SUSY invariant actions with chiral superfields

I) Let's consider the simplest case following the general rule (integral of superfield on superspace):

$$\mathcal{L}^{(0)} = \int d^2\vartheta\, d^2\bar\vartheta \, [\bar\Phi \Phi] = [\bar\Phi\Phi]\Big|_{\vartheta^2\bar\vartheta^2\text{-comp.}}$$

* Right dimension

$\mathrm{Dim}[\Phi] = 1 \qquad \mathrm{Dim}[\vartheta^2] = -1 \qquad \mathrm{Dim}\left[\left(\frac{\partial}{\partial\vartheta}\right)^2\right] = +1$

* Real (note that $\bar\Phi\Phi$ is a real field)

$\Downarrow$

The explicit calculation ($\to$ exercise) yields:

$$\mathcal{L}^{(0)} = \underbrace{\partial_\mu\bar\varphi\,\partial^\mu\varphi}_{\text{(complex) Klein Gordon}} + \underbrace{\frac{i}{2}\left(\partial_\mu\psi\,\sigma^\mu\bar\psi - \psi\,\sigma^\mu\partial_\mu\bar\psi\right)}_{i\,\partial_\mu\psi\,\sigma^\mu\bar\psi \;=\; \text{Dirac (Weyl)}} + \underbrace{\bar F F}_{\text{auxiliary}} + \text{tot. deriv.}$$

e.o.m. for F $\to$ $\boxed{F=0}$

Using the e.o.m. we can eliminate F, in this case $\mathcal{L}^{(0)}$ remains SUSY invariant only if φ & ψ also satisfy their e.o.m. ($\to$ check)

$\Downarrow$

$$\mathcal{L}^{(0)} \equiv \mathcal{L}_{kin}(\varphi) + \mathcal{L}_{kin}(\psi)$$

(- no mass
- no interactions
- $2_F + 2_B$ (on-shell))

It is interesting to derive the e.o.m. directly using Superfields.

We cannot do it from $[\bar{\Phi}\phi]$, since we have to take into account the "chiral-constraint"

* Note that $\int d\vartheta\, f(\vartheta) = \frac{d}{d\vartheta} f(\vartheta)$

$$\Downarrow$$

$$\int d^2\vartheta\, d^2\bar{\vartheta}\; F(\vartheta,\bar{\vartheta}) = \int d^2\bar{\vartheta} \left[D^2 F(\vartheta,\bar{\vartheta}) + \underset{(4\text{ dim})}{\text{total deriv.}} \right]$$

$$\Downarrow$$

$$\int d^2\vartheta\, d^2\bar{\vartheta}\; \bar{\Phi}\phi = \int d^2\bar{\vartheta}\; D^2(\bar{\Phi}\phi) = \int d^2\bar{\vartheta}\; \bar{\Phi} D^2\phi$$

where we have used the chiral constraint $D_\alpha \bar{\Phi} = \emptyset$

$$\Downarrow$$

e.o.m $\boxed{D^2\phi = 0}$ $\longrightarrow$ expanding in components one get's exactly K.G. + Dirac + $F=0$

II) Can we consider more general Lagrangians (using only ϕ & $\bar{\Phi}$) ?

Natural candidate :

$$\mathcal{L}_{(D)} = \int d^2\vartheta\, d^2\bar{\vartheta}\; K(\phi, \bar{\Phi})$$

$\uparrow$ Kähler-potential

K must be
- real
- scalar
- cannot be fuction of $D_\alpha\phi$ or $\bar{D}_{\dot{\alpha}}\bar{\Phi}$ $\rightarrow$ higher deriv. terms

$$\Downarrow$$

$$K(\phi,\bar{\Phi}) = \sum c_{nm}\, \phi^n \bar{\Phi}^m \quad \text{with} \quad c_{nm} = c^*_{mn}$$

but only c_{11} gives renormalizable + non-trivial interactions

$$\left(c_{nm} \sim \Lambda^{+2-n-m} \right)$$

III) A further option is possible, in the case of chiral fields, noting that the F-term of a chiral field transforms under SUSY as a total derivative

$$\bar{D}_{\dot\alpha} \Sigma_i = 0 \quad \rightarrow \quad \int d^2\vartheta \, \Sigma = \Sigma\Big|_F$$ is SUSY-invariant (but tipically not real)

So, let's consider a generic holomorphic function of Φ, $W(\phi)$ $\left(\text{holomorphic} \Rightarrow \frac{\partial W}{\partial \bar\phi} = 0\right)$

If ϕ is chiral $(\bar{D}_{\dot\alpha}\phi = 0) \Rightarrow \bar{D}_{\dot\alpha} W(\phi) = 0$

proof $$\bar{D}_{\dot\alpha} W(\phi) = \frac{\partial W}{\partial \phi} \underbrace{\bar{D}_{\dot\alpha}\phi}_{=0} + \underbrace{\frac{\partial W}{\partial \bar\phi}}_{=0} \bar{D}_{\dot\alpha}\bar\phi = 0$$

$$\Downarrow$$

A potential candidate Lagrangian density is

$$\mathcal{L}_{(F)} = \int d^2\vartheta \, \underbrace{W(\phi)}_{\text{Super-potential}} + \int d^2\bar\vartheta \, \bar{W}(\bar\phi) = \text{hermitian} \oplus \text{SUSY-invariant}$$

N.B: W cannot contains $D_\alpha \phi$, since $D_\alpha\phi$ is NOT a chiral superfield (recall D_α & $\bar{D}_{\dot\alpha}$ do not commute)

The canonical dimension of $W(\phi)$ is 3

$$\Rightarrow \quad W(\phi) = \sum_n a_n \phi^n \qquad n \le 3$$ if we want to have renormalizability

Note that $\mathcal{L}_{(F)}$ selects only the ϑ^2 (or F) component of $W(\Phi)$

$$W(\Phi) = W(\varphi) + \sqrt{2}\,\frac{\partial W}{\partial \varphi}\,\vartheta\psi - \vartheta\vartheta\left(\frac{\partial W}{\partial \varphi}\,F + \frac{1}{2}\frac{\partial^2 W}{\partial\varphi\partial\varphi}\,\psi\psi\right)$$

full - superfield

Scalar comp.

$+$ terms with $\bar\vartheta$

$$\downarrow$$

$$\mathcal{L}_{(F)} = -\frac{\partial W}{\partial \varphi}\,F - \frac{1}{2}\frac{\partial^2 W}{\partial\varphi\partial\varphi}\,\psi\psi + \text{h.c.}$$ (in full generality)

$\underline{\text{IV}}$) Let's now consider the most general Lagrangian built with a single chiral field

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_F = \int d^2\vartheta\, d^2\bar\vartheta\; K(\Phi,\bar\Phi) + \int d^2\vartheta\, W(\Phi) + \int d^2\bar\vartheta\, \bar W(\bar\Phi)$$

$\downarrow$ imposing renormalizability

$$\mathcal{L}^{\text{ren.}}_{D+F} = \partial_\mu\bar\phi\,\partial^\mu\phi + i\,\partial_\mu\psi\,\sigma^\mu\bar\psi + \bar F F - \left(\frac{\partial W}{\partial \varphi}\,F + \frac{1}{2}\frac{\partial^2 W}{\partial\varphi\partial\varphi}\,\psi\psi + \text{h.c.}\right)$$

$\Rightarrow$ e.o.m. $\quad \bar F = \dfrac{\partial W}{\partial \varphi} \qquad F = \dfrac{\partial \bar W}{\partial \bar\varphi}$

$$\downarrow$$

$$\mathcal{L}^{\text{on-shell}} = \mathcal{L}_{\text{kin}} - \left|\frac{\partial W}{\partial \varphi}\right|^2 - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\varphi\partial\varphi}\,\psi\psi + \text{h.c.}\right)$$

$V(\varphi)$ $\Rightarrow$ naturally bounded scalar potential

V) The previous result can easily be generalised to the case of several chiral fields ϕ_i

$$K(\phi_i, \bar{\Phi}_i) = \bar{\Phi}_i \phi_i \quad \rightarrow \quad \text{kinetic terms}$$

$$W(\phi_i) = \Lambda_i^2 \Phi_i + \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} y_{ijk} \phi_i \phi_j \phi_k$$

$$\Downarrow$$

⇒ Scalar potential:

$$V(\varphi_i, \varphi_j) = \sum_i \left| \frac{\partial W(\varphi_i)}{\partial \varphi_i} \right|^2 = \bar{F}_i F_i$$

$$= m_{ik}^* m_{kj} \varphi_i^* \varphi_j + \frac{1}{2} \left(m_{in} y_{jkn}^* \varphi_i \varphi_j^* \phi_k^* + \text{h.c.} \right)$$

$$+ \frac{1}{4} y_{ijn} y_{kln}^* \varphi_i \varphi_j \varphi_k^* \varphi_l^* + \text{const. terms}$$

$$F_i = \left. \frac{\partial W}{\partial \phi_i} \right|_{\phi_i = \varphi_i} = m_{ij} \varphi_j + \frac{1}{2} y_{ijk} \varphi_j \varphi_k$$

for simplicity I set $\Lambda_i^2 = 0$

⇒ Yukawa-type interactions:

$$\Delta \mathcal{L}_Y = -\frac{1}{2} \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \psi_i \psi_j + \text{h.c.}$$

$$= -\frac{1}{2} y_{ijk} \varphi_i \psi_j \psi_k + \text{h.c.}$$

N.B.: even for $\Lambda_i^2 \neq 0$ there is always a one-to-one relation between quartic terms in the potential (φ^4) and Yukawa couplings