



Mechanics

Exercise Sheet 3

HS 2026
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<https://www.physik.uzh.ch/en/teaching/PHY311/HS2026.html>

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Exercise 1: The Morse potential

The Morse Potential can be used as a simplified model for the potential energy of a diatomic molecule. It takes the form

$$V(r) = A \left(1 - e^{-\alpha(r-r_e)} \right)^2, \quad (1.1)$$

where A , α and r_e are dimensionful constants and r is the distance between two atoms with mass m .

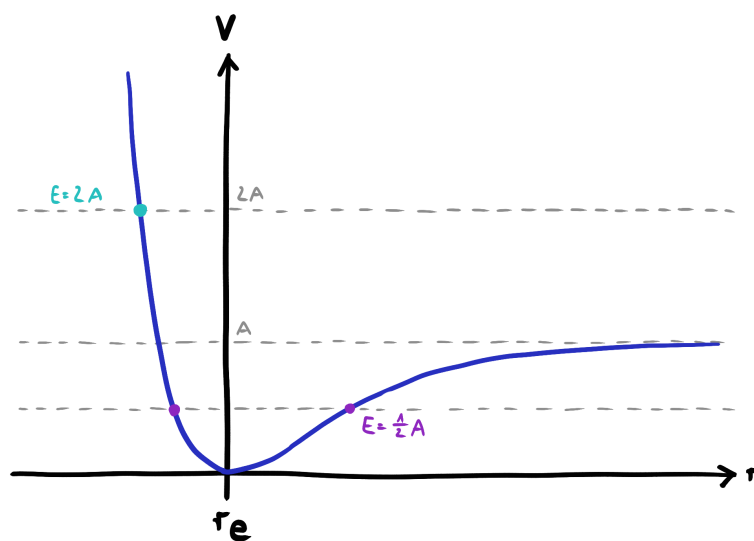
- Sketch the potential as a function of r and describe the motion qualitatively for various values of the molecule's energy E .
- Determine the turning distances for molecules with energies $E = A/2$ and $E = 2A$.
- Calculate the period of the motion for $E = A/2$.

Hint: You might find the following integral useful

$$\int \frac{du}{\left(1 - \frac{u}{\sqrt{2}}\right)\sqrt{1-u^2}} = \sqrt{2} \arctan \left(\frac{\sqrt{2}u - 1}{\sqrt{1-u^2}} \right). \quad (1.2)$$

Solution:

- The potential is given in the figure below.



The potential reaches a minimum at the equilibrium distance r_e . For energies below A the atoms in the molecule can move relative to one another but will always stay in a bound state. For energies $E > A$ the molecule is no longer bound as the atoms become free.

- b) The dots in the figure represent the turning distances for the atoms. We can find them by solving the equations

$$\frac{1}{2}A = A \left(1 - e^{-\alpha(r-r_e)}\right)^2 \quad (1.3)$$

$$2A = A \left(1 - e^{-\alpha(r-r_e)}\right)^2 \quad (1.4)$$

Eq. (1.3) has two solutions

$$r_{1,2} = -\frac{1}{\alpha} \ln(1 \pm \frac{1}{\sqrt{2}}) + r_e, \quad (1.5)$$

meaning that a molecule with energy $\frac{A}{2}$ will have the separation between the two atoms oscillate between these two distances.

Eq. (1.4) only has one real solutions at

$$r_3 = -\frac{1}{\alpha} \ln(\sqrt{2} + 1) + r_e, \quad (1.6)$$

which shows that the atoms will bounce back if they get closer to one another, but will then separate into free states when they move away from one another.

- c) We can derive the time for one period of motion by first looking at the energy of the system as a function of the separation r :

$$E = \frac{1}{2}m\dot{r}^2 + U(r) \quad (1.7)$$

$$\rightarrow \dot{r} = \sqrt{\frac{2(E - U(r))}{m}} = \frac{dr}{dt} \quad (1.8)$$

The total time T for one oscillation can then be found by rearranging and integrating the time and path.

$$\frac{T}{2} = \int_{t_i}^{t_f} dt = \int_{r_i}^{r_f} \sqrt{\frac{m}{2(E - U(r))}} dr \quad (1.9)$$

$$= \sqrt{\frac{m}{A}} \int_{q_i}^{q_f} \sqrt{\frac{1}{1 - 2(1 - e^{-\alpha(r-r_e)})^2}} dr \quad (1.10)$$

We can now perform a change of variables:

$$u = \sqrt{2} \left(1 - e^{-\alpha(r-r_e)}\right), \quad (1.11)$$

with which the integral becomes

$$\frac{T}{2} = \sqrt{\frac{m}{2A}} \frac{1}{\alpha} \int_{u_i}^{u_f} \frac{1}{1 - \frac{u}{\sqrt{2}}} \frac{1}{\sqrt{1 - u^2}} du. \quad (1.12)$$

Using the standard integral in the hint and the fact that $u_i = -1$ and $u_f = +1$ we can evaluate this to be

$$\frac{T}{2} = \sqrt{\frac{m}{A}} \frac{1}{\alpha} (\arctan(\infty) - \arctan -\infty) \quad (1.13)$$

$$= \sqrt{\frac{m}{A}} \frac{\pi}{\alpha} \quad (1.14)$$

Which finally gives as the period of motion:

$$T = \sqrt{\frac{m}{A}} \frac{2\pi}{\alpha} \quad (1.15)$$

Exercise 2: Minimisation of travel time of a bead on a wire

- a) Consider a bead falling along a wire due to gravity. In this exercise you are asked to find the shape of the wire (the curve) which minimises the time taken for the bead to fall from the origin $(0, 0)$ to an arbitrary point (x_0, y_0) . If the path is parameterised by $(x, y(x))$, use conservation of energy to show that the time taken is given by

$$t = \int_0^{x_0} \sqrt{\frac{1 + \dot{y}^2}{2gy}} dx, \quad (2.1)$$

where g is the acceleration due to gravity and y is measured down from the initial point.

- b) Using the minimisation condition (2.11) from the lecture notes, find the differential equation satisfied by $y(x)$ for the extremal path.
- c) Using the fact that

$$\frac{d}{dx}(y + y\dot{y}^2) = \dot{y} + 2y\dot{y}\ddot{y} + \dot{y}^3 \quad (2.2)$$

and the substitution

$$y = C \sin^2 \theta, \quad (2.3)$$

where θ parametrises the path and C is a constant of integration, show that

$$x = \frac{C}{2} (2\theta - \sin 2\theta), \quad (2.4)$$

$$y = \frac{C}{2} (1 - \cos 2\theta), \quad (2.5)$$

is the solution to the differential equation.

- d) Sketch the curve.
- e) From an arbitrary point on the curve (x_1, y_1) , the time taken to reach the bottom $(x_b, y_{\min})$ is given by

$$t = \int_{x_1}^{x_b} \sqrt{\frac{1 + \dot{y}^2}{2g(y - y_1)}} dx. \quad (2.6)$$

Find the time taken to reach $(x_b, y_{\min})$ from (x_1, y_1) . What do you notice?

Solution:

This is the famous brachistochrone problem. Galileo experimented with objects rolling down tracks of different shapes and he may have believed the optimal track would have the shape of a quarter-circle, which is not the right answer. The problem was formally posed by Johann Bernoulli in 1696 as a challenge to the mathematicians of his day. Bernoulli himself found the solution and Newton had a different proof that was the correct one. This problem is widely regarded as the founding problem of the ‘calculus of variations’.

a) If v is the speed along the curve, conservation of energy gives

$$V_0 = \frac{1}{2}mv^2 - mgy + V_0, \quad (2.7)$$

$$v = \sqrt{2gy}. \quad (2.8)$$

where V_0 is the potential energy at the origin. The time taken is then given by

$$t = \int_0^{t_0} dt, \quad (2.9)$$

$$= \int_0^{s_0} \frac{ds}{v}, \quad (2.10)$$

$$= \int_0^{x_0} \frac{\sqrt{1 + \dot{y}^2}}{v} dx, \quad (2.11)$$

$$= \int_0^{x_0} \sqrt{\frac{1 + \dot{y}^2}{2gy}} dx. \quad (2.12)$$

b) The extremal condition (2.11) is

$$\frac{\partial F}{\partial y} = \frac{d}{dx} \left(\frac{\partial F}{\partial \dot{y}} \right), \quad (2.13)$$

where we take

$$F = \sqrt{\frac{1 + \dot{y}^2}{2gy}}. \quad (2.14)$$

Using

$$\frac{\partial F}{\partial y} = -g(1 + \dot{y}^2)^{1/2}(2gy)^{-3/2}, \quad (2.15)$$

$$\frac{\partial F}{\partial \dot{y}} = \dot{y}(1 + \dot{y}^2)^{-1/2}(2gy)^{-1/2}, \quad (2.16)$$

we then find that

$$2y\ddot{y} + \dot{y}^2 + 1 = 0. \quad (2.17)$$

c) Using the hint, we can re-write the differential equation as

$$\frac{d}{dx}(y + y\dot{y}^2) = 0, \quad (2.18)$$

so that

$$y + y\dot{y}^2 = C, \quad (2.19)$$

where C is a constant of integration. This can be re-arranged to give

$$\dot{y} = \sqrt{\frac{C - y}{y}}, \quad (2.20)$$

$$x = \int \sqrt{\frac{y}{C - y}} dy. \quad (2.21)$$

We now use the suggested substitution $y = C \sin^2 \theta$, which has Jacobian $dy/d\theta = 2C \sin \theta \cos \theta$ to solve this integral.

$$x = \int \sqrt{\frac{C \sin^2 \theta}{C(1 - \sin^2 \theta)}} dy, \quad (2.22)$$

$$= 2C \int \tan \theta \sin \theta \cos \theta d\theta, \quad (2.23)$$

$$= 2C \int \sin^2 \theta d\theta, \quad (2.24)$$

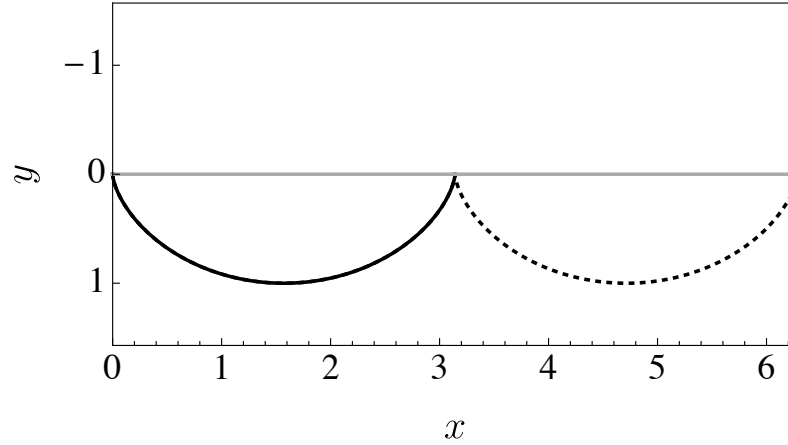
$$= C(\theta - \frac{1}{2} \sin(2\theta)) + D \quad (2.25)$$

The beads starts with a y -coordinate of 0 and end with a co-ordinate of y_0 , thus we can take $\theta_{\min} = 0$ and $D = 0$ giving,

$$x = \frac{C}{2}(2\theta - \sin 2\theta) \quad (2.26)$$

$$y = \frac{C}{2}(1 - \cos 2\theta). \quad (2.27)$$

- d) The curve is given in figure 1. We have continued the curve beyond $x = \pi$ to highlight the fact that this is the curve obtained from fixing a point on a wheel, and tracking the point as the wheel rolls along a surface. Note that the final point (x_0, y_0) is not in general at the bottom of the curve, but can be anywhere in $x \in C(0, \pi]$.



- e) Substituting in the solutions and using double angle formulae we obtain

$$t = \sqrt{\frac{2C}{g}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin \theta}{\sqrt{\cos^2 \theta_0 - \cos^2 \theta}} d\theta \quad (2.28)$$

Making the substitution $u = \cos \theta / \cos \theta_0$ this becomes

$$t = -\sqrt{\frac{2C}{g}} \int_1^0 \frac{1}{\sqrt{1-u^2}} du, \quad (2.29)$$

$$= -\sqrt{\frac{2C}{g}} [\arcsin(u)]_1^0, \quad (2.30)$$

$$= \sqrt{\frac{C}{2g}} \pi. \quad (2.31)$$

The final result is independent of the initial point, so it takes the same time to reach the bottom for any starting point.