2) Representations of $SL(2,\mathbb{C})$ & Weyl spinors

Given the relation between Lorentz & $SL(2,\mathbb{C})$ (or $SU(2)\times SU(2)^*$) we can classify the representations of the Lorentz group starting from those of $SL(2,\mathbb{C})$.

⇒ The basic rep. of $SL(2,\mathbb{C})$ is the spinor (2 component)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta \qquad M \in SL(2,\mathbb{C})$$

and the conjugate representation, that is not equivalent

$$\bar\psi = \begin{pmatrix} \bar\psi_1 \\ \bar\psi_2 \end{pmatrix} \qquad \bar\psi_{\dot\alpha} \to \bar\psi'_{\dot\alpha} = (M^*)_{\dot\alpha}{}^{\dot\beta}\, \bar\psi_{\dot\beta}$$

(inequivalent ⇔ there exists not a C: $M = C M^* C^{-1}$)

$$\boxed{\begin{array}{c}\psi \sim (1/2, 0) \\ \text{LEFT-HANDED} \\ \text{SPINOR}\end{array}} \quad \& \quad \boxed{\begin{array}{c}\bar\psi \sim (0, 1/2) \\ \text{RIGHT-HANDED} \\ \text{SPINOR}\end{array}} \qquad \boxed{\psi_\alpha = \text{Grassman variables}}$$

⇒ Equivalent 2-dim. representations are obteind using the antisymmetric tensors

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\dot\alpha\dot\beta} \qquad \epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad \Rightarrow \quad \psi^\alpha \to \psi^{\alpha\prime} = \psi^\beta (M^{-1})_\beta{}^\alpha$$

$$\bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} \quad \Rightarrow \quad \bar\psi^{\dot\alpha} \to \bar\psi^{\dot\alpha\prime} = \bar\psi^{\dot\beta}(M^{*-1})_{\dot\beta}{}^{\dot\alpha}$$

$\Rightarrow$ Invariant products (contractions):

$$\psi\chi = \psi^\alpha \chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta \chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha \chi_\beta = -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi$$

$$\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \dots \bar\chi\bar\psi$$

$$(\psi\chi)^\dagger = (\psi^\alpha \chi_\alpha)^\dagger = \chi_\alpha^\dagger \psi^{\alpha\dagger} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi = \bar\psi\bar\chi$$

$\Rightarrow$ Introduction of σ^μ:

$$\begin{cases} (\sigma^\mu)_{\alpha\dot\alpha} = (\sigma_0, -\sigma_i)_{\alpha\dot\alpha} \\ (\bar\sigma^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (\sigma_0, +\sigma_i)^{\dot\alpha\alpha} \end{cases}$$

We define dotted & un-dotted indices such that the contractions are always of the type $\psi^\alpha \searrow \chi_\alpha$ $\quad \bar\psi_{\dot\alpha} \nearrow \bar\chi^{\dot\alpha}$

e.g.: $$\begin{cases} \psi\sigma^\mu\bar\chi = \psi^\alpha(\sigma^\mu)_{\alpha\dot\beta}\bar\chi^{\dot\beta} \\ \bar\psi\bar\sigma^\mu\chi = \bar\psi_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\chi_\beta \end{cases}$$

One can proof various useful identities, such as

$$\chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi$$

$$(\chi\sigma^\mu\bar\psi)^\dagger = \psi\sigma^\mu\bar\chi$$

Connections with the Dirac representation :

Dirac matrices in the chiral representation :

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underbrace{\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}}$$

$$\text{equiv. to} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

In this representation

$$\psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \qquad \psi_R = \begin{pmatrix} 0 \\ \bar\chi^{\dot\alpha} \end{pmatrix}$$

Dirac equation :

$$\bar\Psi_L \, i \not\partial \, \Psi_L = \psi_L^\dagger \, i \, \gamma^0 \gamma^\mu \, \partial_\mu \, \psi_L = i \, \bar\psi^{\dot\alpha} (\bar\sigma^\mu)_{\dot\alpha\alpha} \, \partial_\mu \psi^\alpha$$

$$\gamma^0\gamma^\mu \downarrow \begin{pmatrix} \bar\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \qquad = i \left(\psi \, \sigma^\mu \partial_\mu \bar\psi \right)$$

after integr. by parts

$$\bar\Psi_R \, i \not\partial \Psi_R = i \left(\chi \, \sigma \cdot \partial \, \bar\chi \right)$$

Dirac mass term :

$$m_D \left(\bar\Psi_L \Psi_R + \bar\Psi_R \Psi_L \right) = m_D \left(\bar\psi \bar\chi + \chi \psi \right) = m_D \left(\chi\psi + \text{h.c.} \right)$$

Majorana mass term :

$$\frac{1}{2} m_M \left(\psi\psi + \bar\psi\bar\psi \right) \qquad \psi\psi = \psi^\alpha \epsilon_{\alpha\beta} \psi^\beta$$

N.B.

$$\Psi_{Dir} = \begin{pmatrix} \psi \\ \bar\chi \end{pmatrix} \longrightarrow \Psi_{Dir}^c = \begin{pmatrix} \chi \\ \bar\psi \end{pmatrix}$$

$$\Psi_{Maj.} = \begin{pmatrix} \psi \\ \bar\psi \end{pmatrix}$$

Finally, the generators of Lorentz transformations for Dirac spinors are

$$M_{\mu\nu}\Big|_{\substack{\text{Dirac}\\ \text{spinors}}} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$

$$= i\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}$$

which shows that ψ & $\bar\chi$ belongs to two inequivalent representations, with different (independent) generators

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{4}\left[\sigma^\mu_{\alpha\dot\gamma}\,(\bar\sigma^\nu)^{\dot\gamma\alpha} - (\mu\leftrightarrow\nu)\right]$$

$$(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{1}{4}\left[(\bar\sigma^\mu)^{\dot\alpha\gamma}\,(\sigma^\nu)_{\gamma\dot\beta} - (\mu\leftrightarrow\nu)\right]$$