FOURIER TRANSFORMATION

The expansion in Fourier series for a function which is integrable and L periodic is

$$f(x) \sim \sum_{m=-\infty}^{m=+\infty} c_m \, e^{i \frac{2\pi x}{L} m}$$

It would be desirable to have such an expansion for functions that are NOT periodic.

The periodicity condition can be relaxed by taking L very large. Let us observe that the index multiplying x in the Fourier exponent is $\frac{2\pi}{L} m$. When $m \to m+1$ we have $k = \frac{2\pi m}{L} \to \frac{2\pi m}{L} + \frac{2\pi}{L} = k + \Delta k$. When L is large Δk is small

$\Rightarrow$ the values of k become almost continuous

Let us see this in more detail. We have

$$c_m = \frac{1}{L} \int_{-L/2}^{L/2} f(x') \, e^{-i \frac{2\pi}{L} m x'} \, dx' \qquad k = \frac{2\pi}{L} m$$

$$f(x) \sim \sum_{m=-\infty}^{+\infty} c_m \, e^{i \frac{2\pi x}{L} m} = \sum_{m=-\infty}^{+\infty} \frac{1}{L} \int_{-L/2}^{L/2} f(x') \, e^{-i \frac{2\pi}{L} m x'} \, dx' \; e^{i \frac{2\pi}{L} x m}$$

$$= \frac{1}{2\pi} \sum \Delta k \int_{-L/2}^{L/2} f(x') \, dx' \, e^{i \frac{2\pi}{L} m (x - x')}$$

$$\xrightarrow[L\to\infty]{} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \underbrace{\int_{-\infty}^{+\infty} f(x') \, e^{-ikx'} \, dx'}_{\equiv \hat{f}(k)} \; e^{ikx}$$

$\longrightarrow$ The Fourier coefficients become integrals depending on a continuous variable

$c_m \to \hat{f}(k)$ ↗ Fourier transform

$\hat{f} : \mathbb{R} \to \mathbb{C}$

$$\Rightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \, \hat{f}(k) \, e^{ikx}$$

The Fourier transform has a wide range of applications in physics (time $\leftrightarrow$ frequency, position $\leftrightarrow$ momentum; expansion of a signal into its frequency modes ...)

The Fourier transform is an example of a more general class of <u>INTEGRAL TRANSFORMS</u>

An integral transform is a mapping from a function space to another function space which is realised through an integral

$$f(x) \to I(f(x)) = \tilde{f}(k) \qquad \tilde{f}(k) = \int_a^b f(x)\, K(x,k)\, dx$$

kernel of the integral transform

The transformation is linear

$$I(\lambda_1 f_1(x) + \lambda_2 f_2(x)) = \lambda_1 I(f_1(x)) + \lambda_2 I(f_2(x))$$

For the Fourier transform we have $K(x,k) = e^{-ikx}$ $a = -\infty$, $b = +\infty$

[NOTE: In the literature you can find different conventions for the $1/2\pi$ factors!]

<u>Properties</u> : $f \in L^1(\mathbb{R})$

(i) $\alpha, \beta \in \mathbb{C}$ $\qquad \alpha f + \beta g \to \alpha \tilde{f} + \beta \tilde{g}$ (linearity)

(ii) $f_y(x) \equiv f(x-y)$ $\qquad f_y(x) \to e^{-iky}\, \tilde{f}(k)$

(iii) $f \in C^1(\mathbb{R})$ $\qquad \tilde{f'} = ?$

$$\int_{-\infty}^{+\infty} f'(x)\, e^{-ikx} = \cancel{f(x) e^{-ikx}\Big|_{-\infty}^{+\infty}} - \int_{-\infty}^{+\infty} f(x)\,(-ik)\, e^{-ikx}\, dx = ik\, \tilde{f}(k)$$

since $f \in L^1(\mathbb{R})$

$$\Rightarrow \tilde{f'} = ik \tilde{f}$$

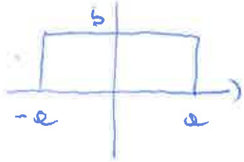
(iv) $\lambda f : x \to f(\lambda x)$ $\qquad \widetilde{\lambda f} = ?$

$\lambda \in \mathbb{R}$

$$\widetilde{\lambda f}(k) = \int_{-\infty}^{+\infty} f(\lambda x)\, e^{-ikx}\, dx = \frac{1}{|\lambda|} \int_{-\infty}^{+\infty} f(t)\, e^{-i\frac{k}{\lambda}t}\, dt$$

$$= \frac{1}{|\lambda|}\, \tilde{f}\left(\frac{k}{\lambda}\right)$$

EXAMPLE



Let us consider the function $f(x) = \begin{cases} b & |x|<a \\ 0 & |x|>a \end{cases}$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx = b \int_{-a}^{a} e^{-ikx} = b \frac{1}{-ik} e^{-ikx}\Big|_{-a}^{a}$$

$$= \frac{ib}{k}\left(e^{-ika} - e^{ika}\right) = \frac{2b}{k} \sin ka$$

1) Consider the limit $a \to \infty$ $\Rightarrow$ the function f becomes a constant

$$\hat{f}(k) = \lim_{a\to\infty} \frac{2b}{k} \sin ka = \lim_{a\to\infty} 2\pi b \frac{\sin ka}{k\pi} = \underbrace{\lim_{a\to\infty} \frac{\sin ka}{k\pi}} \times 2\pi b$$

this function becomes more and more peaked at $k=0$ as a increases

$\frac{\sin ka}{k\pi}$ is what we call a "representation" of the δ function $\left(\int_{-\infty}^{+\infty} \frac{\sin ka}{k\pi} = 1 \right)$

$\delta(k)$ is a special function (actually a distribution) we will discuss later.

Its properties are such that $\delta(k) = 0 \quad k \neq 0$, $\delta(k) = \infty \quad k=0$, $\int_{-\infty}^{+\infty} \delta(k)\, g(k) = g(0)$

$\Rightarrow$ when $a \to \infty$ we have $\tilde{f}(k) = 2\pi b\, \delta(k)$

2) Consider now the case $b \to \infty$, $a \to 0$ such that $2ba = 1$

In this limit the function $f(x)$ is fully localized in the origin and we have

$$f(x) \to \delta(x)$$

For $\tilde{f}(k)$ we get $\tilde{f}(k) = \frac{2b}{k} \sin ka = \frac{2ba}{ka} \sin ka \to 1$

We thus see that, if the function is a constant, its Fourier Transform is localized at $k=0$. On the contrary, if the function is localized, then its Fourier transform is a constant.

EXAMPLE

Fourier transform of a Gaussian $f(x) = a\, e^{-bx^2}$

$$\tilde f(k) = a\int_{-\infty}^{+\infty} e^{-bx^2-ikx}\,dx \qquad \frac{d\tilde f}{dk} = a\int_{-\infty}^{+\infty} e^{-bx^2}(-ix)\,e^{-ikx}\,dx$$

$$= \frac{ai}{2b}\int_{-\infty}^{+\infty}\left(\frac{d}{dx}e^{-bx^2}\right)e^{-ikx} = \frac{ai}{2b}\left[\cancel{e^{-bx^2}e^{-ikx}\Big|_{-\infty}^{+\infty}} - \int_{-\infty}^{+\infty} e^{-bx^2}(-ik)\,e^{-ikx}\right]$$

$$= -\frac{k}{2b}\tilde f(k) \qquad \tilde f(0) = a\int_{-\infty}^{+\infty} e^{-bx^2} = a\sqrt{\frac{\pi}{b}}$$

$\Rightarrow$ we have a first order differential equation for $\hat f(k)$: $\dfrac{d\hat f}{dk} = -\dfrac{k}{2b}\tilde f$

with an initial condition for $\hat f(0)$

We can solve it by separation of variables $\dfrac{d\hat f}{\tilde f} = -\dfrac{k}{2b}\,dk \qquad \ln\tilde f = -\dfrac{k^2}{4b} + \text{const}$

And the solution, by using the initial condition is

$$\boxed{\hat f(k) = a\sqrt{\frac{\pi}{b}}\; e^{-\frac{k^2}{4b}}}$$

Note that the width of the original gaussian is $\sim \dfrac{1}{\sqrt b}$

After Fourier transform we get another gaussian but with width $\sim \sqrt b$

So, as in the previous example, a wide distribution (small b) in x is turned into a narrow distribution in k and viceversa.

Fourier transform in n-dimensions

$f \in L^1(\mathbb{R}^n) \qquad \hat f(k) = \displaystyle\int_{\mathbb{R}^n} f(x)\, e^{-ikx}\,dx \qquad$ where $kx = \displaystyle\sum_{i=1}^{n} k_i x_i \qquad dx = dx_1 \cdots dx_n$

The case of a Gaussian can be treated by using the result we got in the one dimensional case

$$f(x) = a\, e^{-bx^2} \qquad x \in \mathbb{R}^n \qquad x = (x_1, x_2 \dots x_n) \qquad x^2 = \sum_j x_j^2$$

$$\hat{f}(k) = a \int e^{-bx^2 - ikx}\, dx = a \int e^{-b \sum_j x_j^2 - i \sum_j k_j x_j}\, dx_1 \dots dx_n$$

$$= a \prod_j \int e^{-b x_j^2 - i k_j x_j}\, dx_j = a \prod_j \sqrt{\frac{\pi}{b}}\, e^{-\frac{k_j^2}{4b}}$$

$$= a \left(\frac{\pi}{b}\right)^{\frac{n}{2}} e^{-\frac{k^2}{4b}} \qquad k^2 = \sum_j k_j^2 \qquad k = (k_1, \dots k_n)$$

We can further generalize these results to the case in which the Gaussian is defined through a positive symmetric $n \times n$ matrix.

$$f(x) = e^{-\frac{1}{2}(Ax, x)} \qquad \text{where} \qquad (Ax, x) \equiv \sum_{j,e} A_{je}\, x_j x_e$$

We have

$$\hat{f}(k) = \int_{\mathbb{R}^n} e^{-\frac{1}{2}(Ax,x) - ikx}\, dx$$

since A is symmetric and positive there exists a matrix R orthogonal $(R^T = R^{-1})$ such that $R^T A R = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \equiv \Lambda$ $\quad \lambda_i > 0$

$\Rightarrow$ we set $x = Ry$ and we write

$$\tilde{f}(k) = \int_{\mathbb{R}^n} e^{-\frac{1}{2}(\underbrace{ARy}_{= R\Lambda y},\, Ry) - ikRy}\, \underbrace{\det R}_{=1}\, dy$$

$$= \int_{\mathbb{R}^n} e^{-\frac{1}{2}(\Lambda y, y) - i(R^T k, y)}\, dy = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_j \lambda_j y_j^2 - i \sum_j (R^T k)_j\, y_j}\, dy$$

$$= \prod_j \int_{-\infty}^{+\infty} e^{-\frac{\lambda_j y_j^2}{2} - i(R^T k)_j\, y_j}\, dy_j = \prod_j \left(\frac{2\pi}{\lambda_j}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(R^T k)_j^2 / \lambda_j}$$

Now since

$$R\Lambda^{-1}R^T = R(R^TAR)^{-1}R^T = A^{-1}$$

$$\Rightarrow \quad \boxed{\hat{f}(k) = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}} e^{-\frac{1}{2}(A^{-1}k,k)}}$$

Compare with the one dimensional case:

$$x \in \mathbb{R} \qquad f(x) = e^{-\frac{1}{2}\lambda x^2} \quad \Rightarrow \quad \hat{f}(k) = \sqrt{\frac{2\pi}{\lambda}}\, e^{-\frac{k^2}{2\lambda}}$$

$$x \in \mathbb{R}^n \qquad f(x) = e^{-\frac{1}{2}(Ax,x)} \quad \Rightarrow \quad \hat{f}(k) = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}} e^{-\frac{1}{2}(A^{-1}k,k)}$$

We now look at some explicit examples, first considering $n=1$

EXAMPLE

$$f(x) = e^{-m|x|} \qquad \hat{f}(k) = \int_0^{\infty} e^{-mx-ikx}\,dx + \int_{-\infty}^0 e^{mx-ikx}$$

$m>0$

$$= \frac{1}{m+ik} + \frac{1}{m-ik} = \frac{2m}{m^2+k^2}.$$

EXAMPLE: Hermite functions and polynomials

We have seen that if $f(x) = e^{-\frac{\lambda}{2}x^2} \Rightarrow \hat{f}(k) = \sqrt{\frac{2\pi}{\lambda}}\, e^{-\frac{k^2}{2\lambda}}$

Let us consider the case $\lambda = 1$:

$$\phi_0(x) = e^{-\frac{x^2}{2}} \rightarrow \sqrt{2\pi}\, e^{-\frac{k^2}{2}} \quad \Rightarrow$$

$\phi_0(x)$ is an EIGENVECTOR of the operator $f \xrightarrow{F} \tilde{f}$!

$$\tilde{\phi}_0(x) = \sqrt{2\pi}\,\phi_0(x)$$

Let us define

$$\boxed{\phi_n(x) = (-1)^n e^{\frac{x^2}{2}} \left(\frac{d}{dx}\right)^n e^{-x^2}}$$

Hermite functions

We have

$$\phi_0(x) = e^{-\frac{x^2}{2}} \qquad \phi_1(x) = 2x\, e^{-\frac{x^2}{2}} \qquad \phi_2(x) = (4x^2-2)\, e^{-\frac{x^2}{2}} \quad \ldots$$

ϕ_n are polynomials multiplied by the $e^{-\frac{x^2}{2}}$. We can define

$$\boxed{h_n(x) = \phi_n(x)\, e^{\frac{x^2}{2}}}$$

Hermite polynomials

basis of $L^2(-\infty,+\infty)$ (see later)

arise in eigenstates of quantum harmonic oscillator

$$h_0(x) = 1 \qquad h_1(x) = 2x \qquad h_2(x) = 4x^2-2 \quad \ldots$$

We have

$$\boxed{\hat{\phi}_n(k) = (-i)^n (2\pi)^{1/2} \phi_n(k)}$$

Indeed

$$\hat{\phi}_n(k) = \int_{-\infty}^{+\infty} (-1)^n e^{\frac{x^2}{2}} \left(\frac{d^n}{dx^n} e^{-x^2}\right) e^{-ikx}\, dx$$

$$= \int_{-\infty}^{+\infty} e^{-x^2} \left(\frac{d^n}{dx^n} e^{\frac{x^2}{2} - ikx}\right) dx$$

integrating n-times by parts

$$= e^{\frac{k^2}{2}} \int_{-\infty}^{+\infty} e^{-x^2} \left(\frac{d^n}{dx^n} e^{\frac{1}{2}(x-ik)^2}\right) = i^n e^{\frac{k^2}{2}} \int_{-\infty}^{+\infty} e^{-x^2} \left(\frac{d^n}{dk^n} e^{\frac{1}{2}(x-ik)^2}\right)$$

since here $\frac{d}{dx} = i\frac{d}{dk}$

$$= i^n e^{\frac{k^2}{2}} \frac{d^n}{dk^n} \int_{-\infty}^{+\infty} e^{-x^2 + \frac{x^2}{2} - ikx - \frac{k^2}{2}}$$

$$= i^n e^{\frac{k^2}{2}} \frac{d^n}{dk^n} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2} - ikx}\; e^{-\frac{k^2}{2}} = i^n e^{\frac{k^2}{2}} \frac{d^n}{dk^n} \left(\sqrt{2\pi}\, e^{-\frac{k^2}{2}} \cdot e^{-\frac{k^2}{2}}\right)$$

$= (-i)^m \sqrt{2\pi}\, \phi_m(k)$

$\Rightarrow$ the $\phi_m(x)$ are eigenvectors of the Fourier Transformation operator also for $m \neq 0$

FOURIER INVERSION THEOREM (without proof)

$f : \mathbb{R}^m \to \mathbb{C}$, $f \in L^1(\mathbb{R}^m)$, $\tilde{f} \in L^1(\mathbb{R}^m)$, f continuous in x

$$\Rightarrow \boxed{f(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \tilde{f}(k)\, e^{ikx}}$$

EXAMPLE

we have seen that $f(x) = e^{-m|x|} \Rightarrow \tilde{f}(k) = \frac{2m}{m^2+k^2}$

Both f and $\tilde{f} \in L^1(\mathbb{R})$ f continuous

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2m}{m^2+k^2}\, e^{ikx}\, dk$$

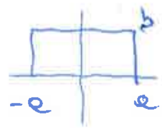
(this result can be checked with the residue theorem $\Rightarrow$ MMP2)

EXAMPLE

Let us consider the characteristic function of the $[-1,1]$ interval $\chi[-1,1]$

$$\chi[-1,1] := \begin{cases} 1 & x \in [-1,1] \\ 0 & x \notin [-1,1] \end{cases}$$

By using the previous results for the Fourier Transform of the function



we have $\tilde{\chi} = \frac{2\sin k}{k}$

$\tilde{\chi} \notin L^1(\mathbb{R})$ (indeed $\int \left| \frac{\sin x}{x} \right| dx$ does not converge)

Despite this we have

$$\chi_{[-1,1]}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\sin k}{k}\, e^{ikx}$$

We now consider the Fourier transform of functions that are INVARIANT UNDER ROTATIONS

We first want to show that if $g: \mathbb{R}^n \to \mathbb{C}$ is invariant under rotations, then it can depend only on the modulus of x, $|x|$, that is $g(x) = f(|x|)$. The reverse is also true.

Proof

Suppose that $g(x) = f(|x|) \Rightarrow$ since $|x|$ is invariant under rotations, $g(x)$ is invariant.

Conversely, suppose that $g(x)$ is invariant under rotations, and define $f(|x|) \equiv g(|x|, 0, 0 \dots 0)$. For every $x \in \mathbb{R}^n$ $\exists R \mid x = R(|x|, 0, 0 \dots 0)$

$\Rightarrow g(x) = g(|x|, 0 \dots 0) = f(|x|)$ ■

Having shown this, we consider a vector $x \in \mathbb{R}^n$ and write it as $x = r y$ with $|y| = 1$

We can write

$$\begin{cases} x_1 = \pm\sqrt{1 - y_2^2 \dots - y_n^2} \cdot r \\ x_2 = r y_2 \\ \vdots \\ x_n = r y_n \end{cases}$$

$\uparrow\ e$

$$(x_1, x_2 \dots x_n) \downarrow (r, y_2 \dots y_n)$$

The Jacobian of this transformation is $J = \det \dfrac{\partial(x_1 \dots x_n)}{\partial(r, y_2 \dots y_n)}$

$$\frac{\partial(x_1 \dots x_n)}{\partial(r, y_2 \dots y_n)} = \begin{pmatrix} \pm e & \mp \frac{r}{e} y_2 & \cdots & \mp \frac{r}{e} y_n \\ y_2 & r & 0 \cdots & 0 \\ y_3 & 0 & r \cdots & 0 \\ \vdots & & & \\ y_n & 0 \cdots & & r \end{pmatrix}$$

determinant of the (1,3) minor

$$J = \left| \pm e r^{n-1} \mp \frac{r y_2^2}{e} r^{n-2} (-1) \mp \frac{r}{e} y_3 \cdot (-y_3 r^{n-2}) + \cdots \right| = \frac{1}{e} r^{n-1} \underbrace{(e^2 + y_2^2 + \dots y_n^2)}_{=1}$$

↑ due to $(-1)^{1+2}$

We can thus write $dx_1 dx_2 \dots dx_M = \rho^{M-1} d\rho \dfrac{dy_2 \dots dy_M}{\sqrt{1-y_2^2 \dots y_M^2}} \equiv \rho^{M-1} d\rho \, d\Omega(y)$

This change of variable is very useful when we have to compute integrals of rotationally invariant functions, as it allows us to factorize the radial dependence from the angular dependence.

EXAMPLE

We can compute the SURFACE OF THE SPHERE in M dimensions

$$S_{M-1} \equiv \int d\Omega(y) = \int dy_1 \dots dy_M \; \delta(1-|y|)$$

integration constraint setting $|y|=1$

We can show that $\boxed{S_{M-1} = \dfrac{2\pi^{M/2}}{\Gamma(M/2)}}$ where $\Gamma(s) \equiv \int_0^\infty t^{s-1} e^{-t} dt$

The function $\Gamma(s)$ is the EULER GAMMA FUNCTION (see MMP2)

For $m \in \mathbb{N}$ we have $\Gamma(m+1) = m!$ $\Rightarrow$ $\Gamma(s)$ is an extension of the factorial to non integers!

We have that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\left(\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi} \right)$$

we also have $\Gamma(s+1) = s\,\Gamma(s)$ $\rightarrow$ analogous to the factorial!

$$\left(\Gamma(s+1) = \int_0^\infty t^s e^{-t} dt = -t^s e^{-t}\Big|_0^\infty + \int_0^\infty s\, t^{s-1} e^{-t} dt = s\,\Gamma(s) \right)$$

To show that the surface of the sphere is indeed related to the Γ function we start from the Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \quad \Rightarrow \quad \int_{\mathbb{R}^m} e^{-(x_1^2 + \dots x_m^2)} dx_1 \dots dx_m = \pi^{m/2}$$

$$= \int_0^\infty e^{-r^2} r^{m-1} \, dr \int d\Omega(y)$$

set $r^2 = t$

$2r\,dr = dt$

$$= \frac{1}{2} \int_0^\infty e^{-t} \, t^{\frac{m}{2}-1} \, dt \cdot S_{m-1} = \frac{1}{2} \Gamma\left(\frac{m}{2}\right) S_{m-1}$$

$$\Rightarrow \quad S_{m-1} = \frac{2\pi^{m/2}}{\Gamma(m/2)} \qquad \blacksquare$$

We have $S_1 = 2\pi$ → length of the circumference of unit radius

$$S_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{\frac{1}{2}\Gamma(\frac{1}{2})} = 4\pi$$ → surface of the sphere of unit radius

We now want to go back to the computation of the Fourier Transform of functions <u>invariant under rotations</u>. We first observe that if $f(x): \mathbb{R}^n \to \mathbb{C}$ is invariant under rotations, also its FT is invariant. Indeed

$$\hat{f}(Rk) = \int_{\mathbb{R}^n} f(x)\, e^{-i(Rk)\cdot x} = \int_{\mathbb{R}^n} f(Rx')\, e^{-i(Rk)\cdot(Rx')}\, dx' \underbrace{\det R}_{=1}$$

$$= \int_{\mathbb{R}^n} f(x')\, e^{-ikx'}\, dx' = \hat{f}(k)$$

We now show that the FT of a function $f : x \in \mathbb{R}^n \to \mathbb{C}$ invariant under rotations can be computed through a one dimensional integral involving special functions known as BESSEL FUNCTIONS

$$\hat{f}(k) = \int_{\mathbb{R}^n} f(|x|)\, e^{-ik\cdot x}\, dx = \int_0^\infty f(r)\, r^{n-1}\, dr \int e^{-irky}\, d\Omega(y)$$

using the variables introduced before

now since $\hat{f}(k)$ is also invariant we have

$$\hat{f}(k) = \hat{f}(|k|, 0 \dots 0) = \int_0^\infty f(r)\, r^{n-1}\, G_n(r|k|)\, dr$$

where

$$G_n(r|k|) \equiv \int e^{-ir|k|y_1}\, d\Omega(y)$$

We now show that $G_n(p)$ obeys a specific second order differential equation.

We have $G_M(|k|) = \int e^{-iky}\, d\Omega(y)$

we define $\Delta = \frac{\partial^2}{\partial k_1^2} + \dots \frac{\partial^2}{\partial k_M^2}$ LAPLACE OPERATOR

$$\Delta G_M = \int (-y_1^2 \dots -y_M^2)\, e^{-iky}\, d\Omega(y) = -G_M$$

Let us look for the explicit expression of ΔG_M. We have $(|k| \equiv p)$

$$\Delta G_M = \sum_{j=1}^{M} \partial_j \partial_j G_M = \sum_{j=1}^{M} \partial_j \left(\frac{k_j}{p} \frac{dG}{dp} \right)$$

$\left(\partial_i |k| = \frac{k_i}{|k|} \right.$ since $|k| = \sqrt{k_1^2 + \dots k_M^2}$

$$= \frac{M}{p} \frac{dG_M}{dp} - \sum_{j=1}^{M} \frac{1}{p^2} \frac{k_j}{p} k_j \frac{dG}{dp} + \sum_{j=1}^{M} \frac{k_j}{p} \frac{k_j}{p} \frac{d^2 G}{dp^2}$$

$$= \frac{M-1}{p} \frac{dG_M}{dp} + \frac{d^2 G_M}{dp^2}$$

$\Rightarrow$ G_M obeys the equation $\boxed{G_M'' + \frac{M-1}{p} G_M' + G_M = 0}$

It is possible to rewrite the function G_M in terms of special functions called <u>BESSEL FUNCTIONS</u>

$$G_M(p) \equiv (2\pi)^{M/2}\, p^{1-\frac{M}{2}}\, J_{M/2-1}(p)$$

$$G_M'(p) = (2\pi)^{M/2} \left(p^{-M/2} \left(1-\frac{M}{2}\right) J_{M/2-1}(p) + p^{1-\frac{M}{2}} J_{M/2-1}'(p) \right)$$

$$G_M''(p) = (2\pi)^{M/2} \left(\left(1-\frac{M}{2}\right)\left(-\frac{M}{2}\right) p^{-\frac{M}{2}-1} J_{M/2-1}(p) + 2\left(1-\frac{M}{2}\right) p^{-\frac{M}{2}} J_{M/2-1}'(p) + p^{1-\frac{M}{2}} J_{M/2-1}''(p) \right)$$

$\Rightarrow$ The differential equation can be recast in the form:

$$\rho^{1-\frac{n}{2}} \Big[J''_{\frac{n}{2}-1}(\rho) + \left(\frac{2}{\rho}\left(1-\frac{n}{2}\right) + \frac{n-1}{\rho} \right) J'_{\frac{n}{2}-1}(\rho)$$

$$+ \left(\frac{n-1}{\rho^2}\left(1-\frac{n}{2}\right) + 1 - \frac{n}{2}\left(1-\frac{n}{2}\right)\frac{1}{\rho^2} \right) J_{\frac{n}{2}-1}(\rho) \Big] = 0$$

$$\Rightarrow \quad J''_{\frac{n}{2}-1}(\rho) + \frac{1}{\rho} J'_{\frac{n}{2}-1}(\rho) + \left(1 - \frac{\left(1-\frac{n}{2}\right)^2}{\rho^2} \right) J_{\frac{n}{2}-1}(\rho) = 0$$

This equation (with $\alpha = \frac{n}{2}-1$) is called BESSEL DIFFERENTIAL EQUATION

$$\boxed{J''_{\alpha}(\rho) + \frac{1}{\rho} J'_{\alpha}(\rho) + \left(1 - \frac{\alpha^2}{\rho^2} \right) J_{\alpha}(\rho) = 0}$$

α can be generalized to arbitrary complex values (see MMP 2)

- If J_α is a solution $\Rightarrow$ $J_{-\alpha}$ is also a solution

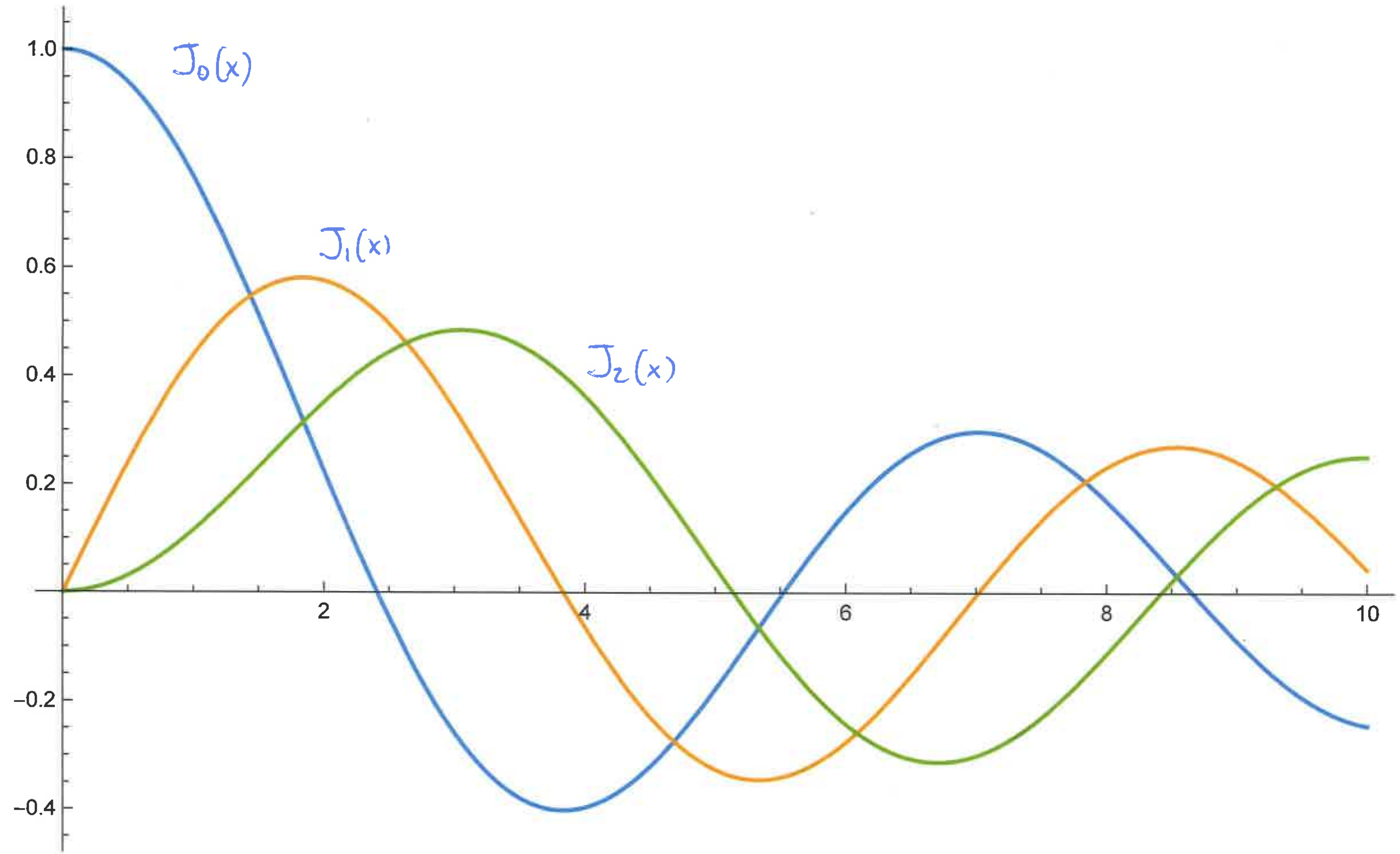
- α integer $\rightarrow$ cilindrical Bessel function

  α semiinteger $\rightarrow$ spherical Bessel functions

- Case $n=2$ $\qquad G_2(k) = \int_{S_1} e^{-iky}\, d\Omega(y) = \int_0^{2\pi} e^{-ik\cos\theta}\, d\theta$

$$= 2\pi J_0(k) \qquad\qquad J_0(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\cos\theta}\, d\theta$$

General solution of the equation can be found by using Frobenius (power series) method $\rightarrow$ MMP 2


$J_0(x)$
$J_1(x)$
$J_2(x)$
1.0
0.8
0.6
0.4
0.2
-0.2
-0.4
2
4
6
8
10

– APPLICATIONS OF FOURIER TRANSFORMATION: THE PROTON STRUCTURE

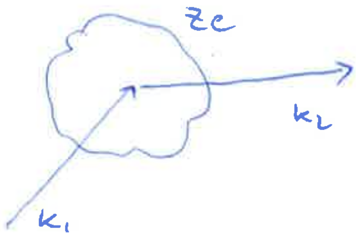
Wave-particle duality:

particle with momentum p $\iff$ wave with wavelength $\lambda = \frac{h}{p}$ → Planck constant

$$e^{-i\frac{2\pi}{\lambda}x} = e^{-i\frac{2\pi}{h}px} = e^{-i\frac{p}{\hbar}x} \rightarrow e^{-ikx} \qquad \hbar = \frac{h}{2\pi}$$

↳ "wave function"

Let us consider the scattering of a particle with $\underline{k}_1$ over a charge distribution



$$Ze = \int \rho(x)\, d^3x$$

↑ total charge

The potential generated by the charge distribution is $V(\underline{x}) = \int \frac{\rho(\underline{x}')}{4\pi|\underline{x}-\underline{x}'|} d^3x'$

The number of scattered particles in the unit angular region is

$dN \sim |T|^2 d\Omega$ where T is the <u>QUANTUM AMPLITUDE</u>

The quantum amplitude for the scattering process can be computed by "sandwiching" the potential between the initial and final states

$$\Rightarrow \quad T \sim \langle f|V|i\rangle = \int e^{i\underline{k}_2\underline{x}}\, V(x)\, e^{-i\underline{k}_1\underline{x}}\, d^3x$$

$$= \tilde{V}(q) \qquad \underline{q} = \underline{k}_1 - \underline{k}_2$$

⇒ <u>The quantum amplitude is obtained as the Fourier Transform of the potential</u>

Let us compute $\tilde{V}(q)$

$$\int V(x)\, e^{-i\underline{q}\underline{x}} = \int d^3x' \frac{\rho(x')}{4\pi|\underline{x}-\underline{x}'|} e^{-i\underline{q}\underline{x}} d^3x = \int d^3x'\, \rho(x') \int d^3x \frac{e^{-iq(x-x'+x')}}{4\pi|\underline{x}-\underline{x}'|}$$

$$= -\int d^3x'\, \rho(x')\, e^{-i\underline{q}\cdot\underline{x}'} \frac{1}{\underline{q}^2}$$ From the exercise sheet

$$= -\frac{1}{\underline{q}^2} Ze\, F(\underline{q}) \qquad F(\underline{q}) = \frac{1}{Ze}\int \rho(x')\, e^{-i\underline{q}\underline{x}'} d^3x'$$

"FORM FACTOR"

$F(0) = 1$

By measuring the distribution of the scattered particles we can get information of the charge distribution

Suppose $\underline{q}$ not too large

$$\Rightarrow F(\underline{q}) = \frac{1}{Ze}\int \rho(x)\, e^{-i\underline{q}\underline{x}} d^3x \approx \frac{1}{Ze}\int \rho(x)\left(1 - i\underline{q}\cdot\underline{x} - \frac{1}{2}(\underline{q}\cdot\underline{x})^2 + \cdots\right)$$

If ρ is spherically symmetric $\Rightarrow$ the linear term in $\underline{q}$ does not contribute and we get

$$F(\underline{q}) \approx 1 - \frac{1}{6}|\underline{q}|^2 \langle r^2 \rangle$$

mean square radius

rough information: total charge

$\Rightarrow$ small scattering angles probe the mean square radius of the charge cloud!

In 1967 SLAC started to use electrons up to a maximum of 22 GeV and to collide them on fixed target protons

The experimentally observed $|q|^2$ dependence was

$$F(q) \approx \left(1 + \frac{|q|^2}{0.71}\right)^{-2}$$

$$\Rightarrow \langle r^2 \rangle \approx (0.8 \cdot 10^{-15} m)^2 \quad \Rightarrow$$

The proton is not pointlike but its size is about $1\,fm = 10^{-15}\,m$