MMP I
Tutorial 11

HS 2017
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Exercise 1: Perturbation theory (8 Pts.)
A linear operator $\mathcal{H} = \mathcal{H}_0 + V$ and a vector $|y\rangle$ in $\mathbb{C}^2$ are represented in the canonical basis by

$$\mathcal{H}_0 = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \; V = \begin{pmatrix} \sin(a\gamma) & b_1\gamma^2 e^{b_2\gamma} \\ b_1\gamma^2 e^{b_2\gamma} & c_1\gamma^3 \end{pmatrix}, \; |y\rangle = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1.1}$$

with $h_i, a, b_i, c_i, \gamma \in \mathbb{R}$. It often happens that the 'perturbation' V contains a small parameter γ. This allows for the construction of approximate solutions as an expansion in γ. To this end the resolvent is written as $R = R_0 \sum_{n=0}^{\infty}(-VR_0)^n$ with $R \equiv R_{\mathcal{H}}(z)$and $R_0 \equiv R_{\mathcal{H}_0}(z)$.

a) Solve $(\mathcal{H}_0 - z)|x\rangle = |y\rangle$ for $z \notin \{h_1, h_2\}$.

b) Solve $(\mathcal{H} - z)|x\rangle = |y\rangle$ approximately, taking into account all terms up to $\mathcal{O}(\gamma)$.

c) Solve $(\mathcal{H} - z)|x\rangle = |y\rangle$ approximately, taking into account all terms up to $\mathcal{O}(\gamma^2)$.

d) Show that the eigenvalues λ_i of $\mathcal{H}$ satisfy:

$$\lambda_i = h_i + \frac{\langle \phi_i^{(0)} | V | \phi_i \rangle}{\langle \phi_i^{(0)} | \phi_i \rangle} \tag{1.2}$$

where $|\phi_i^{(0)}\rangle$ and $|\phi_i\rangle$ are the eigenvectors of $\mathcal{H}_0$ and $\mathcal{H}$, respectively.

e) Compute approximately the eigenvalues of $\mathcal{H}$ taking into account all the terms up to $\mathcal{O}(\gamma)$.

– please turn over –

Exercise 2: Differential operator (6 Pts.)
Consider the Hilbert space $L^2[0,1]$ and let D be the set of functions $|x\rangle \sim x(t) \in L^2[0,1]$ that are differentiable (almost everywhere) and $x'(t) \in L^2[0,1]$ (more precisely: absolutely continuous functions). We now consider four operators with $A_3 \subset A_{1/2} \subset A_0$:

$$\begin{aligned} A_0 &= i\frac{\mathrm{d}}{\mathrm{d}t},\ D_A = D \\ A_1 &= i\frac{\mathrm{d}}{\mathrm{d}t},\ D_{A_1} = \{x|x \in D \text{ and } x(0) = x(1)\} \\ A_2 &= i\frac{\mathrm{d}}{\mathrm{d}t},\ D_{A_2} = \{x|x \in D \text{ and } x(0) = -x(1)\} \\ A_3 &= i\frac{\mathrm{d}}{\mathrm{d}t},\ D_{A_3} = \{x|x \in D \text{ and } x(0) = x(1) = 0\} \end{aligned} \tag{2.1}$$

Among these operators at least one is hermitian (hence also symmetric), at least one is symmetric (but not hermitian) and at least one is not even symmetric. Which is which and why? What about the operator $\mathcal{H} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}$, $D_{\mathcal{H}} = \{x|x \in D \text{ and } x(0) = x(1) = 0\}$?