MMP I
Tutorial 10

HS 2017
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Exercise 1: Adjoint operators (4 Pts.)
Find the adjoint operators of

a) any $H : T = |z\rangle\langle w|$ with $z, w \in H$.

b) $H = L^2[a, b] : |x\rangle \sim x(s) \xrightarrow{T} (Tx)(t) = \int_a^b k(t, s)x(s)\mathrm{d}s$.

c) $H = l^2 : |x\rangle = (x_1, x_2, x_3 \cdots) \xrightarrow{T} T|x\rangle = (\lambda_1 x_1, 0, \lambda_3 x_3, 0, \cdots)$ with $\lambda_i \in \mathbb{C}$ and $\lim_{i\to\infty} |\lambda_i| \to 0$.

d) $H = L^2[0, 1] : |x\rangle \sim x(s) \xrightarrow{T} (Tx)(t) = \int_0^t x(s)\mathrm{d}s$.

Exercise 2: Norm of linear operators (4 Pts.)
The norm $||T||$ of a bounded linear operator T is defined as:

$$||T|| = \sup_{x\neq 0} \frac{||Tx||}{||x||} = \sup_{||x||=1} ||Tx||. \tag{2.1}$$

The aim of the exercise is to show that for a self-adjoint bounded operator $S = S^\dagger$ we can also write the norm $||S||$ as

$$\sup_{x\neq 0} \frac{|\langle x|Sx\rangle|}{||x||^2} = \sup_{||x||=1} |\langle x|Sx\rangle| \equiv \sigma. \tag{2.2}$$

a) For $S = S^\dagger$ show $||S|| \geq \sigma$.

b) For $k \in \mathbb{R}$ define

$$|v_\pm\rangle \equiv k|x\rangle \pm \frac{1}{k} S|x\rangle. \tag{2.3}$$

and prove the following chain of (in)equalities

$$||Sx||^2 = \frac{1}{4}(\langle Sv_+|v_+\rangle - \langle Sv_-|v_-\rangle) \leq \frac{1}{4}\sigma(||v_+||^2 + ||v_-||^2) = \frac{1}{2}\sigma(k^2||x||^2 + k^{-2}||Sx||^2). \tag{2.4}$$

Minimise the right hand side w.r.t. k^2 to show $||S|| \leq \sigma$.

Exercise 3: Orthonormal basis of a Hilbert space (4 Pts.)
Consider $H = L^2[0, \pi]$ and the operator given by $T(f) = gf$ with $g(x) = x$ for $0 \leq x \leq 1$; 1 for $1 \leq x \leq \pi$.

a) Let $u(x) = 1$ in $L^2[0, \pi]$ and consider the functions $\phi_n(x) = T^n(u)$, $n = 0, 1 \cdots$. Are they a complete set in $L^2[0, \pi]$?

b) Find eigenvalues and eigenvectors of T and discuss their degeneracy.