MMP I
Tutorial 7

HS 2019
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`https://www.physik.uzh.ch/en/teaching/PHY312/HS2019.html`

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Exercise 1: Surface minimization (4 Pts.)

Given two points P_1 and P_2 in the plane, use variational calculus to find a curve $y(x)$ from P_1 to P_2 such that the surface of revolution obtained by revolving the curve about the x-axis has minimum surface area.

Exercise 2: Cost minimization (6 Pts.)

An airplane is to travel from point $A = (0,0)$ to point $B = (d,0)$. In this problem we assume that the surface of the Earth is actually a plane and the trajectory $y(x)$ will be a curve in the $x - y$ plane. An airplane costs more money to fly at a lower altitude than at a higher one. We wish to minimize the cost of a trajectory between the points A and B. The cost of travelling a distance $\mathrm{d}s$ at an altitude y is given by $e^{-y/h}\mathrm{d}s$, with h a given constant.

a) Give an expression for the cost of the voyage between the points A and B and express the problem of minimizing this cost as a variational problem. Show that the associated Euler-Lagrange equation can be written as:
$hy''(x) + (y'(x))^2 + 1 = 0$.

b) Find the general solution to this Euler-Lagrange equation, as well as the solution satisfying the boundary conditions for the trip from A to B.

Exercise 3: Brachistochrone problem (5 Pts.)

A bead with mass m and zero initial velocity moves on a frictionless wire placed in a (homogeneous) gravitational field from $P_1 = (x_1, 0)$ to $P_2 = (x_2, y_2)$. Find the curve $y(x)$ which minimizes the time T, i.e. for which the integral

$$T[y] = \int_{x_1}^{x_2} \mathrm{d}x\ F[y(x), y'(x), x] \tag{3.1}$$

is extremal. What is the shape of the wire?
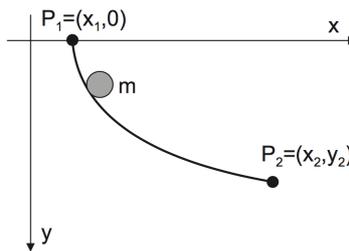
Hint: Express T as a path-integral over the inverse velocity which can be found by looking at the total energy of the wire (which is conserved). Use the substitution $y = \kappa \sin^2(\phi/2)$ to solve the integral (κ is not a new constant but should be chosen conveniently as a function of the constants already introduced).