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Applications of General Relativity

Solutions 3

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Exercise 1 [Metric of a static star]

We start from the general static, spherical symmetric metric

$$\mathrm{d}s^2 = \exp\left[2\alpha(r)\right]\mathrm{d}t^2 - \exp\left[2\beta(r)\right]\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1}$$

From the normalization of the velocity in the fluid restframe we obtain $u_t = \sqrt{g_{tt}}$, which then straightforwardly leads to the energy-momentum tensor

$$T_{\mu\nu} = \mathrm{diag}\left\{\exp\left[2\alpha\right]\rho, \exp\left[2\beta\right]p, r^2\,p, r^2\sin^2\left(\theta\right)\ p\right\}. \tag{2}$$

The Ricci scalar for the above metric ansatz read as

$$R = 2\exp\left[-2\beta\right]\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}(1-\exp\left[2\beta\right])\right). \tag{3}$$

For the time component of the Einstein equations we thus have

$$R_{tt} - \frac{1}{2}Rg_{tt} = 8\pi G T_{tt} \tag{4}$$

$$-\exp\left[2(\alpha-\beta)\right]\left(-\frac{2}{r}\partial_r\beta + \frac{1}{r^2}(1-\exp\left[2\beta\right])\right) = 8\pi G\rho\exp\left[2\alpha\right] \tag{5}$$

$$\frac{1}{r^2}\partial_r m(r) = 4\pi\rho, \tag{6}$$

where $m(r)$ is defined in Eq. (3) of the exercise sheet. This differential equation can be integrated to obtain the mass within a given radius r

$$m(r) = 4\pi\int_0^r \rho(r')r'^2\mathrm{d}r'. \tag{7}$$

Note that it is different from the integrated energy density, which is obtained integrating the density over the volume element $\sqrt{\gamma}\mathrm{d}^3x$. Here γ is the determinant of the spatial metric. In contrast to the above, the integrated energy density accounts for the gravitational binding energy as well.
With the replacement of $m(r)$ we can immediately write down the radial metric element

$$\exp\left[2\beta\right] = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}, \tag{8}$$

which has remarkable similarity with the metric element of the Schwarzschild solution. For the radial component we have

$$R_{rr} - \frac{1}{2}Rg_{rr} = 8\pi G T_{rr} \tag{9}$$

$$\frac{2}{r}\partial_r\alpha - \frac{2G}{r^3}\exp\left[2\beta\right]m(r) = 8\pi G p \tag{10}$$

thus

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = \frac{4\pi G r^3 p + G m(r)}{r\left[r - 2Gm(r)\right]} \tag{11}$$

The Bianchi identy or energy-momentum conservation reads as

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\rho} T^{\rho\nu} + \Gamma^\nu_{\mu\rho} T^{\mu\rho}. \tag{12}$$

While the $\nu = t$ equation is trivial, the $\nu = r$ equation leads to

$$\nabla_\mu T^{\mu r} = \partial_r(\exp\left[-2\beta\right]p) + T^{rr}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r} + \Gamma^t_{tr}) \tag{13}$$

$$+ (\Gamma^r_{tt} T^{tt} + \Gamma^r_{rr} T^{rr} + \Gamma^r_{\phi\phi} T^{\phi\phi} + \Gamma^r_{\theta\theta} T^{\theta\theta}) \tag{14}$$

$$= \exp\left[-2\beta\right]\left\{(\partial_r p - 2p\partial_r\beta) + p(\partial_r\beta + 2/r + \partial_r\alpha) + (\rho\partial_r\alpha + \partial_r\beta p - 2p/r)\right\} \tag{15}$$

$$= \exp\left[-2\beta\right]\left\{(\rho + p)\partial_r\alpha + \partial_r p\right\} \overset{!}{=} 0 \tag{16}$$

From this we obtain

$$(\rho + p)\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}. \tag{17}$$

Plugging in Eq. (11) we finally have

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)\left[Gm(r) + 4\pi G r^3 p\right]}{r\left[r - 2Gm(r)\right]}. \tag{18}$$

Now we assume $\rho = \rho_* =$ const. out to the surface of the star yielding

$$m(r) = \frac{4\pi}{3}\rho_* r^3 = M\frac{r^3}{R^3}, \tag{19}$$

where M is the total mass of the star and from now on R will denote the radius of the star. From Eq. (18) we obtain by separation of variables

$$\frac{\mathrm{d}p}{p^2 + \frac{4}{3}\rho_* p + \frac{\rho_*^2}{3}} = \frac{4\pi G r \mathrm{d}r}{\frac{8\pi G}{3} r^2 \rho_* - 1} \tag{20}$$

$$\frac{\mathrm{d}p}{\left(p + \frac{2}{3}\rho_*\right)^2 - \frac{\rho_*^2}{9}} = \frac{3}{4\rho_*}\frac{\frac{16\pi G}{3} r\rho_* \mathrm{d}r}{\frac{8\pi G}{3} r^2 \rho_* - 1} \tag{21}$$

Using that the pressure vanishes at the surface of the star we can integrate to obtain

$$-\frac{1}{2}\frac{3}{\rho_*}\ln\left[\frac{p' + \frac{2}{3}\rho_* + \frac{1}{3}\rho_*}{p' + \frac{2}{3}\rho_* - \frac{1}{3}\rho_*}\right]^0_{p'=p} = \frac{3}{4\rho_*}\ln\left[\frac{8\pi G}{3} r'^2 \rho_* - 1\right]^R_{r'=r} \tag{22}$$

$$\ln\left[\frac{p + \rho_*}{3p + \rho_*}\right] = \frac{1}{2}\ln\left[\frac{2GMR^2 - R^3}{2GMr^2 - R^3}\right]. \tag{23}$$

Solving for $p(r)$ yields the radial pressure distribution

$$p(r) = \rho_* \frac{\sqrt{R^3 - 2GMR^2} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3\sqrt{R^3 - 2GMR^2}}. \tag{24}$$

For the star of maximum mass the pressure diverges at $r = 0$ and there is no static solution, i. e. the star collapses. The maximum mass corresponds to the case where the denominator vanishes at $r = 0$

$$R = 9(R - 2GM), \tag{25}$$

leading to

$$M < \frac{4}{9}\frac{R}{G}. \tag{26}$$

This result remains true for more general equations of state and is known as Buchdahls theorem. Now it remains to calculate the missing metric coefficients. From Eq. (11) we have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}\ln(p+\rho_*)}{\mathrm{d}r}. \tag{27}$$

The boundary conditions can be set up at the surface of the star, where the interior solution goes over into the exterior Schwarzschild solution, thus $\rho + p|_{r=R} = \rho_*$ and

$$\exp\left[\alpha(R)\right] = \sqrt{1 - \frac{2GM}{R}}. \tag{28}$$

In summary, we have for the metric elements

$$\exp\left[\alpha\right] = \frac{3}{2}\sqrt{1 - \frac{2GM}{R}} - \frac{1}{2}\sqrt{1 - \frac{2GMr^2}{R^3}}, \tag{29}$$

$$\exp\left[-\beta\right] = \sqrt{1 - \frac{2Gm(r)}{r}} = \sqrt{1 - \frac{2GMr^2}{R^3}}. \tag{30}$$