetic radiation [5] and neutrinos [6]. Consequently, this type of memory is often referred to as *null memory*, indicating that particles are moving along null trajectories. In contrast, memory generated by the flow of non-null matter is called *ordinary memory* or *boost memory*. This classification reveals that null memory is closely related to BMS symmetries, which we will demonstrate in the next section. Here, we will illustrate an alternative splitting of memory components and highlight their sources.

We stick to the notation introduced in Sec. 2. It turns out that memory components may be conveniently split into so-called *E*- and *B*-modes, similar to the decomposition of electromagnetic radiation. We will describe this division in terms of linearised gravity and show how this applies to the nature of the memory, following [47].

When presenting the Bondi-Sachs formalism, we introduced the tensor  $C_{AB}$  (16), which describes the distortions of a unit 2-sphere under gravitational waves. This tensor is identical to the regular gravitational wave amplitude  $h_{ij} = h_{ij}(r, u, x^A)$ , with an addition that the former is rescaled so that  $C_{AB} = C_{AB}(u, x^C)$ . Similarly, the electromagnetic radiation at  $\mathcal{I}^+$  can be rescaled into  $E_A(u, x^B)$ . The *E*- and *B*-modes are then defined from the decomposition of the electromagnetic field into a gradient and a dual of the gradient of some real field, corresponding to the electric and magnetic parts, respectively:

$$E_A = D_A \Phi_e + \epsilon_{AB} D^B \Phi_b,$$

where  $D_A$  is the covariant derivative, and  $\epsilon_{AB}$  is the antisymmetric unit sphere surface area tensor, defined as before. Again, introducing the dyads, the electromagnetic radiation can be represented by the spin-weight 1 field:

$$q^A E_A = q^A (D_A \Phi_e + \epsilon_A^B D_B \Phi_b) = \eth(\Phi_e + i\Phi_b).$$

$\eth$  here corresponds to the Newman-Penrose spin-weight raising operator [55, 49]. It converts the scalar field into the spin-weight 1 one, hence the name.

An analogous decomposition can be introduced for the gravitational field. We first note that the strain can be represented in terms of a displacement vector  $s^A$ :

$$C_{AB} = D_{(A} s_{B)} - \frac{1}{4} q_{AB} D^C s_C.$$

With two scalar fields  $\Sigma_e$  and  $\Sigma_b$  forming a complex scalar field  $\Sigma := \Sigma_e + i\Sigma_b$ , a decomposition  $s_A = D_A \Sigma_e + \epsilon_{BA} D^B \Sigma_b$  yields a spin-weight 2 representation of the strain:

$$\sigma_0 = q^A q^B C_{AB} = q^A q^B D_A D_B (\Sigma_e + i\Sigma_b) = \eth^2(\Sigma_e + i\Sigma_b) = \eth^2 \Sigma.$$

Thus, analogously to the electromagnetic case,  $\Sigma_e$  and  $\Sigma_b$  represent the ‘electric’ and ‘magnetic’ parts of the strain. Therefore,  $\Sigma$  is a spin-weight 0 potential generating a spin-weight 2 field via the spin-weight raising operator  $\eth$ .

In electromagnetism,  $E^a$  and  $B^a$  radiation fields are orthogonal and have equal magnitudes. Similarly, in an inertial frame characterised by a time-like vector  $T^a$ , gravitational radiation fields  $\mathfrak{E}$  and  $\mathfrak{B}$  are found from the Weyl tensor and its dual as follows [47]:

$$\mathfrak{E} = T^a T^b q^A q^B C_{aAbB}, \quad \text{and} \quad \mathfrak{B} = -\frac{1}{2} T^a T^b q^A q^B \epsilon_{aAcC} C^{cC}{}_{bB},$$

with  $\epsilon_{0123} = 1$  being the Levi-Civita symbol. These spin-weight 2 fields are also of equal magnitude, and  $\mathfrak{E} \rightarrow \mathfrak{B}$  under the spin rotation  $q^A \rightarrow e^{i\frac{\pi}{4}} q^A$  [47]. Expanded at  $\mathcal{I}^+$  to the leading order in the Weyl tensor, these are related to the strain as:

$$E(u, x^A) = \partial_0^2 \sigma(u, x^A), \quad \text{and} \quad B(u, x^A) = i \partial_0^2 \sigma(u, x^A).$$

The Weyl tensor representation is gauge invariant, whereas the strain has a gauge freedom:

$$\sigma_0 \rightarrow \sigma_0 + \delta^2 \alpha$$

under a transformation  $u \rightarrow u + \alpha(x^A)$ , which is the supertranslation freedom of the BMS symmetry group, as mentioned in Sec. 2.5.3. Thus, the  $E$ -mode of the strain can be gauged away during any stationary epoch. The  $E$ -mode of the memory, defined as the difference (41), is gauge invariant and represents a supertranslation shift between the two preferred gauges for the strain at  $u = \pm\infty$  [47].

We define the reduced stress-energy tensor:

$$\rho_{ab} := T_{ab} - \frac{1}{2} \eta_{ab} T^{ab},$$

with  $\eta_{ab}$  being the background Minkowski metric and  $T_{ab}$  a standard energy-momentum tensor. Solving the linearised field equations with all quantities decomposed in  $\frac{1}{r}$ , the 00-component of the field equations at  $\mathcal{I}^+$  reads [47]:

$$\partial_0 M = \frac{1}{4} \partial_0 \{ \delta \bar{\delta} (\delta \bar{\delta} + 2) \Sigma_e \} - 4\pi \rho_{uu}^{[2]}, \quad (47)$$

where  $M$  is the Bondi mass aspect, and  $\rho_{uu}^{[2]}$  is the leading-order coefficient of the reduced stress-energy tensor decomposition at  $r^{-2}$ . Integration over retarded time from  $u = -\infty$  to  $+\infty$  results in an expression for a change (denoted by  $\Delta$ ) in the electric component of the field:

$$\Delta \{ \delta \bar{\delta} (\delta \bar{\delta} + 2) \Sigma_e \} = 4\Delta M + 16\pi \int_{-\infty}^{+\infty} \rho_{uu}^{[2]} du. \quad (48)$$

The contribution from the reduced energy-momentum tensor represents the energy transport to infinity, i.e., null memory. Without it, the  $E$ -mode memory is thus related to the change in the mass aspect. This is exactly the ordinary memory [28]. Hence, the null and ordinary memory ‘live’ in the  $E$ -mode of the radiation.

It can also be shown [47] that  $\Delta \{ \partial_0 L^e \} = 2\Delta M$ , which relates null memory to the change in the dipole-moment aspect, an ‘electric’ counterpart of the angular momentum aspect introduced in Sec. 2.4.5.

In a similar way, for the  $B$ -mode,

$$\Delta \{ \bar{\partial} \bar{\partial} (\bar{\partial} \bar{\partial} + 2) \Sigma_b \} = -2\Delta \{ \partial_0 L^b \},$$

which is related to the change in the angular momentum aspect itself. One can see that there is no null memory analogue for the magnetic part. Instead, sourceless memory can be shown to ‘reside’ in the  $B$ -mode. Note that conventional sources can produce the  $B$ -mode gravitational waves, but only the ones with vanishing memory [46].

In fact, both  $\Delta \{ \partial_0 L^e \}$  and  $\Delta \{ \partial_0 L^b \}$  are related to anisotropic stress in the system [47], which coincides with what we obtained in Sec. 4.2.

## 4.4 Relation to BMS Symmetries

### 4.4.1 New Coordinates

We shall demonstrate the tight interplays between the BMS symmetries and the gravitational memory. To simplify our equations, we will slightly change the notion of the coordinate system, according to [61]. We introduce the angular coordinates  $z = e^{i\phi} \tan \frac{\theta}{2}$  and the corresponding complex conjugate  $\bar{z}$ . Then, the Bondi metric (2) expanded in  $\frac{1}{r}$  takes the asymptotic form:

$$\begin{aligned} ds^2 = & -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} + \\ & + 2 \frac{M}{r} du^2 + r(C_{zz} dz^2 + C_{\bar{z}\bar{z}} d\bar{z} d\bar{z}) + (D^z C_{zz} dudz + D^{\bar{z}} C_{\bar{z}\bar{z}} dud\bar{z}) + \\ & + \frac{1}{4r^2} C_{zz} C^{\bar{z}\bar{z}} dudr + \gamma_{z\bar{z}} C_{zz} C^{\bar{z}\bar{z}} dz d\bar{z} + \dots \end{aligned} \quad (49)$$

where  $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$  is the unit metric on  $S^2$  and  $D_z$  is its covariant derivative. The field equation (47) at the future null infinity writes:

$$\partial_0 M = \frac{1}{4} [D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}] - T_{00}, \quad (50)$$

with  $T_{00}$  being the total energy flux through a given point at  $\mathcal{I}^+$ . It relates to the matter energy-momentum tensor as follows:

$$T_{00} = \frac{1}{4} N_{zz} N^{zz} + 4\pi \lim_{r \rightarrow \infty} [r^2 T_{00}^M]. \quad (51)$$

Note that  $N_{zz} = \partial_0 C_{zz}$  is the new form of the news tensor (21).

The asymptotic isometry group of this metric (40) written explicitly ( $\alpha(x^A) \rightarrow \alpha(z, \bar{z})$ ):

$$\begin{aligned} \zeta = \zeta(z, \bar{z}), \quad u & \rightarrow u - \zeta, \quad r \rightarrow r - D^z D_z \zeta, \\ z & \rightarrow z + \frac{1}{r} D^z \zeta, \quad \bar{z} \rightarrow \bar{z} + \frac{1}{r} D^{\bar{z}} \zeta, \end{aligned}$$

whose generating vector fields:

$$\xi_\zeta = \zeta \partial_0 + D_z D^z \zeta \partial_1 - \frac{1}{r} (D^{\bar{z}} \zeta \partial_{\bar{z}} + D^z \zeta \partial_z).$$

The Lie derivatives acting on the asymptotic data:

$$\mathcal{L}_\zeta M = \zeta \partial_0 M, \quad (52a)$$

$$\mathcal{L}_\zeta C_{zz} = \zeta N_{zz} - 2D_z^2 \zeta. \quad (52b)$$

#### 4.4.2 BMS Vacuum Transitions

When radiation passes through future null infinity, the vacuum is changed by the BMS transformation. This effect will later be associated with memory. In this section, we follow [61] and deduce the transformation relating the initial and final vacua. For simplicity, we:

- (1) exclude cases with non-vanishing initial or final ADM momentum;
- (2) only consider a short burst of waves, neglecting tails, and hence the radiation flux, outside an intermediate interval  $u_i < u < u_f$ .

The first assumption allows us to consider spacetimes that are asymptotically Schwarzschild both before ( $u < u_i$ ) and after ( $u > u_f$ ) the passage of the wave, with  $M = M_i = \text{const}$  and  $M = M_f = \text{const}$  in the two epochs, respectively. The second assumption means that  $T_{uu}$  and/or  $N_{zz}$  is non-zero on  $\mathcal{I}^+$  during this interval, whereas before  $u_i$  and after  $u_f$  the regions are vacuum, i.e.,  $N_{zz} = 0$ . D. Christodoulou and S. Klainerman [22] considered  $M_f = 0$  spacetimes, where the interval must be large enough to capture most of the long tails. For non-zero  $M_f$ , the late-time geometry can, for example, represent a stable star or black hole.

According to [10], the vacuum is not unique. However, it must be characterised by

$$D_z^2 C_{zz} - D_{\bar{z}}^2 C_{\bar{z}\bar{z}} = 0. \quad (53)$$

Thus, we can consider  $C_{zz} = 0$  for  $u < u_i$  and  $C_{zz} \neq 0$  after  $u_f$ . The general solution to the above equation is of the form:

$$C_{zz} = -D_z^2 C(z, \bar{z}).$$

Comparison with (52b) reveals that different vacua are related by supertranslations  $C \rightarrow C + \zeta$ . The supertranslation may be found by the integration of (50) over  $(u_i, u_f)$ , corresponding to (48) with  $u_i \rightarrow -\infty$  and  $u_f \rightarrow +\infty$ :

$$D_z^2 \Delta C^{zz} = 2\Delta M + 2 \int_{u_i}^{u_f} T_{00} du, \quad (54)$$

where we used (53) and defined  $\Delta X = X(u_f) - X(u_i)$ . Should we ignore condition (1) and allow for the momentum and energy loss, the right-hand side will additionally include a change of an  $l = 1$  mode of angular momentum. The supertranslation  $\Delta C$  generating such a  $\Delta C_{zz}$  is obtained by inverting  $D_z^2 D_{\bar{z}}^2$  [61]:

$$\Delta C(z, \bar{z}) = 2 \int d^2 z' \gamma_{z' \bar{z}'} G(z, \bar{z}; z' \bar{z}') \left[ \Delta M + \int_{u_i}^{u_f} T_{00}(z', \bar{z}') du \right] \quad (55)$$

with

$$G(z, \bar{z}; z', \bar{z}') = -\frac{1}{\pi} \sin^2 \left[ \frac{|z - z'|^2}{(1 + z\bar{z})(1 + z'\bar{z}')} \right] \ln \sin^2 \left[ \frac{|z - z'|^2}{(1 + z\bar{z})(1 + z'\bar{z}')} \right], \quad (56a)$$

$$D_z^2 D_{\bar{z}}^2 G(z, \bar{z}; z', \bar{z}') = -\gamma_{z\bar{z}} \delta^{(2)}(z - z') + \dots \quad (56b)$$

Inserting (56a) into (55) and acting with  $D_z^2 D_{\bar{z}}^2$  using  $\partial_z \partial_{\bar{z}} \ln |z - z'|^2 = 2\pi \delta^{(2)}(z - z')$ , the delta-function will produce the right-hand side of (54), whereas all other terms integrate out due to energy-momentum conservation.  $C(z, \bar{z})$  is unique up to 4 global spacetime translations:

$$\zeta_{\text{global}} = c_0 + \frac{1}{1 + z\bar{z}} [c_1(1 - z\bar{z}) + (c_2 + \bar{c}_3)z + c_3\bar{z}],$$

which do not affect  $C_{zz}$ .

Thus, the vacuum transition is given in (55) by an integral of the total radiation flux  $T_{00}$  over the transition interval  $(u_i, u_f)$ . This relation can be generalised for multiple separated radiation intervals or non-vanishing initial and final momentum.

#### 4.4.3 BMS Detector Memory

Finally, we will relate the BMS vacuum transformations to the gravitational memory effect following [61]. We will refer to a so-called *BMS detector*, travelling along worldlines at a large fixed radius  $r_0$  and fixed angular coordinates  $z_0$  and  $\bar{z}_0$ . We will restrict ourselves to non-trivial transformations so that it is meaningful to describe the observations. It is convenient to pick constant  $z$  and choose a detector near  $\mathcal{I}^+$ , as such observations behave simply under the action of BMS.

At large  $r_0$ , the BMS detector is almost inertial [61], and its trajectory follows the geodesic equation up to  $\mathcal{O}(r_0^{-1})$ . The physical difference is that truly inertial detectors do not remain at fixed  $r_0$  or  $z_0$ , and therefore, after a long time period, their radius might become small. Hence, what we present is valid for inertial detectors over small spans of  $u$ .

Let us now introduce two BMS detectors labelled as 1 and 2. Their initial positions are  $z_1$  and  $z_2$ , and their initial separation is defined as

$$s_0 = \frac{2r_0 |z_1 - z_2|}{1 + z_1 \bar{z}_1}.$$

We denote  $z_1 - z_2$  as  $\delta z$  and take it to be of the order  $\frac{1}{r_0}$  with suppressed subleading corrections. As long as the metric undergoes a transition  $\Delta C_{zz}$ , the new distance between  $z_1 = \text{const}$  and  $z_2 = \text{const}$  using the metric (49) is:

$$\Delta s = \frac{r_0}{2s_0} \Delta C_{zz}(z_1, \bar{z}_1) \delta z^2 + \text{c.c.} = \frac{(1 + z_1 \bar{z}_1)^2 s_0}{8 r_0} \left( \Delta C_{zz}(z_1, \bar{z}_1) \frac{\delta z}{\delta \bar{z}} + \text{c.c.} \right). \quad (57)$$

$\Delta C_{zz}(z, \bar{z})$  is defined in terms of (55):

$$\Delta C_{zz}(z, \bar{z}) = \frac{4}{\pi} \int d^2 z' \gamma_{z'\bar{z}'} \frac{\bar{z} - \bar{z}'}{z - z'} \frac{(1 + z'\bar{z})}{(1 + z'\bar{z}')(1 + z\bar{z})^3} \left[ \int_{u_i}^{u_f} du T_{uu}(z', \bar{z}') + \Delta M \right]. \quad (58)$$

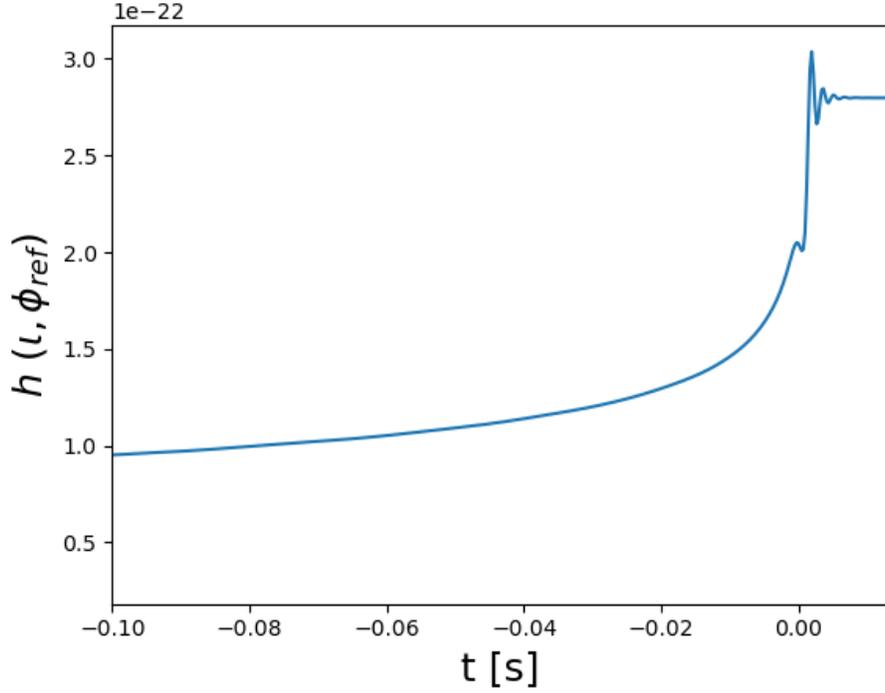


Figure 4: Displacement memory residing in  $h_+^{20}$  mode.

This coincides precisely with (46) and the results of [8, 21, 73] up to a rescale. See Fig. 4 for the gravitational strain of the displacement memory generated with a surrogate model [72].

Not only the distances are shifted. Initially synchronised clocks on the detectors will become desynchronised after the passage of a gravitational wave [61]:

$$\delta_{12}u - \delta_{21}u = D^z \Delta C_{zz} \delta z + c.c.$$

## 5 Introduction III

Establishing a deep connection between asymptotic symmetries and spacetime memories took decades. The BMS formalism emerged as a convenient framework in the 1960s [10], but the BMS group was a ‘failure’ in constructing General Relativity’s equivalent of the Lorentz group. At the same time, memory, though discovered in the 1990s using BMS coordinates [21], had nothing in common with the symmetries. Vacuum transitions, which serve as an elegant connection between the two, were introduced only in 2014 by A. Strominger and A. Zhiboedov [61]. This article, however, brought in a third component — the soft graviton theorem, which relates amplitudes of  $n \rightarrow (m + 1)$ -particle scattering to  $n \rightarrow m$  if the one particle is a soft graviton. The soft theorems are related to asymptotic symmetries via Ward identities and, more straightforwardly, to memories via Fourier transformations<sup>7</sup>.

This work was pivotal for low-energy physics. It turned out that one can construct numerous triangles with the same mathematical relations, and those are ubiquitous. Electric and magnetic memory were related to soft photon and magnetic soft photon theorems and large and complexified gauge symmetries [19, 42, 43, 44, 52, 62], respectively. The soft gluon theorem corresponds to colour memory, linked to the Kac-Moody algebra [54]. Many more triangles are yet to be discovered and completed. This highlights the unified nature of infrared physics.

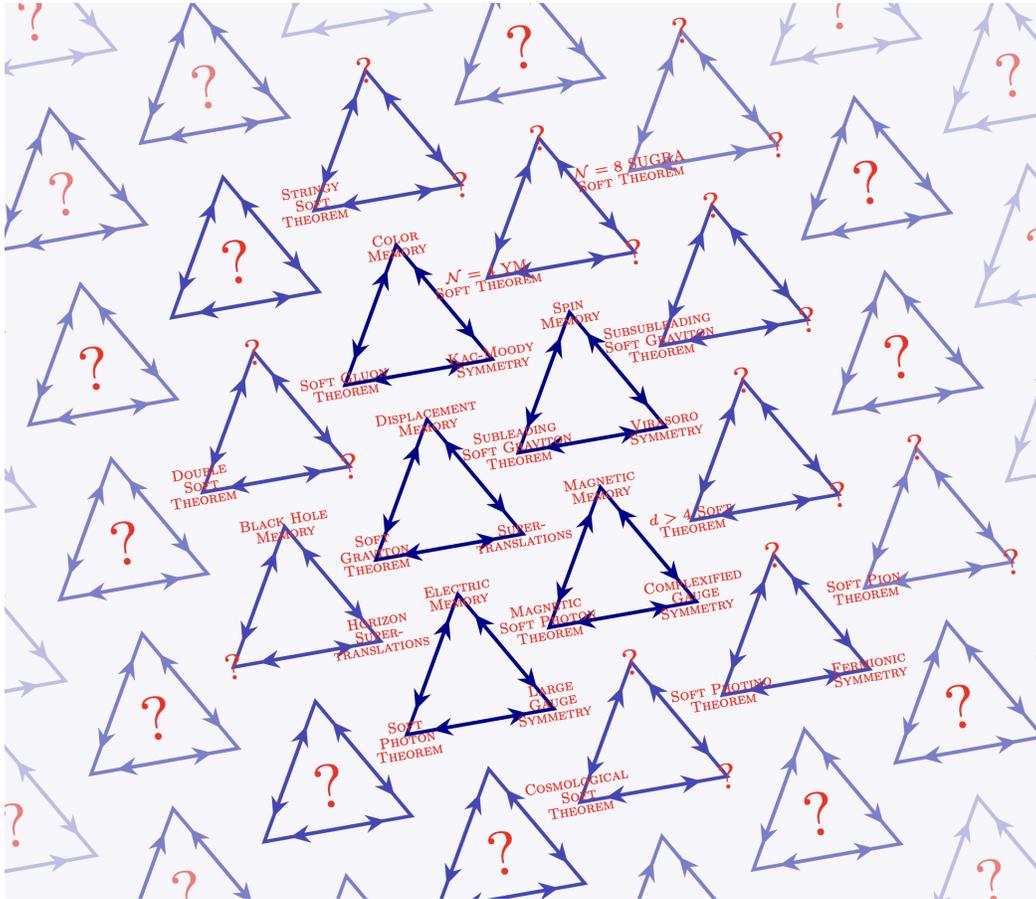


Figure 5: The triangles of infrared physics, taken from [59].

<sup>7</sup>Note that soft gravitons refer to almost-zero-energy gravitons. Therefore, these relations are called ‘Infrared triangles’, highlighting that they manifest interplays in low-energy physics.

## 6 Infrared Triangles

### 6.1 Soft Theorems

Soft theorems are statements that allow for the calculation of the scattering matrix in processes involving soft particles. If there are  $n$  incoming and  $(m + 1)$  outgoing particles, one of which is a low-energy massless boson, the  $\mathcal{S}$ -matrix of the  $n \rightarrow m$  process differs from that of the  $n \rightarrow (m + 1)$  process by a *soft factor*. In this section, we start from the electromagnetic analogy and briefly sketch the derivation of the soft photon theorem following [59]. After that, we proceed to the soft graviton theorem and its connection to the displacement memory effect.

#### 6.1.1 Soft Photon Theorem

Let us start by considering a Feynman diagram of a scattering process with  $n$  incoming particles of momenta  $p_k^{\text{in}}$  and  $m$  outgoing particles with  $p_k^{\text{out}}$ . Now, add an outgoing photon with momentum  $q$  and polarisation  $\varepsilon$  such that  $q_a \varepsilon^a = 0$  as shown in Fig. 6. The derivation for an incoming photon is analogous.

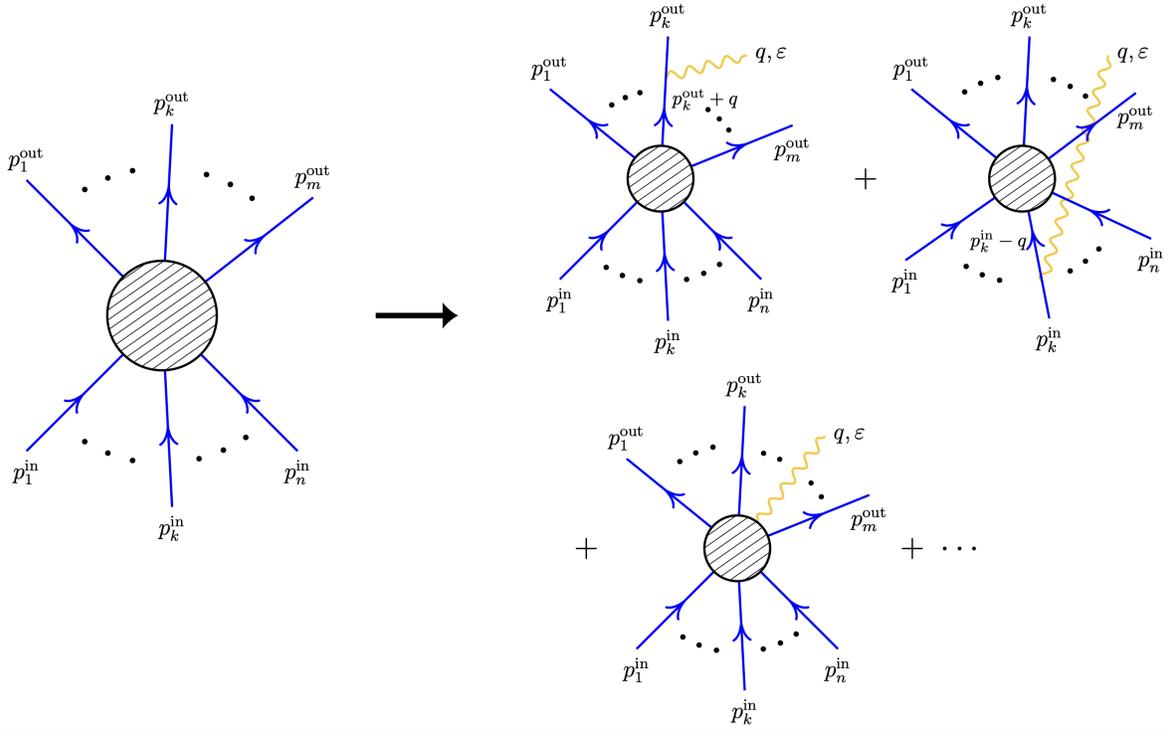


Figure 6:  $n \rightarrow m$  particle scattering (left side) with an added outgoing photon (right side) from [59].

For a soft photon, the amplitude is a sum of terms with the soft photon attached to one of the  $n + m$  external lines (upper part of the right side in Fig. 6) via a vertex or to an internal one (lower part). We will start with the external attachment of the photon. Then, the difference between the ‘before’ and ‘after’ diagrams is exactly one vertex on an external leg and one propagator.

We will use the following expression for the scattering matrix:

$$\mathcal{S} = \mathcal{T} \exp(i\mathcal{L}_{\text{int}}),$$

with an interaction of the form:

$$\mathcal{L}_{\text{int}} = -A^a j_a.$$

Expanding it to the leading order, let our scalar field be of charge  $Q$ . Then, the charge current is:  $j_a = iQ(\phi\partial_a\phi^* - \phi^*\partial_a\phi)$ . The normalisation for single-particle states reads  $\langle p|p'\rangle = 2\omega_p(2\pi)^3\delta^{(3)}(p-p')$ . Thus, the current for a plane-wave writes:

$$j_a \sim 2Qp_a.$$

Under gauge transformations, the current associated with a charge  $Q$  acquires a phase  $e^{iQ\varepsilon}$ . Therefore, the vertex factor, up to an  $\mathcal{O}(q)$  correction, is:

$$A^a j_a = ie\varepsilon^a \cdot 2Qp_a.$$

The propagator of a mass  $m$  scalar particle reads:

$$\frac{-i}{(p+q)^2 + m^2} = \frac{-i}{p^2 + 2p \cdot q + q^2 + m^2} = \frac{-i}{2p \cdot q}, \quad (59)$$

where we used the fact that external lines must be on-shell, yielding  $q^2 = 0$ , and a relation  $p^2 = -m^2$ . Thus, the total contribution is:

$$ie\varepsilon^a \cdot 2Qp_a \cdot \frac{-i}{2p \cdot q} = \frac{eQ\varepsilon \cdot p}{p \cdot q}.$$

For every outgoing particle, there is one such factor; for every incoming, there is one such factor with a minus sign. Together, we get:

$$\sum_{k=1}^m \frac{eQ^{\text{out}}\varepsilon \cdot p^{\text{out}}}{p^{\text{out}} \cdot q} - \sum_{k=1}^n \frac{eQ^{\text{in}}\varepsilon \cdot p^{\text{in}}}{p^{\text{in}} \cdot q}. \quad (60)$$

Now, we consider the terms arising from the addition of the photon to an internal leg. Their propagators are never on-shell ( $p^2 \neq m^2$ ). Thus, these terms will not cancel in the denominator, as in (59). So, if we take  $q^a \rightarrow 0$ , the difference between  $p^2$  and  $m^2$  will dominate, and we will not get a pole. Therefore, in the soft limit, these diagrams do not contribute to the pole and can be neglected, although they are most likely non-vanishing.

Equation (60) is the *soft factor*, for which it holds that:

$$\langle p_1^{\text{out}} \dots p_m^{\text{out}} | a_+^{\text{out}}(q^i) \mathcal{S} | p_1^{\text{in}} \dots p_n^{\text{in}} \rangle = e \left[ \sum_{k=1}^m \frac{Q^{\text{out}}\varepsilon^+ \cdot p^{\text{out}}}{p^{\text{out}} \cdot q} - \sum_{k=1}^n \frac{Q^{\text{in}}\varepsilon^+ \cdot p^{\text{in}}}{p^{\text{in}} \cdot q} \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle + \mathcal{O}(q),$$

with the pole at the leading order. The global charge conservation guarantees the gauge invariance of this soft factor.

### 6.1.2 Soft Graviton Theorem

The graviton soft theorem is straightforwardly deduced using the approach described in Sec. 6.1.1. We now suppose that the newly attached wave in the diagram 6 is a graviton of momentum  $q^a$  rather than a photon. The polarisation tensor satisfies  $\varepsilon_{ab}q^b = 0$  and  $\varepsilon^{ab}\eta_{ab} = 0$ . We specify the interaction:

$$\mathcal{L}_{\text{int}} = \sqrt{8\pi G} h^{ab} T_{ab},$$

which couples the graviton field  $h^{ab}$  in the expansion  $g^{ab} = \eta^{ab} - \sqrt{32\pi G} h^{ab}$ . The normalisation factor  $\sqrt{32\pi G}$  is introduced to avoid factors of  $G$  in the kinetic term.

For a scalar field,

$$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} \eta_{ab} \partial^c \phi \partial_c \phi,$$

or, approximately:

$$T_{ab} \sim 2p_a p_b,$$

where the terms of  $T_{ab}$  proportional to  $\eta_{ab}$  do not contribute to  $\varepsilon_{ab}$ , as the latter is traceless. The propagator is the same,  $\frac{-i}{2p \cdot q}$ . Thus, the product of the vertex and the propagator yields:

$$i\sqrt{32\pi G} \varepsilon^{ab} p_a p_b \cdot \frac{-i}{(p+q)^2 + m^2} \rightarrow \sqrt{8\pi G} \frac{\varepsilon^{ab} p_a p_b}{p \cdot q},$$

which is the result for a single particle. Considering  $m$  outgoing and  $n$  incoming legs, the soft factor is:

$$\sqrt{8\pi G} \left[ \sum_{k=1}^m \frac{\varepsilon^{ab} p_a^{k,\text{out}} p_b^{k,\text{out}}}{p^{k,\text{out}} \cdot q} - \sum_{k=1}^n \frac{\varepsilon^{ab} p_a^{k,\text{in}} p_b^{k,\text{in}}}{p^{k,\text{in}} \cdot q} \right]. \quad (61)$$

Our result looks similar to the gauge theory case up to a replacement  $p^k \rightarrow Q^k$ . Similarly, we cannot get the poles by coupling the soft graviton to the internal lines, so we neglect those diagrams. Thus, the soft graviton theorem reads:

$$\lim_{\omega \rightarrow 0} \mathcal{A}_{m+n+1}(\text{in}; \text{out}; q, \varepsilon^{ab}) = \sqrt{8\pi G} S_{ab} \varepsilon^{ab} \mathcal{A}_{m+n}(\text{in}; \text{out}) + \mathcal{O}(q), \quad (62)$$

with  $\mathcal{A}_n$  denoting scattering amplitudes and

$$S_{ab} = \left[ \sum_{k=1}^m \frac{p_a^{k,\text{out}} p_b^{k,\text{out}}}{p^{k,\text{out}} \cdot q} - \sum_{k=1}^n \frac{p_a^{k,\text{in}} p_b^{k,\text{in}}}{p^{k,\text{in}} \cdot q} \right]^{TT}. \quad (63)$$

The analogue of gauge invariance in the gravity case is the invariance of the soft factor under a shift  $\varepsilon^{ab} \rightarrow \varepsilon^{ab} + \Lambda^a(q) \cdot q^b$ . After we apply it,  $\Lambda^a$  can be pulled out of the sum, meaning that the soft factor shifts by:

$$\Lambda^a \left[ \sum_{k=1}^m p_a^{k,\text{out}} - \sum_{k=1}^n p_a^{k,\text{in}} \right].$$

Therefore, it vanishes if the momentum is conserved. Thus, global charge conservation is required for the soft photon theorem invariance, and global energy-momentum conservation produces the invariance of the soft graviton. Note that we have not used global angular momentum conservation. It turns out this law generates another soft theorem, to the implications of which we will return in Sec. 8.

### 6.1.3 Relation to Memory

As discussed, V. Braginsky and K. Thorne [15] analysed the detection of burst memory produced in collisions and scatterings of massive bodies. They found that such collisions result in a difference in the transverse-traceless part of the asymptotic metric at  $\mathcal{I}^+$ , as given in Eq.(1) of [15]. We rewrite this equation in the  $(-, +, +, +)$  signature, applying the covariant gauge and normalisation:

$$h_{ab} = \frac{1}{\sqrt{32\pi G}}(g_{ab} - \eta_{ab}).$$

Then, the net difference takes the form [61]:

$$\Delta h_{ab}^{TT} = \frac{1}{r_0} \sqrt{\frac{G}{2\pi}} \left[ \sum_{k=1}^m \frac{p_a^{k,\text{out}} p_b^{k,\text{out}}}{p^{k,\text{out}} \cdot q} - \sum_{k=1}^n \frac{p_a^{k,\text{in}} p_b^{k,\text{in}}}{p^{k,\text{in}} \cdot q} \right]^{TT}. \quad (64)$$

Alternatively, one can derive this equation by solving the linearised field equations with a retarded propagator. Using (57), one can easily construct the memory of this collision.

Strikingly, (64) is similar to (63) up to a substitution of massive bodies momenta in [15] with particles momenta in [69]. To make the relation even more understandable, we note that the Fourier transform of  $h_{ab}^{TT}$  at  $\mathcal{I}^+$  can be written by defining  $q = (\omega, \omega \hat{q})$  and using the stationary phase approximation [33] at large  $r$ :

$$h_{ab}^{TT}(\omega, \vec{q}) = 4\pi i \lim_{r \rightarrow \infty} r \int du e^{i\omega u} h_{ab}^{TT}(u, r\hat{q}).$$

Here, we assume that  $\omega r \gg 1$ . Note that  $\hat{q}$  is a null vector pointing from the collision region to  $\mathcal{I}^+$  and serves as a coordinate on the  $S^2$  at null infinity.

Assuming  $h_{ab}^{TT}(u, r\hat{q})$  approaches different finite values as  $u \rightarrow \pm\infty$  and large  $r = r_0$ , (64) is proportional to the coefficient of the pole in  $\omega$ :

$$\Delta h_{ab}^{TT} = \frac{1}{4\pi i r_0} \lim_{\omega \rightarrow 0} [-i\omega h_{ab}^{TT}(\omega, \hat{q})]. \quad (65)$$

Finally, we note that at the linear order, the expectation value of the asymptotic metric fluctuation generated by the  $n \rightarrow m$  scattering obeys:

$$\begin{aligned} \lim_{\omega \rightarrow 0} \omega h_{ab}^{TT}(\omega, \hat{q}) \epsilon^{ab} &= \lim_{\omega \rightarrow 0} \frac{\omega \mathcal{A}_{m+n+1}(\text{in}; \text{out}; \omega, \hat{q}, \epsilon_{ab})}{\mathcal{A}_{m+n}(\text{in}; \text{out})} = \sqrt{8\pi G} \epsilon^{ab} \lim_{\omega \rightarrow 0} \omega S_{ab}(\omega \hat{q}) = \\ &= \sqrt{8\pi G} \epsilon^{ab} \left[ \sum_{k=1}^m \frac{p_a^{k,\text{out}} p_b^{k,\text{out}}}{p^{k,\text{out}} \cdot q} - \sum_{k=1}^n \frac{p_a^{k,\text{in}} p_b^{k,\text{in}}}{p^{k,\text{in}} \cdot q} \right]^{TT}. \end{aligned}$$

Inserting this into (65) leads to a result equivalent to [15]. Hence, the gravitational memory effect gives a physical meaning to the soft graviton theorem. Theoretical soft gravitons are almost impossible to detect, as their almost zero energy requires a long time to measure. Memory, however, can be measured in a finite time. This is because the Fourier transform of the soft graviton pole is a step function (indicating a permanent displacement) in retarded time.

## 6.2 Completing the Triangle

We have already discussed all three vertices of the displacement memory infrared triangle (see Fig. 7). The last untouched piece is the Ward identities, which relate the soft graviton theorem to the BMS supertranslations. In this section, we briefly present this remaining link [59] to highlight the completeness of these correspondences.

### 6.2.1 Ward Identities

Ward identities express the dynamical consequences of conserved charges commuting with the Hamiltonian or, equivalently, with the  $\mathcal{S}$ -matrix, since  $\mathcal{S} \sim \exp(i\mathcal{H}t)$  for  $t \rightarrow \infty$ . They relate any scattering matrix element between a pair of incoming and outgoing states, multiplied by the soft factor, to the same matrix with the insertion of certain soft particle modes.

As before, we represent the amplitudes with  $\mathcal{A} = \langle \text{in} | \mathcal{S} | \text{out} \rangle$ . Then, the conservation of charges can be expressed as:

$$\langle \text{in} | Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^- | \text{out} \rangle. \quad (66)$$

Here,  $Q^+$  and  $Q^-$  are defined on  $\mathcal{I}^+$  and  $\mathcal{I}^-$ , respectively. Thus, the former acts on the out-states and the latter on the in-states. Assuming the in-state is described by  $n$  hard particles coming in at points on the asymptotic sphere described by  $z^{\text{in}}$  (see Sec. 4.4.1 for the definition), the action of  $Q_\varepsilon^-$  on the in-state is:

$$Q_\varepsilon^- |\text{in}\rangle = -2 \int d^2 z \partial_z \partial_{\bar{z}} \varepsilon N^-(z, \bar{z}) |\text{in}\rangle + \sum_{k=1}^n Q_k^{\text{in}} \varepsilon(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) |\text{in}\rangle,$$

where  $N^-(z, \bar{z})$  denotes the incoming soft particle field on  $\mathcal{I}^-$ , the first term is the action of the soft charge, and the second term is the action of the hard charge. Similarly, for the outgoing field:

$$\langle \text{out} | Q_\varepsilon^+ = 2 \int d^2 z \partial_z \partial_{\bar{z}} \varepsilon \langle \text{out} | N^+(z, \bar{z}) + \sum_{k=1}^m Q_k^{\text{out}} \varepsilon(z_k^{\text{out}}, \bar{z}_k^{\text{out}}) \langle \text{out} |.$$

Inserting (66) into the exponent implies that if a state  $X$  evolves into a state  $Y$ , a gauge-transformed  $X$  would evolve into the gauge-transformed  $Y$ . Finally, the Ward identity is:

$$\begin{aligned} 2 \int d^2 z \partial_z \partial_{\bar{z}} \varepsilon \langle \text{out} | N^+(z, \bar{z}) \mathcal{S} - \mathcal{S} N^-(z, \bar{z}) | \text{in} \rangle &= \\ &= \left[ \sum_{k=1}^n Q_k^{\text{in}} \varepsilon(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) - \sum_{k=1}^m Q_k^{\text{out}} \varepsilon(z_k^{\text{out}}, \bar{z}_k^{\text{out}}) \right] \langle \text{in} | \mathcal{S} | \text{out} \rangle. \end{aligned}$$

This is, in fact, an infinite number of Ward identities, one for every function  $\varepsilon$  on the sphere. The BMS charge that generates supertranslations on  $\mathcal{I}^-$  is defined as [60]:

$$Q_\varepsilon^- = \frac{1}{4\pi G} \int_{\mathcal{I}_+^-} d^2 z \gamma_{z\bar{z}} \varepsilon M^- = \frac{1}{4\pi G} \int dv d^2 z \varepsilon [\gamma_{z\bar{z}} T_{00} + \partial_0 \partial_{(z} V_{\bar{z})}], \quad (67)$$

where  $V_x = D^x C_{xx}$ , with  $x \in \{z, \bar{z}\}$  denoted for convenience. Similarly, one defines  $Q_\varepsilon^+$ . These are an infinite number of charges specified by functions  $\varepsilon$ .

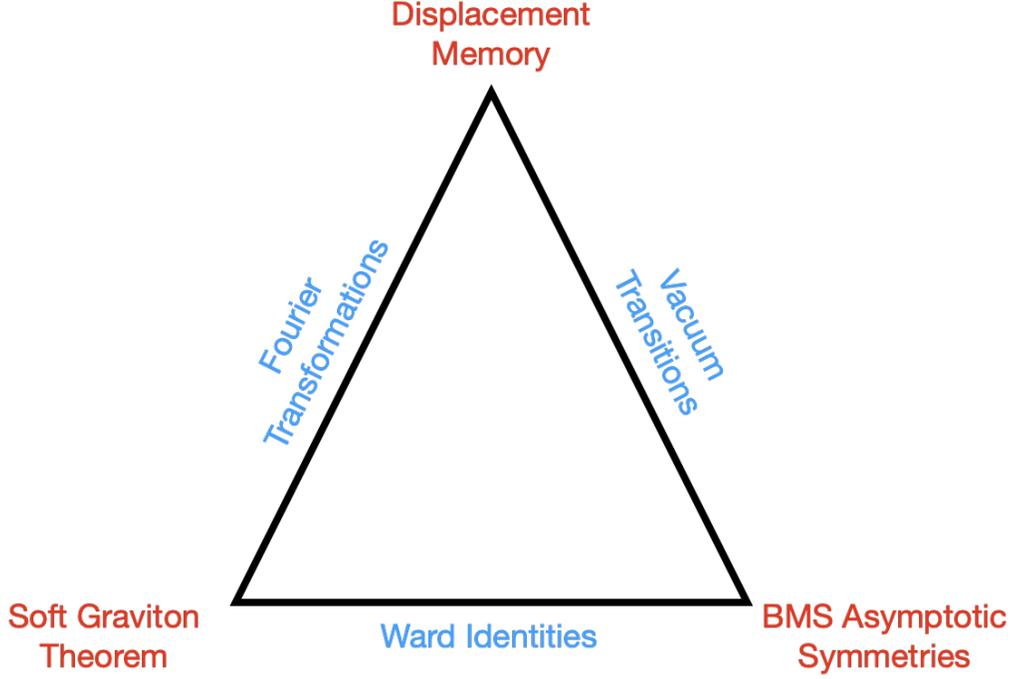


Figure 7: Displacement memory infrared triangle.

### 6.2.2 BMS Symmetries and Soft Gravitons

At late times  $t \rightarrow \infty$ , the gravitational field becomes free and can be approximated by mode expansion [33]:

$$h_{ab}^{\text{out}}(x) = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} [\varepsilon_{ab}^{\alpha*}(\vec{q}) a_{\alpha}^{\text{out}}(\vec{q}) e^{iqx} + \varepsilon_{ab}^{\alpha}(\vec{q}) a_{\alpha}^{\text{out}}(\vec{q})^{\dagger} e^{-iqx}].$$

In retarded Bondi coordinates,  $C_{zz}(u, z, \bar{z}) = \frac{1}{\sqrt{8\pi G}} \lim_{r \rightarrow \infty} \frac{1}{r} h_{zz}^{\text{out}}(r, u, z, \bar{z})$ . Using the mode expansion and the stationary phase approximation to the resulting integral over momentum space, we arrive at:

$$C_{zz} = \frac{-i}{\sqrt{32\pi G}} \frac{1}{4\pi^2(1+z\bar{z})^2} \int_0^{\infty} d\omega_q [a_+^{\text{out}} e^{-i\omega_q u} - (a_-^{\text{out}})^{\dagger} e^{i\omega_q u}].$$

Using this and defining the news  $N_{zz}^{\omega}(z, \bar{z}) = \int du e^{i\omega u} \partial_0 C_{zz}$ , we get:

$$N_{zz}^{\omega}(z, \bar{z}) = \frac{-1}{\sqrt{32\pi G}} \frac{1}{2\pi(1+z\bar{z})^2} \int_0^{\infty} d\omega_q \omega_q [a_+^{\text{out}} \delta(\omega_q - \omega) - (a_-^{\text{out}})^{\dagger} \delta(\omega_q + \omega)].$$

The zero mode can be found in a Hermitian way:

$$N_{zz}^0 = \lim_{\omega \rightarrow 0^+} \frac{1}{2} (N_z z^{+\omega} + N_z z^{-\omega}) = -\frac{1}{\sqrt{32\pi G}} \frac{1}{4\pi(1+z\bar{z})^2} \lim_{\omega \rightarrow 0^+} [\omega a_+^{\text{out}} + \omega (a_-^{\text{out}})^{\dagger}] = D_z^2 N. \quad (68)$$

A parallel construction is possible on  $\mathcal{I}^-$ , which we shall call  $M_{zz} = D_z^2 M$ .

The soft graviton current is defined as the charge (67) difference, namely [33]:

$$P_z := \frac{1}{2G} \left[ \int_{-\infty}^{+\infty} dv \partial_0 V_z - \int_{-\infty}^{+\infty} du \partial_0 U_z \right] = \frac{1}{2\pi} \int dz (Q_\varepsilon^+ - Q_\varepsilon^-). \quad (69)$$

By defining  $\mathfrak{D}_{zz} = D_z^2 N + D_z^2 M$ , we rewrite the current as  $P_z = \frac{1}{4G} \gamma^{z\bar{z}} \partial_{\bar{z}} \mathfrak{D}_{zz}$ .

Finally, we relate the soft theorems to the Ward identities as in [33]. Consider a matrix element  $\langle \text{out} | \mathfrak{D}_{zz} \mathcal{S} | \text{in} \rangle$ . Using (68) and the definition for  $\mathfrak{D}_{zz}$ :

$$\langle \text{out} | \mathfrak{D}_{zz} \mathcal{S} | \text{in} \rangle = -\frac{1}{\sqrt{32\pi G}} \frac{1}{4\pi(1+z\bar{z})^2} \lim_{\omega \rightarrow 0^+} [\omega \langle \text{out} | a_+^{\text{out}} \mathcal{S} | \text{in} \rangle + \omega \langle \text{out} | \mathcal{S} (a_-^{\text{in}})^\dagger | \text{in} \rangle].$$

Here, we also account for the fact that the creation (annihilation) operator destroys the out (in) state at zero frequencies. The first term is the  $\mathcal{S}$ -matrix element with a single outgoing positive helicity soft graviton, and the second is the corresponding incoming negative helicity one with the same spatial momentum. The two amplitudes are equal, which leads to:

$$\langle \text{out} | \mathfrak{D}_{zz} \mathcal{S} | \text{in} \rangle = -\frac{1}{\sqrt{32\pi G}} \frac{1}{2\pi(1+z\bar{z})^2} \lim_{\omega \rightarrow 0^+} \omega \langle \text{out} | a_+^{\text{out}} \mathcal{S} | \text{in} \rangle.$$

The soft graviton theorem (62) with a positive helicity outgoing graviton yields:

$$\langle \text{out} | \mathfrak{D}_{zz} \mathcal{S} | \text{in} \rangle = \frac{8G}{(1+z\bar{z})} \langle \text{out} | \mathcal{S} | \text{in} \rangle.$$

From the relation between  $P_z$  and  $\mathfrak{D}_{zz}$ , we deduce:

$$\langle \text{out} | P_z \mathcal{S} | \text{in} \rangle = \frac{1}{4G} \gamma^{z\bar{z}} \partial_{\bar{z}} \langle \text{out} | \mathfrak{D}_{zz} \mathcal{S} | \text{in} \rangle.$$

Lastly, with the definition for the current (69) and total momentum conservation, we arrive at:

$$\left[ \sum_{k=1}^n Q_k^{\text{in}} \varepsilon(z_k^{\text{in}}, \bar{z}_k^{\text{in}}) - \sum_{k=1}^m Q_k^{\text{out}} \varepsilon(z_k^{\text{out}}, \bar{z}_k^{\text{out}}) \right] \langle \text{in} | \mathcal{S} | \text{out} \rangle$$

with  $\varepsilon = \frac{1}{z-\omega}$  and momenta parameterised in terms of  $z$  and  $\bar{z}$  [59]. This exactly reproduces the supertranslation Ward identity [60]. We can also reverse the above derivation to show that the supertranslation Ward identity implies Weinberg's soft graviton theorem.

## 7 Introduction IV

From the triangular structure described in Sec. 6, it becomes evident that the discovery of asymptotic symmetries immediately implies the existence of a new soft theorem and a new memory effect. This also works the other way around with any vertex. Thus, in principle, one can have as many memories as one wants as long as one finds an underlying symmetry.

Despite great efforts in BMS group constructions, the focus has been on specifying local coordinates and global boundary conditions at null infinity for asymptotically flat 4-spacetimes. However, it was largely overlooked that one should also require a globally well-defined finite transformation. This was highlighted by G. Barnich and C. Troessart [2, 3]. They discovered that factoring out supertranslations yields a symmetry algebra, which is the infinite-dimensional Virasoro algebra rather than the finite-dimensional Lorentz. Thus, the symmetry algebra of asymptotically flat spacetimes can be enlarged to incorporate all symmetry vector fields, including those with analytic singularities. These additional generators were dubbed ‘superrotations’, as they are a sort of generalisation of the angle-dependent rotations and boosts.

Given these new symmetries, one would expect an additional soft graviton theorem. Motivated by this observation, F. Cachazo and A. Strominger proved the subleading soft graviton theorem [17]. Later, it was shown that it is equivalent to the superrotation charge conservation in [18, 38].

With the general relations of the triangle being known, the last vertex was not long to be discovered [53]. The new type of gravitational memory was called ‘spin’ memory. It will be described in detail in the following sections, summarising everything discussed in this thesis.

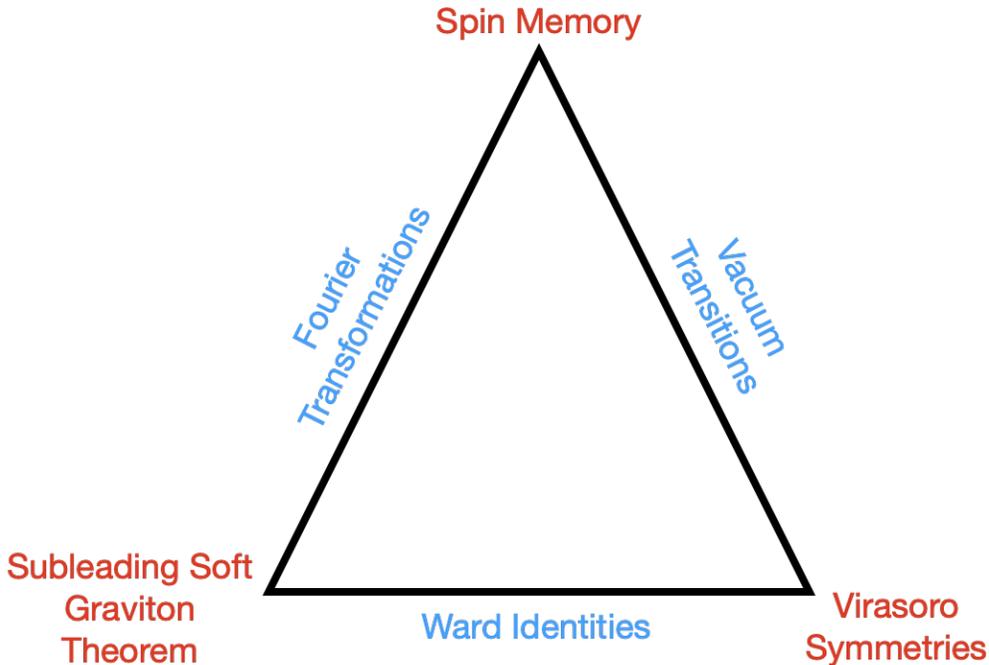


Figure 8: The spin memory triangle.

## 8 Spin Memory

### 8.1 The New Triangle

#### 8.1.1 Asymptotic Symmetries

In the previous sections, we mostly concentrated on the Bondi mass aspect  $M$ , the first correction to the Bondi metric (2) near  $\mathcal{I}^+$ . The next terms in the  $\frac{1}{r}$  expansion include the angular momentum aspect  $L^A$ , introduced in Sec. 2.4.5. In the coordinates of Sec. 4.4.1, we will denote it as  $L_z$ .

The angular momentum is subject to the field equation  $G_{0z} = 8\pi G T_{0z}^M$ , which is, to the leading order (cf. (50) for  $M$ ):

$$\partial_0 L_z = \frac{1}{4} \partial_z (D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u \partial_0 \partial_z M - T_{0z}. \quad (70)$$

Note that  $N_z$  is constrained by momentum density, whilst  $M$  was related to the energy density. We introduce the rescaled momentum density in the  $z$ -direction (cf. (51) for  $T_{00}$ ):

$$T_{0z} := 8\pi G \lim_{r \rightarrow \infty} [r^2 T_{0z}^M] - \frac{1}{4} \partial_z (C_{zz} N^{zz}) - \frac{1}{2} C_{zz} D_z N^{zz}.$$

Thus, once we specify  $M$  and  $C_{zz}$ ,  $L_z$  is immediately defined by (70) up to an integration function. This is fixed by the matching condition  $L_z^{\mathcal{I}^+} = L_z^{\mathcal{I}^-}$ , implying an infinite number of charges, similar to  $Q_\varepsilon$ , but now specified by a vector function  $Y$  [59]:

$$Q_Y^+ = \frac{1}{4\pi G} \int_{\mathcal{I}_+^+} d^2 z Y_{(\bar{z}} L_z) = \frac{1}{4\pi G} \int_{\mathcal{I}_+^-} d^2 z Y_{(\bar{z}} L_z) = Q_Y^-.$$

This is a conservation of superrotation charge, meaning that there must be an infinity of corresponding symmetries. A reasoning leading to superrotations was suggested by G. Barnich and C. Troessart [2, 3]. They stated that several falloff assumptions by H. Bondi et al. [10] are overly restrictive. Let us calculate the Lie derivatives of the metric with respect to the vector field:

$$\mathcal{L}_Y g_{01} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_Y g_{z1} = \mathcal{O}(r^{-1}), \quad \mathcal{L}_Y g_{z\bar{z}} = \mathcal{O}(r), \quad \mathcal{L}_Y g_{00} = \mathcal{O}(r^{-1}), \quad (71a)$$

$$\mathcal{L}_Y g_{\bar{z}\bar{z}} = 2r^2 \gamma_{z\bar{z}} \partial_{\bar{z}} Y^z + \mathcal{O}(r). \quad (71b)$$

The most important equation is (71b). Asymptotic conditions for the Bondi metric are only satisfied if the first term vanishes. It restricts  $Y$  to be a holomorphic vector field, globally confined to the choice  $Y \sim 1, z, z^2$ , generating a Lorentz group with six Killing vectors. All the rest violate (71b) and were therefore discarded in the original paper. However, it happens that this condition is only violated at isolated points. For instance,  $Y^z = \frac{1}{z-\omega}$  only leads to a singularity at  $z = \omega$  and should not be disregarded. Hence, we get infinite symmetries. Interestingly, allowing meromorphic  $Y^z$  was conjectured long ago in 1984 by Belavin et al. [4]. Analytic singularities may, on the one hand, be allowed because the symmetries can then be analysed in a local patch, and singularities outside that patch do not affect local identities. On the other hand, Ward identities of those symmetries provide quick derivations of relations among correlators, which can be independently obtained from other lengthy methods.

### 8.1.2 Subleading Soft Graviton

The subleading soft graviton theorem, similar to the soft graviton theorem in Sec. 6.1.2, is derived from the superrotation charge conservation:

$$\langle \text{out} | Q_Y^+ \mathcal{S} - \mathcal{S} Q_Y^- | \text{in} \rangle = 0.$$

For the full calculations and a diagrammatic proof, refer to [38]. Here, we only present the subleading soft factor for a negative helicity graviton:

$$S^{(1)-} = -i \sum_k \frac{p_a^k \varepsilon^{-ab} q^c J_{cb}^k}{p_k \cdot q},$$

with  $J_{ab}^k = L_{ab}^k + S_{ab}^k$ , where  $L_{ab}^k$  is the orbital angular momentum and  $S_{ab}^k$  is the helicity of the internal spin of the  $k$ -th particle. This subleading soft factor can be obtained from the leading soft factor (61) by replacing  $p^a$  with  $q^a J_{ab}^k$ ; in other words, by replacing translations with rotations about  $q$ . This substitution makes sense because translations were generated by a hard piece  $\varepsilon T_{00}$ , and rotations are generated with  $Y^z T_{0z}$ . One can show that  $p^a \rightarrow q^a J_{ab}^k$  is equivalent to  $\varepsilon T_{00} \rightarrow Y^z T_{0z}$ . This is how the soft formula was guessed before it was properly derived [59].

While we are not proving the formula, we will provide a quick consistency check. For the Weinberg soft theorem, the leading soft factor is:

$$\sum_k \frac{\varepsilon^{ab} p_a^k p_b^k}{p^k \cdot q},$$

which vanishes for pure gauge gravitons, i.e., when  $\varepsilon^{ab} = \Lambda^a q^b$ . With this choice, the soft factor becomes:

$$\sum_k \frac{\Lambda^a q^b p_a^k p_b^k}{p^k \cdot q} = \Lambda^a \sum_k \frac{p_a^k p^k \cdot q}{p^k \cdot q} = \Lambda^a \sum_k p_a^k = 0$$

due to momentum conservation. For consistency, we need the same property to hold for  $p^a \rightarrow q^a J_{ab}^k$ . Since it is not automatically symmetric, we insert

$$\varepsilon^{ab} = q^a \Lambda^b + q^b \Lambda^a$$

and find that

$$q^a \Lambda^b \sum_k J_{ab}^k + \sum_k \frac{p_k \cdot \Lambda q^a q^b J_{ab}^k}{p_k \cdot q} = 0.$$

The first term vanishes due to angular momentum conservation, and the second because of the convolution of the antisymmetric  $J_{ab}^k$  with the symmetric  $q^a q^b$ . Thus, angular momentum for superrotation is analogous to energy-momentum for supertranslations.

### 8.1.3 New Memory Discovery

Finally, the new memory effect related to the superrotational symmetry was discovered [53]. This effect is sourced by asymmetric changes in the angular momentum radiated to null infinity and by changes in the superspin charges, which are the magnetic-parity part of the charges conjugate to the superrotation vector field.

## 8.2 The $h_{30}$

### 8.2.1 Where Memories Dwell

We will now focus on the applications of what was described in Sec. 2 and 4. The first point we address is how to extract memory from the NR-generated waveforms, particularly from the  $h_{lm}$ -decomposition (1). We recall that the full memory effect is stored in the E-mode, described to the linear order by (48). The total expression for the memory in the Bondi framework reads [50]<sup>8</sup>:

$$\Delta \{ \bar{\partial} \bar{\partial} (\bar{\partial} \bar{\partial} + 2) \Sigma_e \} = 4\Delta M + \int_{-\infty}^{+\infty} (16\pi r^2 T_{uu} + \frac{1}{2} N_{AB} N^{AB}) du, \quad (72)$$

with  $\frac{1}{2} N_{AB} N^{AB}$  arising from the non-linear part of the equations, neglected in [46]. The linear memory can be projected out as indicated in [50]. Thus, we only have to decompose the non-linear part into the  $(l, m)$ -modes, taking the multipole moments of (72). We artificially truncate the integral over the retarded time from  $(-\infty; +\infty)$  to some finite time  $u_f$ . This is common in the Post-Newtonian formalism [50]:

$$\Delta \Sigma_e = \frac{(l-2)!}{(l+2)!} \int d^2\Omega \int_{-\infty}^{u_f} du N_{AB} N^{AB} Y_{lm}^*,$$

where  $l \geq 2$ . In Sec. 4.1, we introduced the mass and current radiative moments. We re-express the memory in terms of those moments:

$$\begin{aligned} \Delta \Sigma_{lm}^e &= \frac{1}{2} \frac{(l-2)!}{(l+2)!} \sum_{l', l'', m', m''} C_l(-2, l', m'; 2, l'', m'') \times \\ &\times \int_{-\infty}^{u_f} du \left\{ 2i [1 - (-1)^{l+l'+l''}] \dot{U}_{l'm'} \dot{V}_{l''m''} + [1 + (-1)^{l+l'+l''}] (\dot{U}_{l'm'} \dot{U}_{l''m''} + \dot{V}_{l'm'} \dot{V}_{l''m''}) \right\}, \quad (73) \end{aligned}$$

where

$$\begin{aligned} C_l(s', l', m'; s'', l'', m'') &= (-1)^{l+l'+l''} \sqrt{\frac{(2l'+1)(2l''+1)}{4\pi(2l+1)}} \langle l', s'; l'' s'' | l, s' + s'' \rangle \times \\ &\times \langle l', m'; l'', m'' | l, m' + m'' \rangle. \quad (74) \end{aligned}$$

Here, we have used a bra-ket notation for the Clebsch-Gordon coefficients. Note that for  $l = 0, 1$  the gravitational waves can be gauged away, as they are spin-2. Therefore, the summation is for  $l, l', l'' \geq 2$ . The radiative moments can be related to the  $h_{lm}$ -modes as follows:

$$U_{lm} = \frac{r}{\sqrt{2}} [h_{lm} + (-1)^{-m} h_{l-m}^*], \quad (75a)$$

$$V_{lm} = \frac{ir}{\sqrt{2}} [h_{lm} - (-1)^{-m} h_{l-m}^*]. \quad (75b)$$

<sup>8</sup>Note that the expression by D. Nichols [50] is double the expression we deduced from [46]. In the end, Nichols ends up having a factor 2 discrepancy, which we avoid.

Let us analyse the result. First, it is known that the non-oscillatory part can be completely housed in the  $m = 0$  waveforms by choosing a standard polarisation triad [26]. Therefore, we start to look for the memory from there. Second, due to (75),  $V_{l0}$  are always identically vanishing and only  $U_{l0}$  survive. Hence,  $\dot{V}_{l0} = 0$  and for our memory (73):

$$\Delta\Sigma_{lm}^e = \frac{1}{2} \frac{(l-2)!}{(l+2)!} \sum_{l', l'', m', m''} \mathcal{C}_l(-2, l', m'; 2, l'', m'') \int_{-\infty}^{u_f} du \left\{ [1 + (-1)^{l+l''}] \dot{U}_{l'm'} \dot{U}_{l''m''} \right\}.$$

Third, as verified by M. Favata [26] after lengthy computations, the major contribution to memory is the  $(l, m) = (2, 2)$ -mode. For this dominant contribution, we set  $l' = l'' = 2$  and  $m' = -m'' = 2$  to avoid vanishing results, as only  $m = m' + m''$  terms contribute to the Clebsch-Gordon coefficients. Thus, for any odd  $l$  we get an identically vanishing result, and the displacement memory turns out to only live in  $(2n, 0)$  modes, which are always purely real. Note that due to (43), it is obvious that smaller  $l$  dominate due to the factor  $c^{l+2}$  in the denominator.

A totally analogous expression can be deduced for the spin memory [50], yielding:

$$\begin{aligned} \Delta\Phi_{lm} = & \frac{1}{4\sqrt{l(l+1)}} \frac{(l-2)!}{(l+2)!} \sum_{l', l'', m', m''} \left[ 2\sqrt{(l'-1)(l'+2)} \mathcal{C}_l(-1, l', m'; 2, l'', m'') + \right. \\ & \left. + \sqrt{(l''-2)(l''+3)} \mathcal{C}_l(-2, l', m'; 3, l'', m'') \right] \times \\ & \times \int_{-\infty}^{u_f} du \left\{ i[1 - (-1)^{l+l''}] (U_{l'm'} \dot{U}_{l''m''} - \dot{U}_{l'm'} U_{l''m''} + V_{l'm'} \dot{V}_{l''m''} - \dot{V}_{l'm'} V_{l''m''}) + \right. \\ & \left. + [1 + (-1)^{l+l''}] (U_{l'm'} \dot{V}_{l''m''} + \dot{V}_{l'm'} U_{l''m''} - \dot{U}_{l'm'} V_{l''m''} - V_{l'm'} \dot{U}_{l''m''}) \right\}. \end{aligned}$$

Next, we again erase all current multipoles, as they are vanishing for non-oscillatory waveforms:

$$\begin{aligned} \Delta\Phi_{lm} = & \frac{1}{4\sqrt{l(l+1)}} \frac{(l-2)!}{(l+2)!} \sum_{l', l'', m', m''} \left[ 2\sqrt{(l'-1)(l'+2)} \mathcal{C}_l(-1, l', m'; 2, l'', m'') + \right. \\ & \left. + \sqrt{(l''-2)(l''+3)} \mathcal{C}_l(-2, l', m'; 3, l'', m'') \right] \int_{-\infty}^{u_f} du \left\{ i[1 - (-1)^{l+l''}] (U_{l'm'} \dot{U}_{l''m''} - \dot{U}_{l'm'} U_{l''m''}) \right\}. \end{aligned}$$

Contrary to what we got for the displacement memory, here, only odd  $l$  components survive. Thus, the spin memory dwells in  $(2n+1, 0)$  waveforms, with the biggest contribution residing in the lowest odd mode,  $h_{30}$ .

## 8.2.2 Behavioural Patterns

Let us investigate the behaviour of the corresponding mode. Given the detections of black hole binaries by LIGO-Virgo [1], we will focus on compact binary black holes, the most promising source for memory detection. The waveforms of these binaries are characterised by several intrinsic parameters, such as their proper spins and masses (or, equivalently, the total mass and the mass ratio) and their orbital properties, like eccentricity and the initial frequency of the binary. The extrinsic parameters include the inclination of the binary and the distance to it.

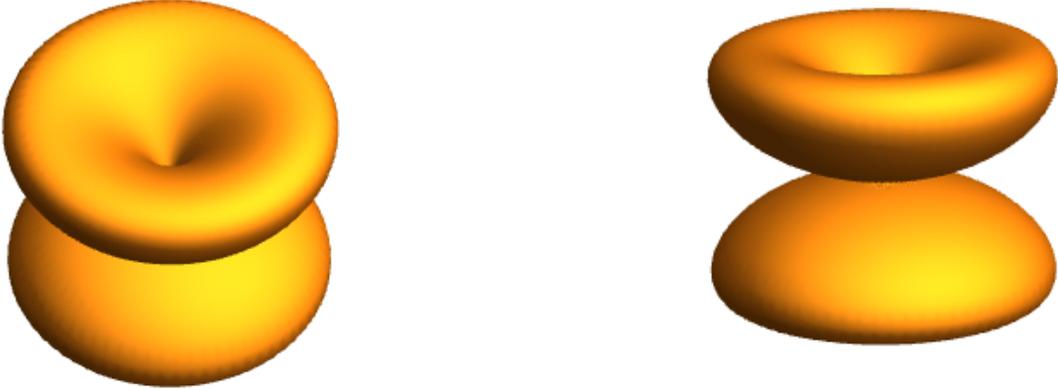


Figure 10:  ${}_{-2}Y_{30}$  spin-weighted spherical harmonic [48].

To model the behaviour of the binaries, we will use the NRHybSur3dq8\_CCE model [72]. It allows for flexible parameter choices; however, it is restricted to proper spins aligned with the angular momentum. We will confine ourselves to non-eccentric orbits with aligned spins  $s_{1z}$  and  $s_{2z}$ , as these otherwise require accounting for several precession contributions. We plan to return to them later.

The angular dependence of the spin memory is entirely encoded in  ${}_{-2}Y_{30}$ , depicted in Fig. 10. We fix some particular angles and compare the displacement memory mode  ${}_{-2}Y_{20}$  with the spin one.

Let us first examine the behaviour of the  $h_{30}$  mode. Like any odd mode, it is purely imaginary, in contrast to the purely real even modes. This suggests that displacement memory is related to the  $h_+$  components, and spin memory to the  $h_\times$ , which we will elaborate on in more detail in Sec. 10.5. A sample plot of this mode is shown in Fig. 11, which is generated for the same system as the displacement memory in Fig. 4, at a distance of 100 Mpc.

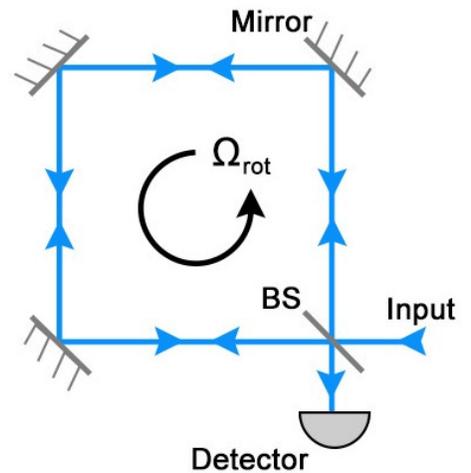


Figure 9: A scheme of Sagnac interferometer response, taken from [32].

In the displacement case, it is clearly seen that the amplitude saturates to a certain value and remains non-zero after the passage of the wave. This is different for the spin memory. The reason behind this is that Michelson-type interferometers are constructed so that their mirrors can be displaced but not rotated. Therefore, they can maintain a displacement after the wave passage but not a rotation; the spin memory then appears as a sudden burst. In contrast, Sagnac-type interferometers (see Fig. 9) are suitable for spin memory detection [50, 53] and can accumulate a

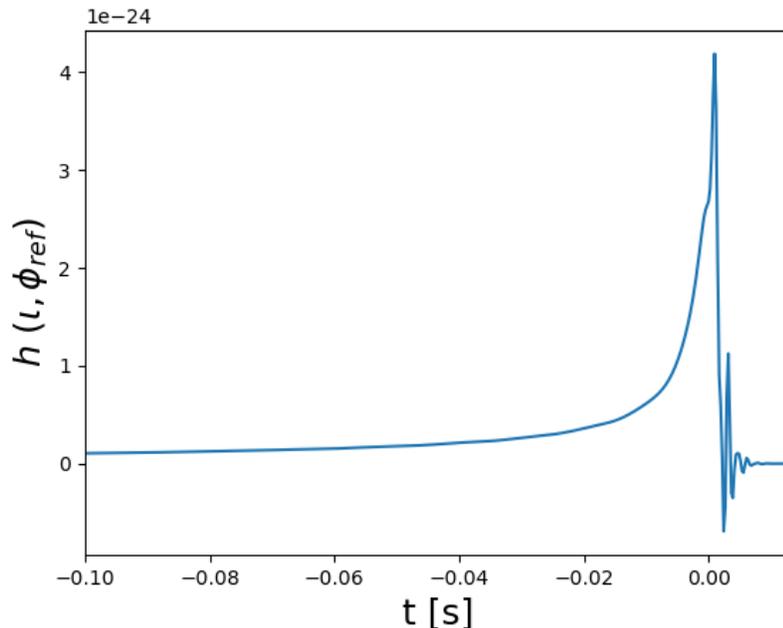


Figure 11: Spin memory residing in  $h_{\times}^{30}$  mode.

rotation angle instead of a linear displacement. When modelling for Sagnac interferometers, a burst with a subsequent decay will represent the displacement memory, while the spin memory will result in a permanent rotation.

We will now perform several amplitude comparisons for the two types of memory. A very simple plot illustrating both memories together can be found in Fig. 12, which indicates that they differ by several orders of magnitude, in accordance with (43).

For a more thorough and explicit comparison, we will explore the parameter dependencies of the two modes. We will model waveforms for binaries with varying  $m_1$  and  $m_2$  from within  $[3, 25]M_{\odot}$ , as long as the model is well-trained on  $m_1/m_2 \in [1, 8]$  [72]. Then, we will plot the maximal values for  $Y_{20}$  and  $Y_{30}$  modes<sup>9</sup> in Fig. 13. In the second plot (Fig. 14), we will vary the  $z$ -components of the spins. The model allows for  $s_z \in [-0.8, 0.8]$  [72], which we will examine for equal mass black holes with  $m_1 = m_2 = 30M_{\odot}$ .

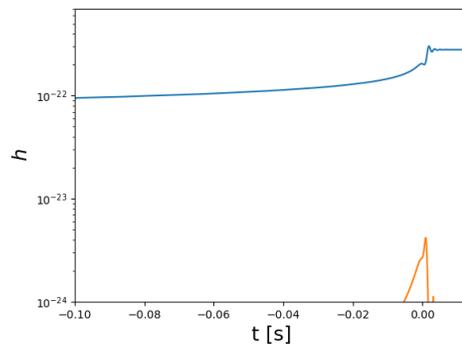


Figure 12: Displacement (blue) and spin (orange) memory magnitude comparison.

As expected, the mass ratio's influence on both memories is identical (see Fig. 13): with higher masses, the system curves the spacetime more strongly. As a result, the gravitational waves have a higher amplitude and produce a greater memory effect. Moreover, we observe that nearly-equal-mass binaries produce more memory, as they are capable of generating greater bursts when passing the peribothron.

<sup>9</sup>For spin memory, the root mean square is a much more illustrative property. However, for displacement memory, it will keep accumulating indefinitely. Therefore, we will use the peak (maximal) values for both.

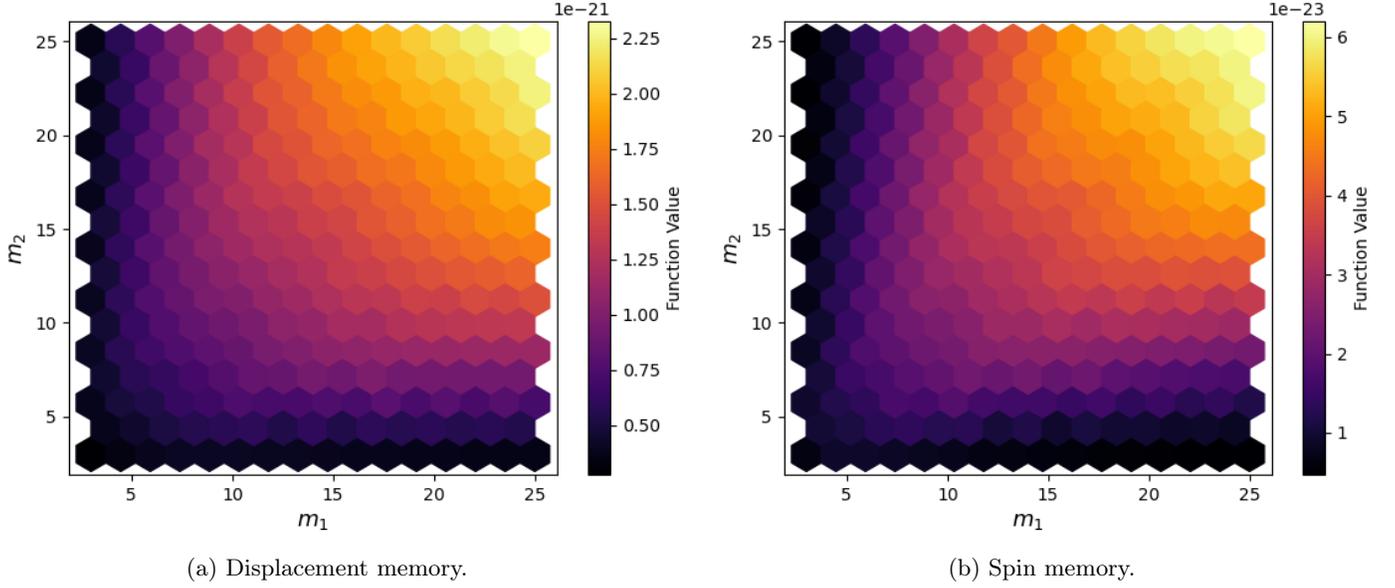


Figure 13: Peak amplitudes comparison for binaries of varying masses for different memories.

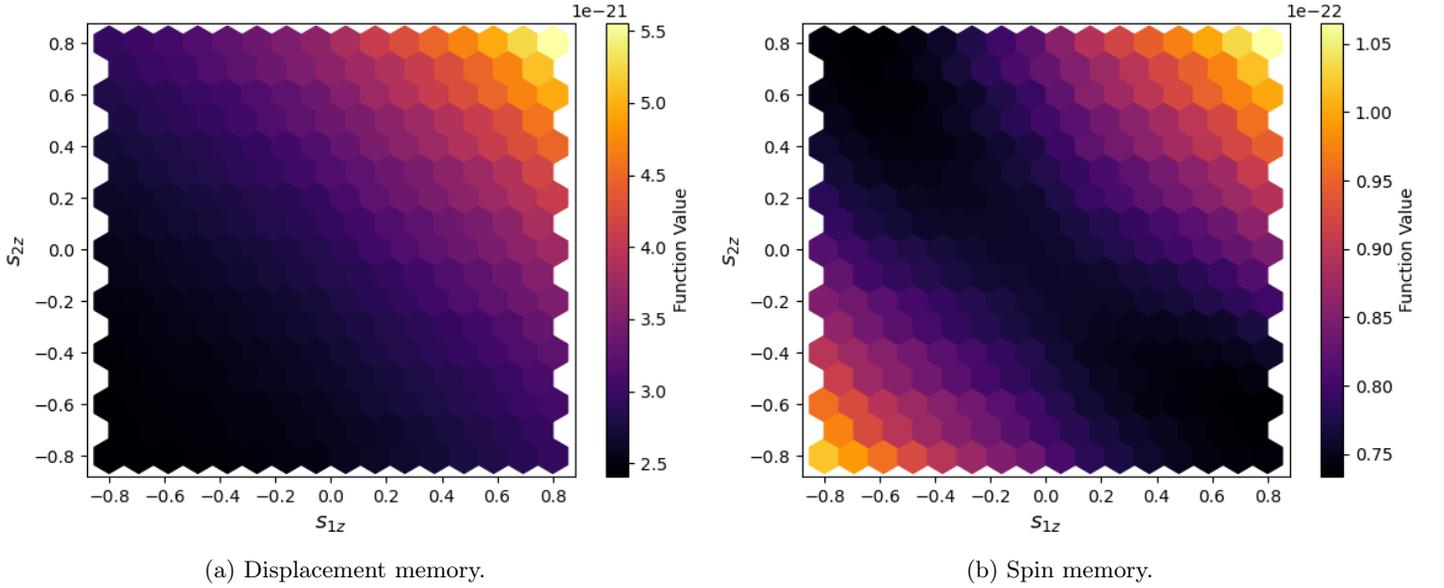


Figure 14: Peak amplitudes comparison for binaries of varying spins for different memories.

The spin dependence, however, significantly distinguishes between the two. This is primarily due to the different generation mechanisms: displacement memory results from energy being carried away, whereas spin memory arises from the emitted angular momentum flux. Positive spins, directed along the orbital angular momentum, enhance the system's total energy and, therefore, lead to stronger wave emissions. Negative proper spins reduce the gravitational wave emissions and produce a smaller displacement memory. At the same time, spin memory grows with any increase in  $|s_z|$ , although it is slightly less for negative spins.

## 9 Introduction V

The spin memory effect causes a system to rotate after a gravitational wave passes. We have also demonstrated that spinning sources produce a larger spin memory. An opposite question arises: what will happen to a gyroscope, i.e., an already spinning object after a gravitational wave hits it?

Recently, this effect was studied by A. Seraj and B. Oblak [57, 58]. They also discovered an analogous electromagnetic memory soon thereafter [51]. By considering the motion of a freely-falling gyroscope, similar to the usual study of the Lense–Thirring effect [36], they pointed out a suitable choice of the tetrad and found that the axis of a gyroscope will tilt, causing it to precess. This effect appears already at the order  $\mathcal{O}(r^{-2})$ , in contrast to the Lense–Thirring precession, which comes in at  $\mathcal{O}(r^{-3})$ . This makes the orientation memory a leading-order correction to the Lense–Thirring effect. Simultaneously, the spin memory contribution described in Sec. 8 was obtained. Thus, gyroscopic memory can be seen as a generalisation or a correction to spin memory.

Physically, this effect will induce a precession in a black hole binary (and an additional incline for spinning black holes) or cause a tilt in a pulsar. However, the effect is incredibly tiny. Spin memory decays faster than displacement memory ( $r^{-2}$  vs  $r^{-1}$ ). A simple order-of-magnitude estimation provided in [58] suggests that the tilt angle

$$\Phi \sim \frac{G^2 M^2}{c^4 r^2} \simeq 2 \times 10^{-39} \left( \frac{M/M_\odot}{r/1 \text{ Mpc}} \right)^2.$$

Common masses and distances (e.g., LIGO observations [1]) are  $M \simeq 30M_\odot$  and  $r \simeq 400 \text{ Mpc}$ .

It is still possible that one day gyroscopic memory will be observed in extreme events, such as supermassive black hole mergers, where angles around  $\Phi \simeq 10^{-26}$  rad are possible [58]. Pulsar tilts are another potent candidate for measuring orientation memory. For instance, if a pulsar is close to a violent merger and gains a tilt of  $\Phi \simeq 10^{-26}$ , an idealised narrow beam emitted  $10^3$  light-years away from the Solar System will shift the position of the resulting light spot on Earth by around  $10^{-5}$  cm.

## 10 Gyroscopic Memory

### 10.1 Response of a Freely-Falling Gyroscope

#### 10.1.1 Precession Equation

We start by studying the motion of a gyroscope near null infinity in asymptotically flat spacetime described by the Bondi metric (2), following [57]. Consider a small<sup>10</sup> gyroscope with proper velocity  $\mathbf{u} = u^a \partial_a$  and proper spin  $\mathbf{S} = S^a \partial_a$ , so that  $S^a u_a = 0$ . A freely falling gyroscope follows a parallel transport equation  $\mathbf{u} \cdot \nabla^a \mathbf{S} = 0$ .

Let an observer carrying the gyroscope measure its orientation relative to some local tetrad  $\mathbf{e}_{\hat{a}} = e_{\hat{a}}^b \partial_b$ . These are orthonormal in the sense that  $\mathbf{e}_{\hat{a}} \mathbf{e}_{\hat{b}} = \eta_{\hat{a}\hat{b}}$ . Thus, we write the spin vector as  $\mathbf{S} = S^{\hat{a}} \mathbf{e}_{\hat{a}}$ , and the parallel transport equation becomes:

$$\frac{dS^{\hat{a}}}{d\tau} = -u^b \omega_b^{\hat{a}\hat{c}} S_{\hat{c}}, \quad (76)$$

with  $u^{\hat{a}} = e_{\hat{b}}^{\hat{a}} u^b$  being the components of the proper velocity in the observer's frame. The spin connection one-form associated with the tetrad  $\{\mathbf{e}_{\hat{a}}\}$  is defined as:

$$\omega_a^{\hat{b}\hat{c}} := e_{\hat{d}}^{\hat{b}} \nabla_a e^{\hat{c}d} = e_{\hat{d}}^{\hat{b}} (\partial_a e^{\hat{c}d} + \Gamma_{ea}^d e^{\hat{c}e}). \quad (77)$$

Equation (76) states that the precession is fully determined by the spin connection. Thus, its rate is independent of the properties of the gyroscope. This may be seen as a manifestation of the equivalence principle, and would not be the case for, e.g., magnetic dipoles [51].

Note that we always choose frames adapted to the observer so that  $\mathbf{e}_{\hat{0}} = \mathbf{u}$ . In that case, the spin vector is purely spatial and (76) becomes:

$$\frac{dS^{\hat{A}}}{d\tau} = \Omega_{\hat{B}}^{\hat{A}} S^{\hat{B}}, \quad \Omega^{\hat{A}\hat{B}} = -u^a \omega_a^{\hat{A}\hat{B}} \quad (78)$$

where we introduce indices  $\mathcal{A}, \mathcal{B}, \dots = 1, 2, 3$  running through spatial coordinates  $(r, x^A)$ . This new equation means that the observer's acceleration does not contribute. The antisymmetric tensor  $\Omega^{\hat{A}\hat{B}}$  thus yields the gyroscope's precession rate. For convenience, we can dualise it into a vector:

$$\boldsymbol{\Omega} = \Omega^{\hat{A}} \mathbf{e}_{\hat{A}} = -\frac{1}{2} \epsilon^{\hat{A}\hat{B}\hat{C}} \Omega_{\hat{B}\hat{C}} e_{\hat{A}},$$

with  $\epsilon^{\hat{A}\hat{B}\hat{C}}$  being the usual Levi-Civita symbol. In this way, (78) is simply a Euclidean cross product:

$$\dot{\mathbf{S}} = \boldsymbol{\Omega} \times \mathbf{S}.$$

To determine the gyroscope's motion, we thus need to choose a velocity  $\mathbf{u}$  and a local frame to compute the spin connection (77). We will build two tetrad sets and explore a freely falling and a Bondi gyroscope, as described in Sec. 4.4.3.

<sup>10</sup>This allows us to neglect self-gravity and utilise the parallel transport equation.

## 10.2 Tetrad Choice

### 10.2.1 Source-oriented Frame

Whether or not the observer follows the geodesics, their proper velocity in Bondi coordinates is:

$$\mathbf{u} = \frac{dx^a}{d\tau} \partial_a = \gamma(\partial_0 + v^1 \partial_1 + v^A \partial_A),$$

where the gamma factor [57]:

$$\gamma = \frac{du}{d\tau} = \left\{ e^{2\beta} \left[ -\frac{V}{r} + 2v^1 \right] - r^2 \gamma_{AB} \left[ v^A - \frac{U^A}{r^2} \right] \left[ v^B - \frac{U^B}{r^2} \right] \right\}^{-\frac{1}{2}}$$

normalises the proper velocity with respect to the metric. For a geodetic observer, one may verify through a lengthy but straightforward calculation (approach of Sec. 4.3 or [57]) that:

$$\gamma = 1 + \frac{M(u_0)}{r} + \frac{\gamma^{(2)}}{r^2}, \quad \gamma^{(2)} = \int_{u_0}^u du' M + \frac{1}{16} \Delta(C_{AB} C^{AB}), \quad (79a)$$

$$v^1 = \frac{\Delta M}{r} - \frac{1}{r^2} \left[ \gamma^{(2)} + \Delta \left\{ \frac{1}{6} D_A L^A + \frac{1}{8} D_B C^{AB} D^C C_{AB} \right\} \right], \quad (79b)$$

$$v^A = -\frac{1}{2r^2} D_B \Delta C^{AB} + \frac{1}{r^3} (D^A \gamma^{(2)} - \frac{2}{3} \Delta L^A + \frac{1}{2} C^{AB} D^C \Delta C_{BC}). \quad (79c)$$

The static observer ‘moves’ with  $\mathbf{u} = \gamma \partial_0$ .

Next, we build a local tetrad  $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_{\hat{A}}\}$  adapted to the observer. We further demand that it is tied to the source, i.e., that  $\mathbf{e}_1$  is aligned with the light rays emitted from the origin. To obtain the radial tetrad vector, recall from Sec. 2.2 that  $k^a = \mathcal{R} \partial_1 x^a$  (equivalently,  $\mathbf{k} = \mathcal{R} \partial_1$ ) is tangent to outgoing null rays. Hence,  $\mathbf{e}_1$  coincides with it up to the  $\mathbf{e}_0$  component, which is projected out:

$$\mathbf{e}_1 = \frac{1}{\gamma} \mathbf{k} - \mathbf{u}.$$

It remains to fix the angular tetrads  $\mathbf{e}_{\hat{A}}$ . We make use of the following facts:

- Orthogonality to  $\mathbf{e}_0$  and  $\mathbf{e}_1$ , for some linearly independent  $q_{\hat{A}}$  tangent to the sphere, implies:

$$\mathbf{e}_{\hat{A}} = \frac{q_{\hat{A}}^B}{r} [\partial_B + \gamma_{BC} (r^2 v^C - U^C) \mathbf{k}].$$

- Mutual orthogonality  $\mathbf{e}_{\hat{A}} \mathbf{e}_{\hat{B}} = \delta_{\hat{A}\hat{B}}$  means that  $q_{\hat{A}}$  form an orthonormal dyad with respect to the metric  $\gamma_{AB}$  so that  $\gamma_{AB} q_{\hat{A}}^A q_{\hat{B}}^B = \delta_{\hat{C}\hat{D}}$ .
- As long as this fixes the angular vectors up to local rotations [57], we further require that the angular tetrad vectors point in the same direction at all radii, i.e.,  $(\mathbf{k} \cdot \nabla \mathbf{e}_{\hat{A}})^B = 0$ . Together with expansion (37), this implies:

$$q_{\hat{A}}^B \sim \bar{q}_{\hat{A}}^C \left[ \delta_C^B - \frac{1}{2r} C_C^B + \frac{1}{8r^2} C_{CD} C^{BD} \right].$$

Here,  $\bar{q}_{\hat{A}}(x^B)$  is a time-independent dyad on the celestial sphere.

### 10.2.2 Frame Tied to Distant Stars

The frame we have built is adequate for many applications, including the study of radiative effects at  $\mathcal{O}(r^{-1})$ . One can readily compute the corresponding spin connections (77) and deduce a precession rate. However, there is a significant drawback that makes the frame unsuitable for our purposes: it mixes the precession contributions from gravitational waves with geodetic precession caused by the motion of the gyroscope on the asymptotic sphere. This precession arises because a source-oriented frame must continuously rotate to align with the radial rays, regardless of the presence of radiation. Nevertheless, we can single out the precession caused by the gravitational waves by transforming to a new frame, ‘tied to distant stars’ [57], whose existence is guaranteed by the asymptotic flatness of spacetime.

The spin connection evaluated at the null infinity of Bondi spacetime reads:

$$\bar{\omega}_{\hat{1}\hat{A}} = -\hat{q}_{\hat{A}B} dx^B, \quad \bar{\omega}_{\hat{A}\hat{B}} = -\hat{q}_{\hat{A}}^C D_D \bar{q}_{\hat{B}C} dx^D,$$

where barred quantities are background; a free gyroscope with angular velocity  $v^A$  thus precesses at a rate  $-v^A \bar{\omega}_{\hat{A}\hat{B}\hat{C}}$  with respect to the source-oriented frame due to the tetrad rotation and makes the spin connections non-zero. Therefore, we aim to construct a tetrad whose spin connection vanishes in flat space, which should hold for generic asymptotic spacetimes. We tie this frame to ‘distant stars’ at an effectively infinite distance from the observer and a large but finite distance from the source.

In practice, the desired frame can be derived from the source-oriented tetrad by a celestially local rotation, cancelling the counterfeit time-dependent rotation of  $\{\mathbf{e}_{\hat{a}}\}$ . Thus,

$$\mathbf{e}_{\hat{0}} := \mathbf{f}_{\hat{0}} = \mathbf{u}, \quad \mathbf{f}_{\hat{A}} := R_{\hat{A}}^{\hat{B}}(x^C) \mathbf{e}_{\hat{B}}.$$

The rotation matrix  $R_{\hat{A}}^{\hat{B}}(x^C)$  only depends on the angular components and may be chosen to be the identity at some reference point, say, the initial angular position of the observer. This change of the tetrads implies a modification of the spin connection, which follows a standard one-form transformation law:

$$\omega'_{\hat{A}\hat{B}} = R_{\hat{A}}^{\hat{C}} R_{\hat{B}}^{\hat{D}} \omega_{\hat{C}\hat{D}} + R_{\hat{A}}^{\hat{C}} dR_{\hat{B}\hat{C}}. \quad (80)$$

A rotation that yields a frame with a vanishing spin connection  $\omega'$  in Minkowski space is given by a path-ordered exponential  $R(x^A) \propto \exp \int_{x_0^A}^{x^A} \bar{\omega}$  [57]. The curve connecting  $x_0^A$  to  $x^A$  is arbitrary, but its choice affects the value of  $R$  due to the non-zero curvature of  $S^2$ . The choice is ultimately irrelevant, as for every pair  $(x_0^A, x^A)$  in the neighbourhood of the observer’s initial position, one has to pick any convenient path. We thus assume a choice for which  $R(x_0^A) = \mathbb{I}$ .

As for the geometric justification, we suppose that the source-oriented frame, centred at  $x_0^A$  at  $u_0$ , points towards three stars at infinity, and one of them is aligned with the source’s direction. With the flow of time, the source-oriented frame rotates time-dependently to keep pointing towards the source. As it rotates, the observer’s position on the celestial sphere changes from  $x_0^A$  to some  $x^A$ . Performing the rotation for  $R(x^A)$  at every tick of their motion, their frame reorients to be aligned with the stars again, regardless of any time-dependent quantities. Hence, we refer to the tetrad  $\{\mathbf{f}_{\hat{a}}\}$  as being ‘tied to distant stars’ at infinity.

### 10.2.3 Spin Connections

We now write down the spin connections for the tetrad tied to distant stars, using the transformation (80). The actual components depend on the choice of the worldline. Using the velocity expressions (79a) at large  $r$ , one derives the spin connections for a geodetic observer [57]:

$$\omega'_{\hat{1}\hat{A}} \sim \bar{q}_{\hat{A}}^B \left[ \frac{1}{4r^2} N_{BC} D_D C^{CD} du + \frac{1}{2r^2} D^C C_{BC} dr + \frac{1}{2} N_{BC} dx^C \right], \quad (81a)$$

$$\omega'_{\hat{A}\hat{B}} \sim -\frac{1}{2r^2} \bar{q}_{\hat{A}}^C \bar{q}_{\hat{B}}^D \left[ D_{[C} D^E C_{D]E} - \frac{1}{2} N_{E[C} C_{D]}^E \right] du + \mathcal{O}(r^{-3}) dr + \mathcal{O}(r^{-1}) dx^C. \quad (81b)$$

We employed an initial condition  $C_{AB}(u \rightarrow -\infty) = 0$  to slightly simplify the expressions. Note that mass and angular momentum aspects do not contribute to the spin connection to this order. The Lense–Thirring effect is hidden in the subsequent, implicitly neglected corrections.

For a static observer with  $\mathbf{u} = \gamma \partial_0$ :

$$\omega'_{\hat{1}\hat{A}} \sim \bar{q}_{\hat{A}}^B \left[ \frac{1}{2r} D^C N_{BC} du + \mathcal{O}(r^{-3}) dr + \frac{1}{2} N_{BC} dx^C \right], \quad (82a)$$

$$\omega'_{\hat{A}\hat{B}} \sim -\frac{1}{2r^2} \bar{q}_{\hat{A}}^C \bar{q}_{\hat{B}}^D \left[ D_{[C} D^E C_{D]E} - \frac{1}{2} N_{E[C} C_{D]}^E \right] du + \mathcal{O}(r^{-3}) dr + \mathcal{O}(r^{-1}) dx^C, \quad (82b)$$

again up to neglected subleading orders. Note that the  $\hat{A}\hat{B}$  components are identical for both cases, whereas the  $\hat{1}\hat{A}$   $u$ -components are at different orders of  $\frac{1}{r}$ .

These results are crucial for gyroscopic memory studies. They determine the precession rate according to (78), where  $u^a$  is either a geodetic (with (81)) or a static (with (82)) observer velocity. In the next sections, we will study these more precisely.

## 10.3 Precession

### 10.3.1 Dual Mass Aspect

We now examine the precession, focusing primarily on freely falling observers. In this scenario, the gyroscope's motion is restricted to a plane transverse to the source's direction.

Let us define a quantity:

$$\tilde{\mathcal{M}} = \frac{1}{4} D_A D_B \tilde{C}^{AB} - \frac{1}{8} N_{AB} \tilde{C}^{AB}. \quad (83)$$

Then, we rewrite:

$$\Omega_{\hat{1}\hat{A}} = \mathcal{O}(r^{-3}), \quad \Omega_{\hat{A}\hat{B}} = \frac{\epsilon_{\hat{A}\hat{B}}}{r^2} \tilde{\mathcal{M}} + \mathcal{O}(r^{-3}). \quad (84)$$

Here,  $\epsilon_{\hat{A}\hat{B}}$  is the antisymmetric surface area tensor on a unit sphere, as in Sec. 2.4. In particular, we defined the dual shear tensor  $\tilde{C}_{AB} = \epsilon_{CA} C_B^C$ .

Equations (84) state that to the leading order, the gyroscope only rotates in the plane, tangent to the celestial sphere at the observer's location. The precession frequency is then defined by  $\tilde{\mathcal{M}}$ , the *dual covariant mass aspect* [29], named after the usual<sup>11</sup> covariant mass aspect

<sup>11</sup>Note that the Weyl scalar  $\Psi_2 = \mathcal{M} + i\tilde{\mathcal{M}}$ .

$\mathcal{M} = M + \frac{1}{8}N_{AB}C^{AB}$ . The term  $\frac{1}{8}N_{AB}C^{AB}$  is, on the one hand, needed so that the (dual) mass aspect transforms under BMS transformations without an inhomogeneous term. On the other hand, it is a generator of the local duality of the gravitational phase space [57]. We also stress that  $\tilde{\mathcal{M}}$  has nothing to do with the mass directly and is instead related to the multipole moments of gravitational angular momentum. Thus, mass and spin are mutually dual in general relativity.

To illustrate the relation between the dual mass aspect and the angular momentum, we decompose the quantities into + and - parity components:

$$L_A = D_A L^+ + \epsilon_{AB} D^B L^-, \quad C_{AB} = D_{(A} D_{B)} C^+ + \epsilon_{C(A} D_{B)} D^C C^-, \quad (85)$$

where superscripts  $\pm$  denote parity eigenvalues of the corresponding functions. The angular brackets refer to a symmetric trace-free projection, i.e.,  $D_{(A} D_{B)} = D_{(A} D_{B)} - \frac{1}{2}q_{AB} D^C D_C$ . Using this, it is straightforward to see that the precession rate only has the odd parity:

$$\tilde{\mathcal{M}} = \frac{1}{8}D^2(D^2 + 2)C^- - \frac{1}{8}N_{AB}\tilde{C}^{AB}.$$

Considering the evolution equation (32), we immediately notice that  $\frac{1}{8}D^2(D^2 + 2)C^- = \partial_0 L^- + \mathcal{G}^-$ , where the latter is defined from  $\mathcal{G}^- = (D^a \mathcal{G}_a)^- = D^a(-\frac{1}{4}D_B[N^{BC}C_{AC}] - \frac{1}{2}D_B N^{BC}C_{AC})^-$  in a decomposition analogous to (85). Thus, the dual mass aspect is finally recast as:

$$\tilde{\mathcal{M}} = \dot{L}^- + \mathcal{G}^- - \frac{1}{8}N_{AB}\tilde{C}^{AB}.$$

An immediate consequence of this form is the absence of precession in non-radiative spacetimes, where the angular momentum aspect is constant, and the fluxes vanish since they are proportional to the news.

## 10.4 Gyroscopic Memory

Any finite burst of radiation leaves a permanent imprint on the gyroscope's orientation. To compute this effect, we start with the precession equation (78) and write the solution as a time-ordered exponential of  $\Omega$  (84) acting on the initial spin vector:

$$\Delta S^{\hat{1}} = \mathcal{O}(r^{-3}), \quad \Delta S^{\hat{A}} = \Phi \epsilon^{\hat{A}\hat{B}} S_{\hat{b}}(u_0) + \mathcal{O}(r^{-3}).$$

The net rotation angle  $\Phi = \bar{\Phi}r^{-2}$  decays as  $r^{-2}$  with:

$$\bar{\Phi} = \frac{1}{4}D_A D_B \int du \tilde{C}^{AB} - \frac{1}{8} \int du N_{AB} \tilde{C}^{AB}. \quad (86)$$

## 10.5 Explicit Calculations

We now provide, to our knowledge for the first time, an explicit calculation of the gyroscopic memory effect, deriving an expression suitable for its modelling. Furthermore, this expression will allow us to perform a straightforward analysis of the 'soft' and 'hard' (linear and non-linear) parts of  $\Phi$  (86).

### 10.5.1 Weighted Levi-Civita $\epsilon^{AB}$

We start with the calculations of the dual shear tensor. We define the dyads<sup>12</sup> as in Sec. 2.4.5,  $q^A = (1, \frac{i}{\sin\theta})$ . The antisymmetric surface area tensor on a unit sphere is defined as (see Sec. 2.4.6):

$$\epsilon^{AB} = i(q^A \bar{q}^B - \bar{q}^A q^B).$$

Due to its symmetries (or by direct computations), we write  $\epsilon^{\theta\theta} = \epsilon^{\phi\phi} = 0$  and  $\epsilon^{\theta\phi} = -\epsilon^{\phi\theta}$ . Hence, we only calculate one component:

$$\epsilon^{\theta\phi} = i(q^\theta \bar{q}^\phi - \bar{q}^\phi q^\theta) = i\left(1 \cdot \frac{-i}{\sin\theta} - 1 \cdot \frac{i}{\sin\theta}\right) = \frac{1}{\sin\theta}.$$

### 10.5.2 Dual Shear $\tilde{C}^{AB}$

By definition, the dual shear tensor equals:

$$\begin{aligned} \tilde{C}^{AB} &= \epsilon^{AX} C_X^B = \epsilon^{AX} q^{BY} C_{XY} = \\ &= \epsilon^{A\theta} q^{BY} C_{\theta Y} + \epsilon^{A\phi} q^{BY} C_{\phi Y} = \epsilon^{A\theta} q^{B\theta} C_{\theta\theta} + \epsilon^{A\theta} q^{B\phi} C_{\theta\phi} + \epsilon^{A\phi} q^{B\theta} C_{\phi\theta} + \epsilon^{A\phi} q^{B\phi} C_{\phi\phi}. \end{aligned}$$

Here, we stress that indices are raised and lowered with the asymptotically flat  $q^{AB}$  metric, which does not involve the non-linear corrections  $\sim \frac{C_{AB}}{r}$ . Now, we calculate the dual shear tensor. We use the explicit components of  $\epsilon^{AB}$  and  $q^{AB}$  to identify vanishing terms immediately. Componentwise:

$$\tilde{C}^{\theta\theta} = \epsilon^{\theta\theta} q^{\theta\theta} C_{\theta\theta} + \epsilon^{\theta\theta} q^{\theta\phi} C_{\theta\phi} + \epsilon^{\theta\phi} q^{\theta\theta} C_{\phi\theta} + \epsilon^{\theta\phi} q^{\theta\phi} C_{\phi\phi} = 0 + 0 + \frac{1}{\sin\theta} \cdot 1 \cdot C_{\phi\theta} + 0 = \frac{1}{\sin\theta} C_{\phi\theta},$$

$$\tilde{C}^{\theta\phi} = \epsilon^{\theta\theta} q^{\phi\theta} C_{\theta\theta} + \epsilon^{\theta\theta} q^{\phi\phi} C_{\theta\phi} + \epsilon^{\theta\phi} q^{\phi\theta} C_{\phi\theta} + \epsilon^{\theta\phi} q^{\phi\phi} C_{\phi\phi} = 0 + 0 + 0 + \frac{1}{\sin\theta} \cdot \frac{1}{\sin^2\theta} C_{\phi\phi} = \frac{1}{\sin^3\theta} C_{\phi\phi},$$

$$\tilde{C}^{\phi\theta} = \epsilon^{\phi\theta} q^{\theta\theta} C_{\theta\theta} + \epsilon^{\phi\theta} q^{\theta\phi} C_{\theta\phi} + \epsilon^{\phi\phi} q^{\theta\theta} C_{\phi\theta} + \epsilon^{\phi\phi} q^{\theta\phi} C_{\phi\phi} = -\frac{1}{\sin\theta} \cdot 1 \cdot C_{\theta\theta} + 0 + 0 + 0 = -\frac{1}{\sin\theta} C_{\theta\theta},$$

$$\tilde{C}^{\phi\phi} = \epsilon^{\phi\theta} q^{\phi\theta} C_{\theta\theta} + \epsilon^{\phi\theta} q^{\phi\phi} C_{\theta\phi} + \epsilon^{\phi\phi} q^{\phi\theta} C_{\phi\theta} + \epsilon^{\phi\phi} q^{\phi\phi} C_{\phi\phi} = 0 - \frac{1}{\sin\theta} \cdot \frac{1}{\sin^2\theta} C_{\theta\phi} + 0 + 0 = -\frac{1}{\sin^3\theta} C_{\theta\phi}.$$

### 10.5.3 Linear Term $D_A D_B \tilde{C}^{AB}$

We will utilise the definition for strain (19) to extract + and  $\times$  components:

$$r(h_+ + ih_\times) = \frac{1}{2}\left(C_{\theta\theta} - \frac{C_{\phi\phi}}{\sin^2\theta}\right) + i\frac{C_{\theta\phi}}{\sin\theta} \quad \Rightarrow \quad \begin{cases} h_+ = \frac{1}{2r}\left(C_{\theta\theta} - \frac{C_{\phi\phi}}{\sin^2\theta}\right), \\ h_\times = \frac{1}{r} \frac{C_{\theta\phi}}{\sin\theta}. \end{cases} \quad (87)$$

We now calculate the covariant derivatives of the dual shear tensor with respect to the angular metric  $q^{AB}$ . The covariant derivative of a (2,0)-tensor is defined as:

$$D_A C^{BC} = \partial_A C^{BC} + \Gamma_{AX}^B C^{XC} + \Gamma_{AX}^C C^{BX}.$$

<sup>12</sup>We set the normalisation  $\chi = 1$  for simplicity, but it can be assigned an arbitrary value.

In the case of the convolution (rearrange and relabel the indices):

$$D_A C^{AB} = \partial_A C^{AB} + \Gamma_{AX}^A C^{XB} + \Gamma_{AX}^B C^{AX}.$$

The full computation of the Christoffel symbols can be found in the appendix A. Double second derivative acts as:

$$\begin{aligned} D_A D_B \tilde{C}^{AB} &= D_A (\partial_B \tilde{C}^{AB} + \Gamma_{BX}^A \tilde{C}^{XB} + \Gamma_{BX}^B \tilde{C}^{AX}) = \\ &= \partial_A (\partial_B \tilde{C}^{AB} + \Gamma_{BX}^A \tilde{C}^{XB} + \Gamma_{BX}^B \tilde{C}^{AX}) + \Gamma_{AY}^A (\partial_B \tilde{C}^{YB} + \Gamma_{BX}^Y \tilde{C}^{XB} + \Gamma_{BX}^B \tilde{C}^{YX}) = \\ &= \partial_A \partial_B \tilde{C}^{AB} + \partial_A (\Gamma_{BX}^A \tilde{C}^{XB}) + \partial_A (\Gamma_{BX}^B \tilde{C}^{AX}) + \Gamma_{AY}^A \partial_B \tilde{C}^{YB} + \Gamma_{AY}^A \Gamma_{BX}^Y \tilde{C}^{XB} + \Gamma_{AY}^A \Gamma_{BX}^B \tilde{C}^{YX}. \end{aligned}$$

We transform each term, plugging in the expression for the dual shear tensor and the Christoffel symbols, which are mostly vanishing. After that, we regroup the resulting terms and plug in (87). Finally, we account that  $h_+$  and  $h_\times$  are independent of  $(\theta, \phi)$ .

1.

$$\begin{aligned} \partial_A \partial_B \tilde{C}^{AB} &= \partial_\theta \partial_\theta \tilde{C}^{\theta\theta} + \partial_\phi \partial_\theta \tilde{C}^{\phi\theta} + \partial_\theta \partial_\phi \tilde{C}^{\theta\phi} + \partial_\phi \partial_\phi \tilde{C}^{\phi\phi} = \\ &= \partial_\theta \partial_\theta \left[ \frac{1}{\sin \theta} C_{\phi\theta} \right] - \partial_\phi \partial_\theta \left[ \frac{1}{\sin \theta} C_{\theta\theta} \right] + \partial_\theta \partial_\phi \left[ \frac{1}{\sin^3 \theta} C_{\phi\phi} \right] - \partial_\phi \partial_\phi \left[ \frac{1}{\sin^3 \theta} C_{\theta\phi} \right] = \\ &= \partial_\theta \partial_\theta [rh_\times] - \partial_\phi \partial_\theta \left[ \frac{1}{\sin \theta} \cdot 2rh_+ \right] - \partial_\phi \partial_\phi \left[ \frac{1}{\sin^2 \theta} rh_\times \right] = 0. \end{aligned}$$

2.

$$\begin{aligned} \partial_A (\Gamma_{BX}^A \tilde{C}^{XB}) &= \partial_\theta (\Gamma_{BX}^\theta \tilde{C}^{XB}) + \partial_\phi (\Gamma_{BX}^\phi \tilde{C}^{XB}) = \partial_\theta (\Gamma_{\phi\phi}^\theta \tilde{C}^{\phi\phi}) + \partial_\phi (\Gamma_{\theta\theta}^\phi \tilde{C}^{\theta\theta}) + \partial_\phi (\Gamma_{\phi\theta}^\phi \tilde{C}^{\theta\phi}) = \\ &= -\partial_\theta \left[ (-\sin \theta \cos \theta) \frac{1}{\sin^3 \theta} C_{\theta\phi} \right] - \partial_\phi \left[ \cot \theta \frac{1}{\sin \theta} C_{\theta\theta} \right] + \partial_\phi \left[ \cot \theta \frac{1}{\sin^3 \theta} C_{\phi\phi} \right] = \\ &= \partial_\theta \left[ \cot \theta \frac{1}{\sin \theta} C_{\theta\phi} \right] - \partial_\phi \left[ \cot \theta \frac{1}{\sin \theta} \left( C_{\theta\theta} - \frac{1}{\sin^2 \theta} C_{\phi\phi} \right) \right] = \\ &= \partial_\theta [\cot \theta rh_\times] - \partial_\phi \left[ \cot \theta \frac{1}{\sin \theta} 2rh_+ \right] = rh_\times \partial_\theta \cot \theta - 0 = -\frac{1}{\sin^2 \theta} rh_\times. \end{aligned}$$

3.

$$\begin{aligned} \partial_A (\Gamma_{BX}^B \tilde{C}^{AX}) &= \partial_A (\Gamma_{\phi X}^\phi \tilde{C}^{AX}) + 0 = \partial_A (\Gamma_{\phi\theta}^\phi \tilde{C}^{A\theta}) = \\ &= \partial_A (\cot \theta \tilde{C}^{A\theta}) = \partial_\theta (\cot \theta \tilde{C}^{\theta\theta}) + \partial_\phi (\cot \theta \tilde{C}^{\phi\theta}) = \partial_\theta \left( \cot \theta \frac{1}{\sin \theta} C_{\phi\theta} \right) - \cot \theta \frac{1}{\sin \theta} \partial_\phi C_{\theta\theta} = \\ &= \partial_\theta (\cot \theta rh_\times) - 0 = rh_\times \partial_\theta \cot \theta = -\frac{1}{\sin^2 \theta} rh_\times. \end{aligned}$$

4.

$$\begin{aligned} \Gamma_{AY}^A \partial_B \tilde{C}^{YB} &= \Gamma_{\phi\theta}^\phi \partial_B \tilde{C}^{\theta B} = \cot \theta \partial_B \tilde{C}^{\theta B} = \cot \theta \partial_\theta \tilde{C}^{\theta\theta} + \cot \theta \partial_\phi \tilde{C}^{\theta\phi} = \\ &= \cot \theta \partial_\theta \left[ \frac{1}{\sin \theta} C_{\phi\theta} \right] + \cot \theta \partial_\phi \left[ \frac{1}{\sin^3 \theta} C_{\phi\phi} \right] = \cot \theta \partial_\theta [rh_\times] + 0 = 0. \end{aligned}$$

5.

$$\begin{aligned}\Gamma_{AY}^A \Gamma_{BX}^Y \tilde{C}^{XB} &= \Gamma_{\phi\theta}^\phi \Gamma_{\phi\phi}^\theta \tilde{C}^{\phi\phi} = \cot\theta(-\sin\theta\cos\theta)\tilde{C}^{\phi\phi} = -\cos^2\theta\tilde{C}^{\phi\phi} = \\ &= -\cos^2\theta\left(-\frac{1}{\sin^3\theta}C_{\theta\phi}\right) = \cot^2\theta\left(\frac{1}{\sin\theta}C_{\theta\phi}\right) = \cot^2\theta rh_\times.\end{aligned}$$

6.

$$\begin{aligned}\Gamma_{AY}^A \Gamma_{BX}^B \tilde{C}^{YX} &= \Gamma_{\theta Y}^\theta \Gamma_{BX}^B \tilde{C}^{YX} + \Gamma_{\phi Y}^\phi \Gamma_{BX}^B \tilde{C}^{YX} = 0 + \Gamma_{\phi Y}^\phi \Gamma_{BX}^B \tilde{C}^{YX} = \Gamma_{\phi Y}^\phi \Gamma_{BX}^B \tilde{C}^{YX} = \\ &= \Gamma_{\phi\theta}^\phi \Gamma_{BX}^B \tilde{C}^{\theta X} + \Gamma_{\phi\phi}^\phi \Gamma_{BX}^B \tilde{C}^{\phi X} = \cot\theta \Gamma_{BX}^B \tilde{C}^{\theta X} + 0 = \cot\theta \Gamma_{\phi\theta}^\phi \tilde{C}^{\theta\theta} + 0 = \cot^2\theta \tilde{C}^{\theta\theta} = \\ &= \cot^2\theta \frac{1}{\sin\theta} C_{\phi\theta} = \cot^2\theta rh_\times.\end{aligned}$$

Putting it all together:

$$D_A D_B \tilde{C}^{AB} = 2 \cot^2\theta rh_\times - 2 \frac{1}{\sin^2\theta} rh_\times = 2r \left[ \cot^2\theta - \frac{1}{\sin^2\theta} \right] h_\times = -2rh_\times. \quad (88)$$

#### 10.5.4 Non-Linear Term $N_{AB} \tilde{C}^{AB}$

By definition (21)<sup>13</sup>,  $N_{AB} = \dot{C}_{AB}$ . Hence:

$$\begin{aligned}\dot{C}_{AB} \tilde{C}^{AB} &= \dot{C}_{\theta\theta} \tilde{C}^{\theta\theta} + \dot{C}_{\theta\phi} \tilde{C}^{\theta\phi} + \dot{C}_{\phi\theta} \tilde{C}^{\phi\theta} + \dot{C}_{\phi\phi} \tilde{C}^{\phi\phi} = \\ &= \dot{C}_{\theta\theta} \frac{1}{\sin\theta} C_{\phi\theta} + \dot{C}_{\theta\phi} \frac{1}{\sin^3\theta} C_{\phi\phi} - \dot{C}_{\phi\theta} \frac{1}{\sin\theta} C_{\theta\theta} - \dot{C}_{\phi\phi} \frac{1}{\sin^3\theta} C_{\theta\phi} = \\ &= \dot{C}_{\theta\theta} rh_\times + r \dot{h}_\times \frac{1}{\sin^2\theta} C_{\phi\phi} - r \dot{h}_\times C_{\theta\theta} - \dot{C}_{\phi\phi} \frac{1}{\sin^2\theta} rh_\times = \\ &= rh_\times \left( \dot{C}_{\theta\theta} - \dot{C}_{\phi\phi} \frac{1}{\sin^2\theta} \right) + r \dot{h}_\times \left( \frac{1}{\sin^2\theta} C_{\phi\phi} - C_{\theta\theta} \right) = \\ &= rh_\times \cdot 2r \dot{h}_+ - r \dot{h}_\times \cdot 2rh_+ = 2r^2(h_\times \dot{h}_+ - \dot{h}_\times h_+). \quad (89)\end{aligned}$$

#### 10.5.5 Dual Mass Aspect $\tilde{\mathcal{M}}$

From the definition (83), while combining (88) and (89), we deduce the expression for the dual mass aspect:

$$\begin{aligned}\tilde{\mathcal{M}} &= \frac{1}{4} D_A D_B \tilde{C}^{AB} - \frac{1}{8} N_{AB} \tilde{C}^{AB} = \frac{1}{4} (-2rh_\times) - \frac{1}{8} \cdot 2r^2(h_\times \dot{h}_+ - \dot{h}_\times h_+) = \\ &= -\frac{r}{2} h_\times + \frac{r^2}{4} (\dot{h}_\times h_+ - h_\times \dot{h}_+). \quad (90)\end{aligned}$$

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<sup>13</sup>We proceed without the factor  $\frac{1}{2}$  to match the notation by [57].

## 10.6 Memory Angle $\Phi$

Equation (86) provides us with the expression for the rotation angle. We will now look at its observable, which scales as  $r^{-2}$ . Using (90), we have:

$$\Phi = \int du \frac{\tilde{\mathcal{M}}}{r^2} = -\frac{1}{2r} \int du h_{\times} + \frac{1}{4} \left[ \int du \dot{h}_{\times} h_{+} - \int du h_{\times} \dot{h}_{+} \right]. \quad (91)$$

The integrals are assumed for  $u \in (-\infty; +\infty)$ . Here, we see two parts: the linear ('soft') and the non-linear ('hard') one. The soft part identically matches the expression given in [50] for the spin memory. The non-linear term, however, is the new result, representing the *gyroscopic memory*. Considering the implicit  $r^{-1}$  dependence of  $h$ , both contributions scale as  $r^{-2}$ , as expected for spin memory.

This expression is valuable because it allows for direct computations of the observable angle. However, it also points out a possible parity violation. Indeed, for the right and left helicity of the gravitational waves, one writes:

$$h_{+} = \frac{h_R + ih_L}{\sqrt{2}},$$

$$h_{-} = \frac{h_R - ih_L}{\sqrt{2}},$$

indicating an asymmetry in our resulting equation (91). Thus, the existence of spin memory or the gyroscopic correction immediately implies parity violations.

## 11 Conclusions

In this thesis, we provided a comprehensive overview of gravitational memory, particularly focusing on gyroscopic and spin memory. We demonstrated that the passage of gravitational waves can lead to measurable precession. Through explicit calculation, we derived a formula (91), readily available for computing the observable rotation angle given the waveform. This reduces to the known linear-order results [50] and passes consistency checks [57], making it a plausible result. From a theoretical perspective, this indicates possible parity violations for gravitational wave helicities.

Furthermore, we investigated the dependencies of spin memory on various source parameters and compared it to the well-studied displacement memory. Despite being tiny, the spin memory is almost comparable to the displacement memory for rapidly rotating black holes (Fig. 14). The effect could potentially be measured with Sagnac interferometers, given a sufficiently violent event. Alternatively, a compelling idea of measuring the tilt from some pulsar located close to an extreme event was proposed.

Finally, we provided an extensive description of memory effects. We reviewed the Bondi–Metzner–Sachs formalism and the BMS group and described the interplays between asymptotic symmetries, memory effects, and soft theorems of infrared physics. We stressed that these are indeed related phenomena connected through mathematical transformations. This review is unique, as it combines all topics related to memory and unifies their notation, making it the most complete overview of gravitational memory known to the author.

We also highlighted several possible directions for future studies. Apart from those generally related to memories, we plan to focus on further studies of binary black hole precession, encompassing non-aligned spins of the black holes with the orbital spin. This is crucial for observational science, as binaries are naturally versatile, and restricting ourselves to only certain types risks missing events. Furthermore, it could compensate for limited detector sensitivity and help reconstruct comparatively large amplitude waveforms.

The obtained angular observable calculations may be extended in at least three directions. First, non-linearities can be added to obtain higher-order gyroscopic corrections and investigate their magnitudes. Second, the calculation can be carried out for the asymptotically flat Kerr spacetime. The rotation of the central body might enhance the effect, yielding higher chances to observe the tilt. Third, gravitational self-force corrections can be included in the gyroscope’s geodesics so that, for instance, Kerr black holes become valid gyroscopes. It would be interesting to study the effect of the gyroscope itself on its tilt.

## A Christoffel Symbols for the 2-Sphere Metric

By definition, Christoffel symbols are calculated as follows:

$$\Gamma_{BC}^A = \frac{1}{2}q^{AX}(q_{XB,C} + q_{XC,B} - q_{BC,X})$$

Explicit component calculations:

$$\Gamma_{\theta\theta}^\theta = \frac{1}{2}q^{\theta X}(2q_{X\theta,\theta} - q_{\theta\theta,X}) = \frac{1}{2}q^{\theta\theta}(2q_{\theta\theta,\theta} - 0) = 0,$$

$$\Gamma_{\phi\theta}^\theta = \Gamma_{\theta\phi}^\theta = \frac{1}{2}q^{\theta X}(q_{X\theta,\phi} + q_{X\phi,\theta} - q_{\phi\theta,X}) = \frac{1}{2}q^{\theta\theta}(q_{\theta\theta,\phi} + 0 - 0) = 0,$$

$$\Gamma_{\phi\phi}^\theta = \frac{1}{2}q^{\theta X}(2q_{X\phi,\phi} - q_{\phi\phi,X}) = \frac{1}{2}q^{\theta\theta}(0 - q_{\phi\phi,\theta}) = -\frac{1}{2} \cdot 1 \cdot \sin\theta \cos\theta = -\sin\theta \cos\theta,$$

$$\Gamma_{\theta\theta}^\phi = \frac{1}{2}q^{\phi X}(2q_{X\theta,\theta} - q_{\theta\theta,X}) = \frac{1}{2}q^{\phi\phi}(0 - 0) = 0,$$

$$\Gamma_{\phi\theta}^\phi = \Gamma_{\theta\phi}^\phi = \frac{1}{2}q^{\phi X}(q_{X\theta,\phi} + q_{X\phi,\theta} - q_{\phi\theta,X}) = \frac{1}{2}q^{\phi\phi}(0 + q_{\phi\phi,\theta} - 0) = \frac{1}{2} \frac{1}{\sin^2\theta} \cdot 2 \sin\theta \cos\theta = \cot\theta,$$

$$\Gamma_{\phi\phi}^\phi = \frac{1}{2}q^{\phi X}(2q_{X\phi,\phi} - q_{\phi\phi,X}) = \frac{1}{2}q^{\phi\phi}(0 - 0) = 0.$$

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