6.2 Hamilton's equations

Given a system with N generalised coordinates q_i and a Lagrangian $L(q_i, \dot{q}_i, t)$ we define the Hamiltonian of the system as the Legendre transformation of L with respect to the $\dot{q}_i$ variables:

$$H(q_i, p_i, t) = \sum_{i=1}^{N} p_i \dot{q}_i(q_i, p_i) - L(q_i, \dot{q}_i(q_i, p_i), t) \tag{6.6}$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = p_i(q_i, \dot{q}_i) \longrightarrow \dot{q}_i = \dot{q}_i(q_i, p_i) \tag{6.7}$$

By construction, given the properties of Legendre transformations, (see 6.1) we have

$$dH = \dot{q}_i(q_i, p_i)\, dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \tag{6.8}$$

On the other hand, we can re-write this equation as

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt \tag{6.9}$$

Equating (6.8) and (6.9) we deduce that

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Finally, using Euler-Lagrange eqs., namely $\dot{p}_i = \frac{\partial L}{\partial q_i}$, we arrive to

$$\begin{cases} \dot{q}_i = +\dfrac{\partial H}{\partial p_i}(q_i, p_i) \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}(q_i, p_i) \end{cases} \tag{6.10}$$

These are known as **Hamilton's equations**.

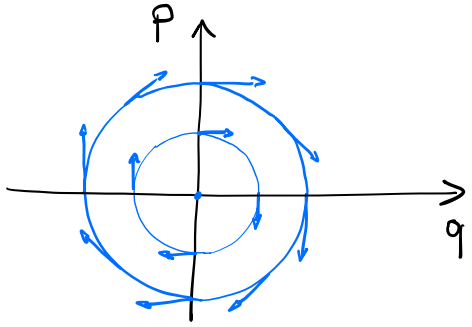
As anticipated, they are differential 1st order equations and describe the flow in phase space

Example: one-dim. harmonic oscillator

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2$$

$$p = \frac{\partial L}{\partial \dot{q}} = m \dot{q}$$

$$H = \frac{1}{2m} p^2 + \frac{1}{2} k q^2 \longrightarrow \begin{cases} \dot{q} = \frac{1}{m} p \\ \dot{p} = -k q \end{cases}$$



Example: point-like particle in a central field potential

Using radial coordinates:

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2\vartheta \, \dot{\phi}^2 \right) - U(r)$$

$$p_r = m \dot{r} \qquad p_\vartheta = m r^2 \dot{\vartheta} \qquad p_\phi = m r^2 \sin^2\vartheta \, \dot{\phi}$$

$$H = \frac{1}{2m} \left(p_r^2 + \frac{1}{r^2} p_\vartheta^2 + \frac{1}{r^2 \sin^2\vartheta} p_\phi^2 \right) + U(r)$$

$$\begin{cases} \dot{r} = \frac{1}{m} p_r \\ \dot{\vartheta} = \frac{1}{m r^2} p_\vartheta \\ \dot{\phi} = \frac{1}{m r^2 \vartheta^2} p_\phi \end{cases} \qquad \begin{aligned} \dot{p}_r &= \frac{1}{m r^3} p_\vartheta^2 + \frac{1}{m r^3 \sin^2\vartheta} p_\phi^2 - \frac{\partial U}{\partial r}(r) \\ \dot{p}_\vartheta &= \frac{1}{m r^2} \frac{\cos\vartheta}{\sin^3\vartheta} p_\phi^2 \\ \dot{p}_\phi &= \not{0} \qquad \text{(cyclic)} \end{aligned}$$

Note that the functional expression of H changes if we uses different variables. For instance with cartesian coordinates

$$L = \frac{1}{2} m \, \dot{\vec{x}} \cdot \dot{\vec{x}} - U(|\vec{x}|) \qquad \vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m \dot{\vec{x}}$$

$$H = \frac{1}{2m} |\vec{p}|^2 + U(|\vec{x}|)$$

In both examples H coincides with the total energy of the system, although conceptually these are two different things. Physical differences arise only in systems where the energy is not conserved. However, the subtle connection between the two can already be apreciated by the next example.

Example: point-like particle moving into an electromagnetic field

$$L = T - V = \underbrace{\frac{1}{2} m (\dot{\vec{x}})^2}_{T} - \underbrace{q\left[\Phi - \vec{A}\cdot\dot{\vec{x}}\right]}_{V}$$

- q = charge
- ϕ = scalar potential $\qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$
- $\vec{A}$ = vector " $\qquad \vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{F} = q\left(\vec{E} + \dot{\vec{x}}\times\vec{B}\right) = q\left[-\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} + \dot{\vec{x}}\times\left(\vec{\nabla}\times\vec{A}\right)\right]$$

$$= -q\,\vec{\nabla}\left[\phi - \vec{A}\,\dot{\vec{x}}\right] \qquad \text{assuming } \frac{d}{dt}\vec{A} = \not{0}$$

The momentum is

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + q\vec{A}$$

↳ new term due to the e.m. field

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi$$

As can be seen, in this case $H \neq T + V$. However, H describes indeed correctly the energy of the system, since only ϕ contribute to the potential energy

6.2.1 Simple conservation laws in Hamiltonian systems

The conservation of H in the limit where $\frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = 0$ directly follows from the expression of dH:

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = -\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i = 0 \qquad (6.11)$$

($\frac{\partial H}{\partial t} \to 0$)

An interesting case is that of cyclic variables, i.e. variables for which the corresponding momentum is conserved:

$$\frac{\partial L}{\partial q_1} = 0 \quad \to \quad p_1 = \frac{\partial L}{\partial \dot{q}_1} = \text{const} \equiv \alpha_1 \qquad (6.12)$$

In the Lagrangian formulation eq. (6.12) can be viewed as a constraint on the remaining N-1 differential equations:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}(q_1, \dots q_N, \dot{q}_1, \dots \dot{q}_N, t)\right] - \frac{\partial L}{\partial q_i}(q_1, \dots q_N, \dot{q}_1 \dots \dot{q}_N, t) = 0 \qquad i = 2 \dots N$$

In the Hamiltonian formalism a cyclic coordinate can be completely eliminated (it does not contributes to the flow):

By construction, if L does not depend on q_1, also H does not depend on q_1

$$H = H(q_2, \dots q_N, p_1, \dots p_N, t) = H(q_2, \dots q_N, \alpha_1, p_2, \dots p_N, t)$$

$$\equiv H'(q_2, \dots q_N, p_2, \dots p_N, t)$$

⇒ Hamilton's equations depend only on the remaining $2\times(N-1)$ variables

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q_i, p_i, t) \qquad \dot{p}_i = -\frac{\partial H'}{\partial q_i}(q_i, p_i, t) \qquad i = 2, \dots N$$

Hence we acchieved a "dimensional reduction" of the original problem.

6.2.2 Hamilton's eq.s from a variational principle

Hamilton equations can also be derived by the principle of Least Action, provided we extend it to the whole phase space:

$$S\left[\{q_i(t)\}\right] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$

Lagrangian formalism

$$\downarrow$$

$$S\left[\{q_i(t), p_i(t)\}\right] = \int_{t_1}^{t_2} \left[p_i\, \dot{q}_i(p_i, q_i) - H(p_i, q_i)\right] dt$$

Hamiltonian formalism

↑ functional on the phase space

We now impose the condition $\delta S = 0$, under a variation of both $q_i(t)$ & $p_i(t)$, under the constraint $\delta q_i(t_1) = \delta q_i(t_2) = \phi$

$$\delta S = \int_{t_1}^{t_2} \left[\delta p_i\, \dot{q}_i + p_i\, \delta \dot{q}_i - \frac{\partial H}{\partial p_i}\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i\right] dt$$

$$\hookrightarrow \frac{d}{dt}(p_i\, \delta q_i) - \dot{p}_i\, \delta q_i$$

$$= \int_{t_1}^{t_2} \left\{\left[\dot{q}_i - \frac{\partial H}{\partial p_i}\right]\delta p_i - \left[\dot{p}_i + \frac{\partial H}{\partial q_i}\right]\delta q_i\right\} dt + \left[p_i\, \delta q_i\right]_{t_1}^{t_2}$$

$$\Downarrow \qquad\qquad \hookrightarrow \phi$$

$$\delta S = 0 \iff \text{Hamilton's eq.s}$$

Note an important subtle point: there was no need to impose $\delta p_i = 0$ at the boundaries, as one would have expected from a generic minimization procedure.
This is an indication of the redundancy of the Hamiltonian formalism in this context.

6.2.3 Simplectic representation of Hamilton's eq.s

Hamilton's equations bring q_i and p_i essentially at the same level. Still, the eq.s are not completely symmetric in these two sets of variables. A formulation that brings them to an even more symmetric level, that is useful in certain applications, is the so-called symplectic representation.

The idea is to define a $2N$ vector

$$\eta_i = q_i \quad i=1\dots N \qquad \eta_{N+i} = p_i \quad i=1\dots N \qquad \Rightarrow \quad \eta = \begin{pmatrix} q_i \\ p_i \end{pmatrix} \tag{6.13}$$

In terms of η Hamilton's eq.s can be written as

$$\frac{d}{dt}\eta_i = J_{ij}\frac{\partial H}{\partial \eta_j} \qquad i,j = 1\dots 2N \tag{6.14}$$

where

$$J_{ij} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{6.15}$$

$N\times N$ identity matrix

The $2N\times 2N$ matrix J satisfies

$$J^2 = -\mathbb{1}_{2N\times 2N} \qquad J^T = -J = J^{-1} \qquad \text{and} \qquad \det J = 1 \tag{6.16}$$