Applications of General Relativity
Sheet 3

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Exercise 1 [Metric of a static star]
The exterior spacetime of extended spherical objects can be approximated by the Schwarzschild metric. In this exercise we will derive the interior metric modeling the star by a perfect fluid with energy-momentum tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$. As a first step we will consider the general static, spherically symmetric metric

$$\mathrm{d}s^2 = \exp\left[2\alpha(r)\right]\mathrm{d}t^2 - \exp\left[2\beta(r)\right]\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1}$$

The corresponding non-vanishing Christoffel symbols are (up to permutations of the two lower indices)

$$\begin{aligned}
&\Gamma^t_{t\,r} = \partial_r\alpha \\
&\Gamma^r_{t\,t} = \partial_r\alpha\exp\left[2(\alpha-\beta)\right] \quad \Gamma^r_{r\,r} = \partial_r\beta \quad \Gamma^r_{\theta\,\theta} = -r\exp\left[-2\beta\right] \quad \Gamma^r_{\phi\,\phi} = -r\sin^2(\theta)\exp\left[-2\beta\right] \\
&\Gamma^\theta_{\theta\,r} = r^{-1} \quad \Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta) \\
&\Gamma^\phi_{\phi\,r} = r^{-1} \quad \Gamma^\phi_{\phi\,\theta} = \cot(\theta),
\end{aligned}$$

while the Ricci tensor takes the form

$$R_{tt} = \exp\left[2(\alpha-\beta)\right]\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right) \tag{2}$$

$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \tag{3}$$

$$R_{\theta\theta} = 1 - \exp\left[-2\beta\right] + (\partial_r\beta - \partial_r\alpha)\, r\exp\left[-2\beta\right] \tag{4}$$

$$R_{\phi\phi} = \sin^2(\theta) R_{\theta\theta}. \tag{5}$$

(i) In the fluid rest frame the velocity is pointing in the timelike direction. Using this and the normalization $u_\mu u^\mu = 1$ show that the energy momentum tensor can be written as

$$T_{\mu\nu} = \mathrm{diag}\left\{\exp\left[2\alpha\right]\rho, \exp\left[2\beta\right]p, r^2\,p, r^2\sin^2(\theta)\;p\right\}. \tag{6}$$

(ii) Write down the Einstein equations inside the star.

It will be convenient to define

$$m(r) = \frac{1}{2G}\big(r - r\exp\left[-2\beta\right]\big) \tag{7}$$

and to rewrite the Einstein equations in terms of $m(r)$.

(iii) Use the energy-momentum conservation $\nabla_\mu T^{\mu\nu}$ to show

$$(\rho + p)\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}. \tag{8}$$

Use the above equation together with the rr-component of the Einstein equations to derive the Tolman-Oppenheimer-Volkoff equation of a star

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)\left[Gm(r) + 4\pi G r^3 p\right]}{r\left[r - 2Gm(r)\right]}. \tag{9}$$

This is the relativistic version of the equation of hydrostatic equilibrium for a star.

(iv) Assuming an incompressible fluid with constant density ρ_* out to the surface of the star, solve for the pressure $p(r)$ and write down the metric. What is the maximum mass a star of a given radius R can have?