

Website: <http://www.physik.uzh.ch/en/teaching/PHY519/>

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**Exercise 1** [The cosmological constant in the black hole context]

a) In vacuum,  $T_{\mu\nu} = 0$ . Taking the trace of the field equations we get

$$R = -4\Lambda. \tag{1}$$

Substituting this back in to the field equations gives

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2}$$

b) We follow the same procedure here as we did for the Schwarzschild solution. From expressions for the components of the Ricci tensor for a Schwarzschild ansatz, (see the notes from the previous semester, or Weinberg), we have that

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA} \left( \frac{A'}{A} + \frac{B'}{B} \right). \tag{3}$$

This leads to

$$\frac{d}{dr} \ln(AB) = 0 \implies AB = \text{const.} = 1. \tag{4}$$

Considering our third equation,  $R_{\theta\theta} + \Lambda g_{\theta\theta} = 0$ , we have

$$R_{\theta\theta} + \Lambda g_{\theta\theta} = 1 + \frac{r}{2A} \left( \frac{A'}{A} - \frac{B'}{B} \right) - \frac{1}{A} - \Lambda r^2 = 0, \tag{5}$$

which simplifies to a first-order linear differential equation for  $B(r)$ ,

$$1 - rB' - B - \Lambda r^2 = 0, \tag{6}$$

which is easily solved by the ansatz  $B(r) = 1 - \frac{2GM}{r} + B_\Lambda(r)$ . The result is

$$B_\Lambda(r) = -\frac{\Lambda}{3}r^2. \tag{7}$$

c) As discussed in GRI, the bending angle can be written as

$$\Delta\phi = 2|\phi(r_0) - \phi_\infty| - \pi, \tag{8}$$

where  $r_0$  is the distance of closest approach. For a photon in a static isotropic metric we have the formula (GRI 23.19)

$$\phi(r) - \phi_\infty = \int_r^\infty \frac{\sqrt{B^{-1}}}{\sqrt{\left(\frac{r}{r_0}\right)^2 \frac{B(r_0)}{B(r)} - 1}} \frac{dr}{r} \tag{9}$$

$$= \int_r^\infty \sqrt{\frac{r_0^2}{r^2 B(r_0) - r_0^2 B(r)}} \frac{dr}{r}. \tag{10}$$

It turns out that, because the term  $B_\Lambda(r)$  is  $\propto r^2$ , the terms exactly cancel each other. This means that light deflection does not depend on  $\Lambda$ , and therefore we cannot use the deflection of light as a tool to determine  $\Lambda$ , an important observational consequence.

**Exercise 2** [Age of the universe]

From the lecture you know that the Friedmann equation for a flat Universe can be written as

$$H(t)^2 = \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho_m(t) + \frac{\Lambda}{3}. \quad (11)$$

Using that the matter density scales as  $\rho_m(t) = \rho_{m,0}a(t)^{-3}$  and the definition of the present day density parameters

$$\Omega_{m,0} = \frac{8\pi G}{3H_0^2}\rho_{m,0} \qquad \Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2} \quad (12)$$

one can easily derive

$$H(t)^2 = H_0^2 (\Omega_{m,0}a(t)^{-3} + \Omega_{\Lambda,0}). \quad (13)$$

The age of the Universe is now given by

$$t = \int_0^{t_0} dt = \int_0^1 \frac{da}{\dot{a}} = \int_0^1 \frac{da}{aH(a)} = 1.3870 \times 10^{10} \text{ y}, \quad (14)$$

where we used that that  $1/H_0 = 1.4386 \times 10^{10} \text{ y}$  and evaluated the integral numerically.

The comoving distance traveled by a photon on a radial null geodesic is given by ( $ds = 0 \Rightarrow c dt = a dr$ )

$$r = c \int_0^{t_0} \frac{dt}{a} = c \int_0^1 \frac{da}{a\dot{a}} = c \int_0^1 \frac{da}{a^2 H(a)} = 14\,571 \text{ Mpc}. \quad (15)$$

Note that the Hubble length  $c/H_0 \approx 4409 \text{ Mpc}$  is a typical length scale for causality in our expanding Universe.