

PHY 117 HS2026

Today:

Cross product

Torque

Static equilibrium

Center of mass

Stability vs. rotation

Moment of inertia

Week 4, Lecture 1

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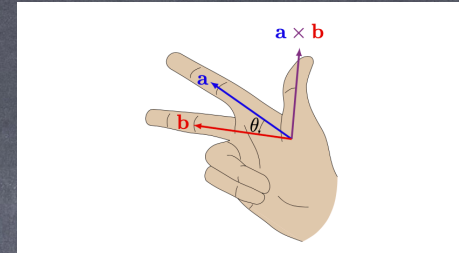
vector product or cross product

$$\vec{c} = \vec{a} \times \vec{b}$$

$\vec{c}$ is $\perp$ to both $\vec{a}$ & $\vec{b}$

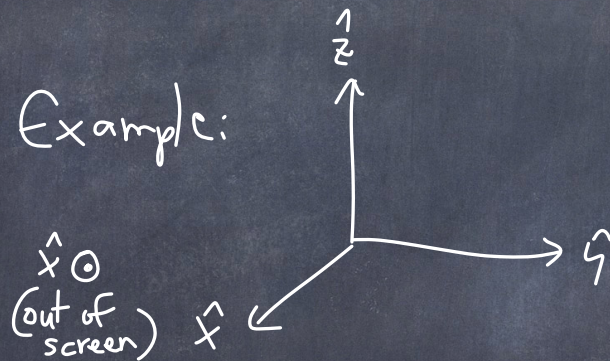
$$\vec{c} = \vec{a} \times \vec{b}$$

$$|\vec{c}| = |\vec{a}| |\vec{b}| \sin \theta_{ab}$$



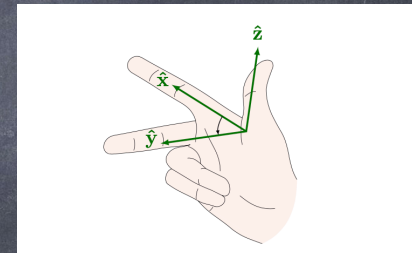
θ_{ab} is the angle from $\vec{a}$ to $\vec{b}$
The angle has a direction

Example:



what is

$$\begin{aligned} \hat{x} \times \hat{y} &= \hat{z} \\ \hat{y} \times \hat{z} &= \hat{x} \\ \hat{z} \times \hat{x} &= \hat{y} \end{aligned}$$



Note: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

- Torque :
- a force applied at a distance from a fixed point
 - tends to cause an object to rotate.
 - symbol, $\vec{\tau}$ vector
 - units: $\tau \cdot x = [N \cdot m]$



$$\text{torque} = \vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = |\vec{r}| |\vec{F}| \sin \theta_{rF}$$

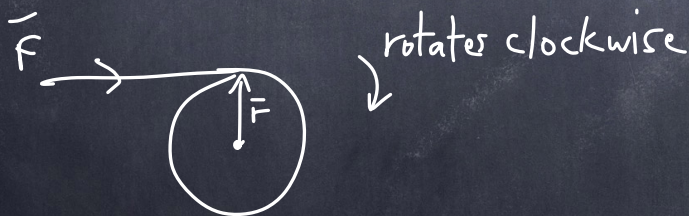
$$= \underbrace{r}_{\text{magnitudes}} \underbrace{F}_{\text{magnitudes}} \underbrace{\sin \theta_{rF}}_{\text{angle from } \vec{r} \text{ to } \vec{F}}$$

can be (+) or (-)

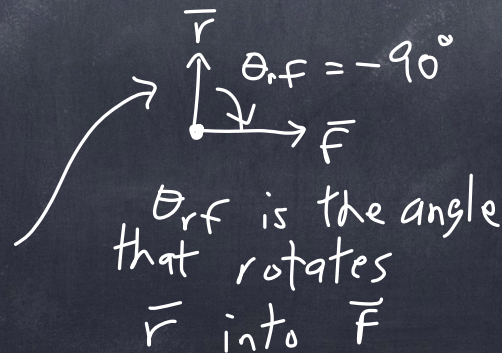
Right-hand:

(-) = clockwise = $\otimes$

(+) = counter-clockwise = $\odot$



Draw the $\vec{r}$ + $\vec{F}$ vectors so that they start at the same point



θ_{rF} is the angle that rotates $\vec{r}$ into $\vec{F}$

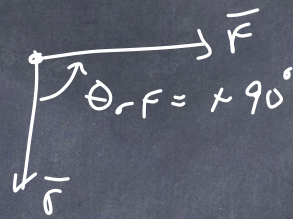
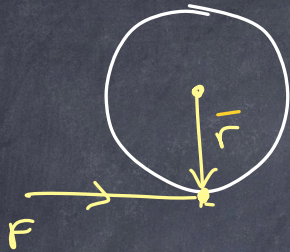
$$\tau = r F \sin(-90^\circ)$$

$$= -r F$$

(-) = clockwise

calculate the torque + direction for each case

1)

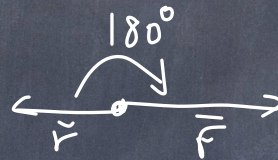
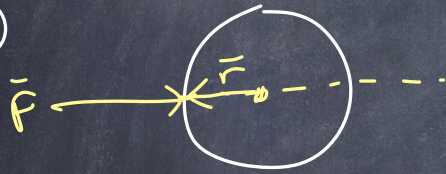


$$\tau = rF \sin 90^\circ = rF(+1) = +rF$$

(+) direction or counter-clockwise (CCW)

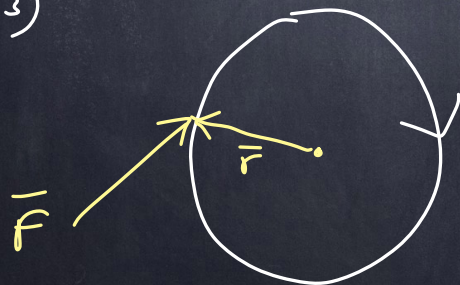
$$\vec{\tau} = \odot$$

2)



$$\tau = rF \sin(180^\circ) = 0$$

3)



θ_{rF} is (-)

$$\vec{\tau} = \otimes$$

$$\tau = rF \sin \theta_{rF}$$

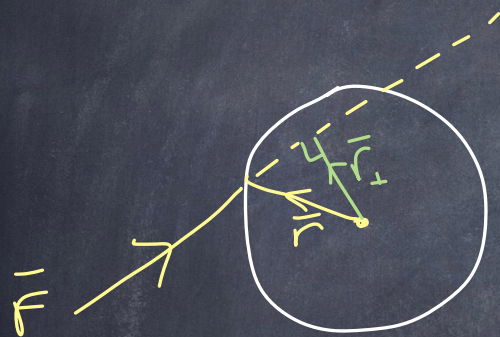
(-)

(-) direction or clockwise direction (CW)

Torque can be also thought of as the product of the force with the "lever arm"
 $= r_{\perp}$ = the part or component of $\vec{r}$ that is perpendicular to the force.

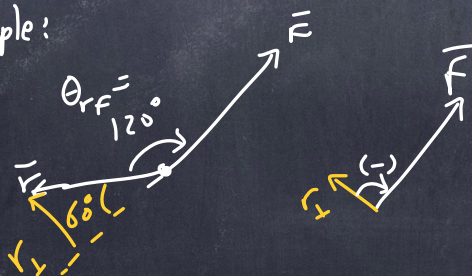
$$\tau = r_{\perp} F \leftarrow (\text{then add } + \text{ or } - \text{ sign})$$

$r_{\perp}$: component of $\vec{r} \perp$ to $\vec{F}$



Note: The trigonometry can get tricky with $r_{\perp}$:

example:



$$\sin(-60^\circ) = \sin(-120^\circ)$$

we know

$$\tau = r F \sin(-120^\circ)$$

cw

(+)

but also

$$\tau = r_{\perp} F$$

assign
(+)

cw

(+)

$$\tau = r \sin(-60^\circ) F$$

in $\otimes$ direction

static equilibrium!

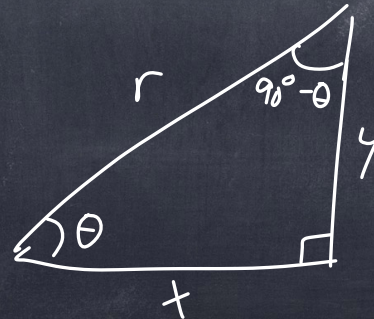
2 conditions: $\sum \vec{F} = 0$ (so no acceleration)

$$\sum \vec{\tau} = 0 \quad \tau_{cw} = \tau_{ccw}$$

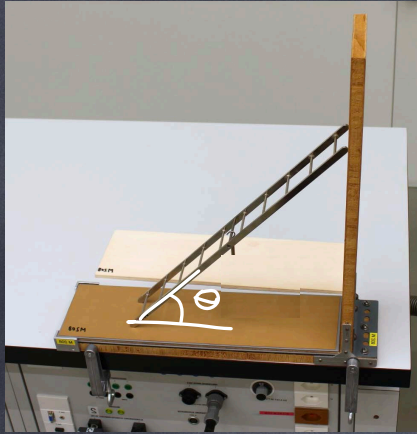
Trick: The point of rotation matters
(the right point makes the problems easier to solve)

Also, good to know that

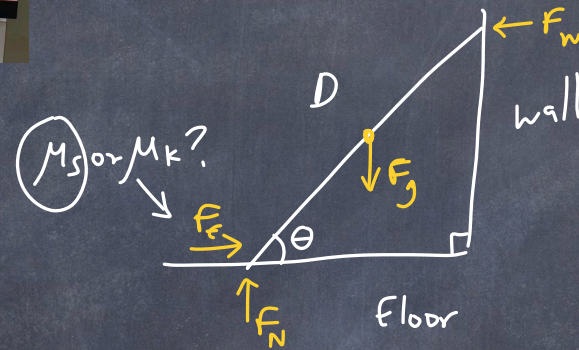
$$\sin \theta = \cos(90^\circ - \theta)$$



$$\sin \theta = \frac{y}{r}$$
$$\cos(90^\circ - \theta) = \frac{y}{r}$$



What is the smallest angle, $\theta_{\min}$, such that the ladder does not fall?
 Ladder has mass, M
 length, D
 angle, θ
 (Consider floor friction, but not wall friction)



+y Define coordinate system.
 +x

First, consider the forces: In equilibrium, $\sum \vec{F} = 0$

y-direction:

$$\sum F_y = 0 = F_N + (-F_g)$$

$$F_g = F_N$$

x-direction:

$$F_f = \mu_s F_N = \mu_s Mg$$

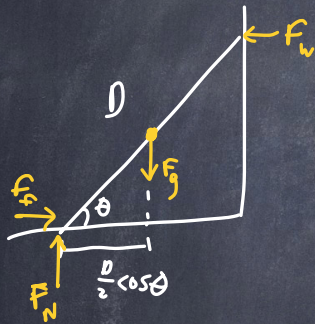
$$\sum F_x = F_f - F_w = 0$$

$$F_w = F_f = \mu_s Mg \quad (1)$$

Next, we consider $\sum \tau = 0$
equilibrium condition
what should be the
rotation point?

$$\tau_{cw} = \tau_{ccw}$$

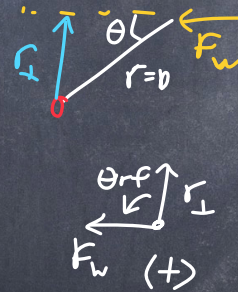
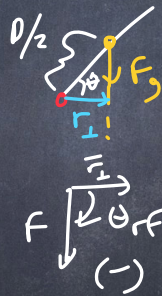
✱
Good choice



bad choice:
all 4 forces
create torque

No torque for F_f or F_N
since they point through
the axis of rotation.

$$\sum \tau_o = \tau_g + \tau_w + \tau_N + \tau_f$$



↑
0

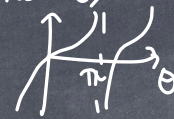
↑
0

$$0 = -Mg \frac{D}{2} \cos \theta + F_w D \sin \theta + 0 + 0$$

$$\frac{\sin \theta}{\cos \theta} = \frac{Mg}{2F_w} \Rightarrow \tan \theta = \frac{Mg}{2F_w} \quad (2)$$

insert ① in ② $\Rightarrow \tan \theta = \frac{Mg}{2\mu_s Mg}$
 $\tan \theta = \frac{1}{2\mu_s} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2\mu_s}\right)$

Reminder: $\tan^{-1}(x) \equiv \arctan(x)$ = undoes tangent: $\tan(\theta)$
 so: $\tan(45^\circ) = 1$
 $\tan^{-1}(1) = 45^\circ$

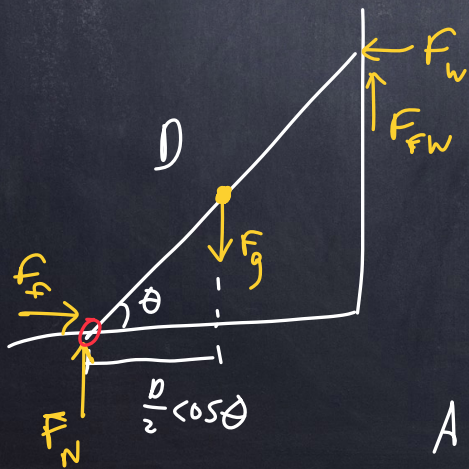


OK, we want $\theta_{\min} \Rightarrow \theta_{\min} = \tan^{-1}\left(\frac{1}{2\mu_s}\right)$
 when θ is at a minimum, so is $\tan\theta$

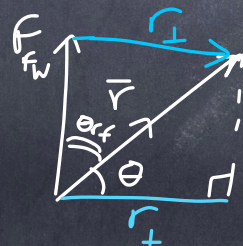
If $\mu_s = 0.3 \Rightarrow$ then $\theta_{\min} = 59^\circ$

If $\theta < 59^\circ$, then the ladder falls

What if the wall has friction?
 Does $\theta_{\min}$ get smaller or larger?



Note:
 $\theta = 90^\circ - \theta_{rf}$



Need to add
 one more torque

$$\tau = r F \sin\theta_{rf}$$

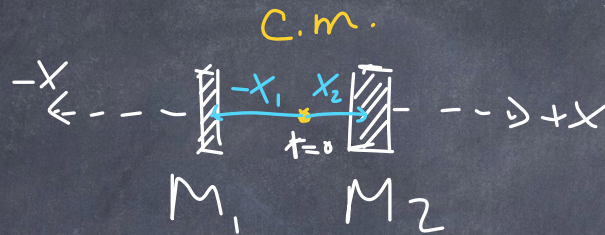
$$= D F_{fw} \sin\theta_{rf}$$

↗ same

$\tau_{ccw} = r_{\perp} F_{fw} = (D \cos\theta) F_{fw}$
 Adding τ_{ccw} makes it more stable, $\theta_{\min}$ can be smaller

Center of mass

Consider 2 objects of mass:



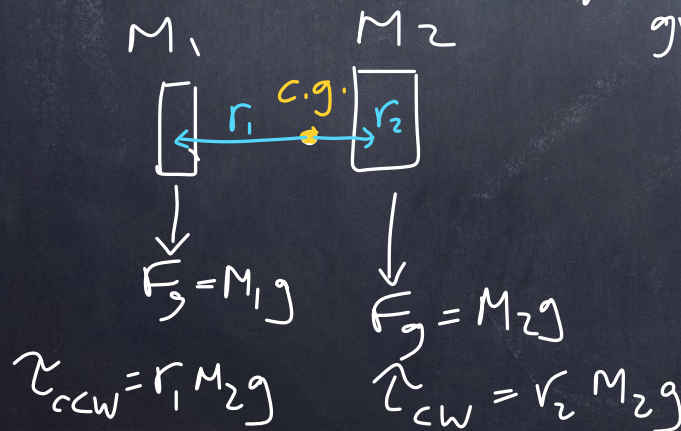
We choose the center of mass as the origin ($x=0$). The mass-weighted positions about this point balance.

$$\text{Then } M_2 x_2 + M_1 (-x_1) = 0$$

If many masses, then
$$\sum_i M_i x_i = 0$$

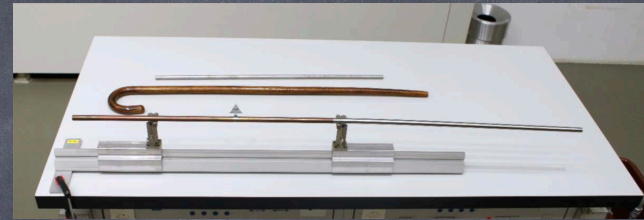
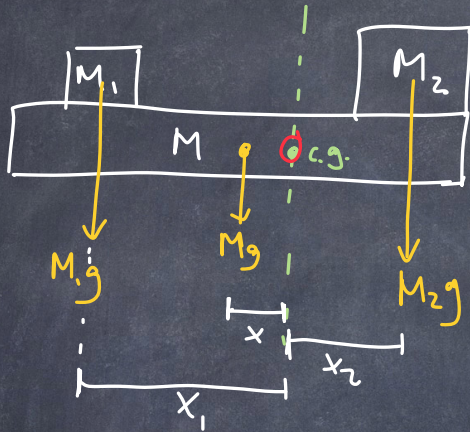
Center of gravity:

point at which the torques due to gravity sum to zero.



If $\bar{g}$ is uniform over the object, center of gravity = center of mass

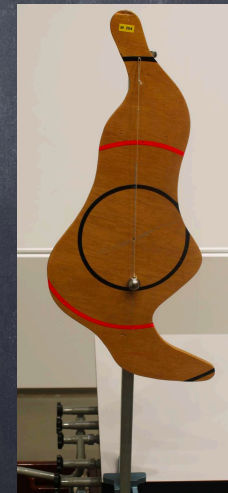
Considering C.G.: choose the axis of rotation through the center of gravity. Then $\sum_i \tau_i = 0$



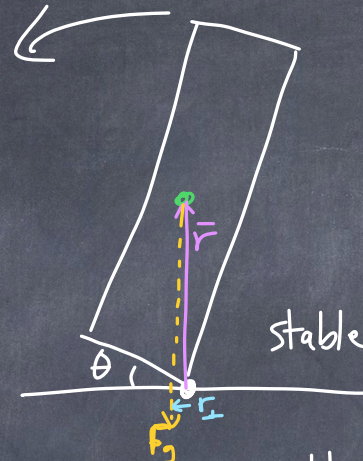
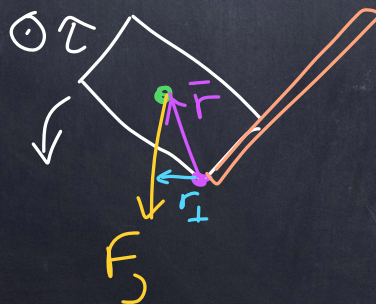
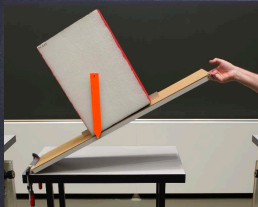
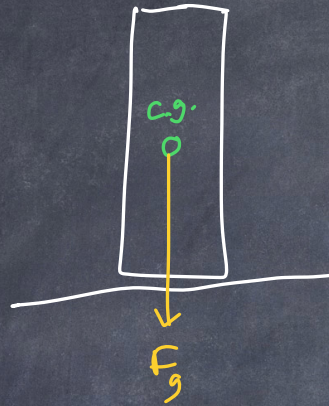
when origin is at the center of gravity
then $\sum \tau_{cw} = \sum \tau_{ccw}$
and so

$$x_2 M_2 g = x_1 M_1 g + x M g$$

IF object is supported at center of gravity, then it is balanced.



Stability



τ is ccw $\odot$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = r_\perp F \quad (+) \text{ direction ccw}$$

An object is stable when the torque due to gravity tends to restore the object to its equilibrium position.

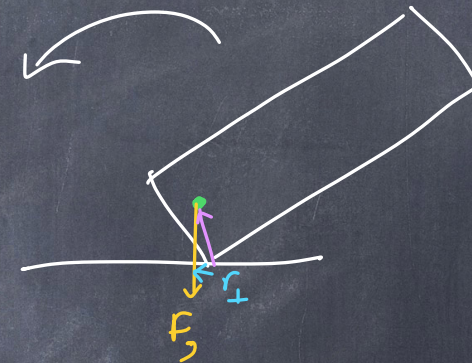
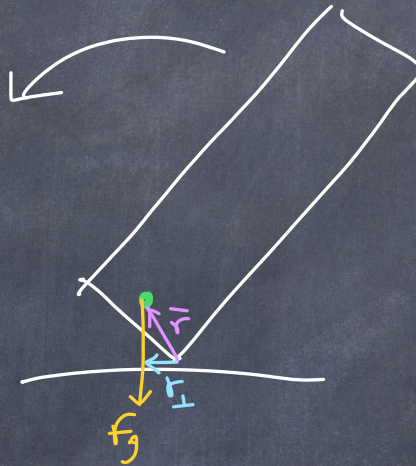
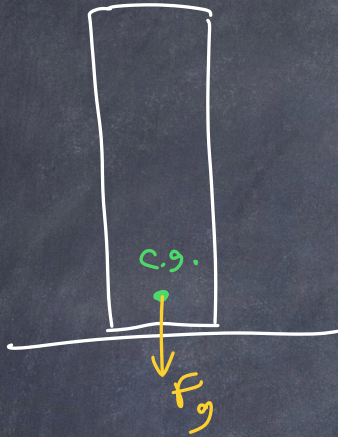
This depends on the direction of the torque with respect to the pivot point.

example:

Torque is counter clockwise, causing object to fall.

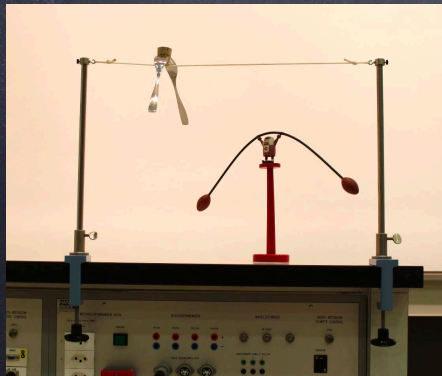
Not stable.

To improve stability, lower the center of gravity,
(heavier at the bottom)



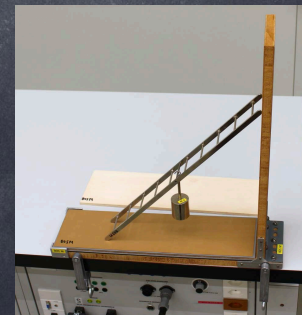
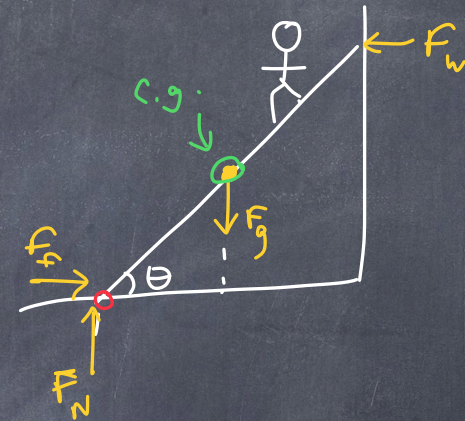
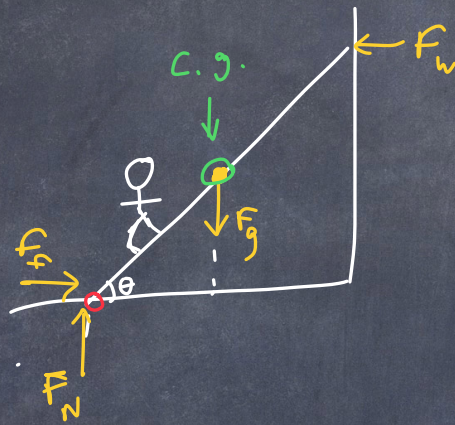
both stable:

↶ rotates the object back
to the initial position.

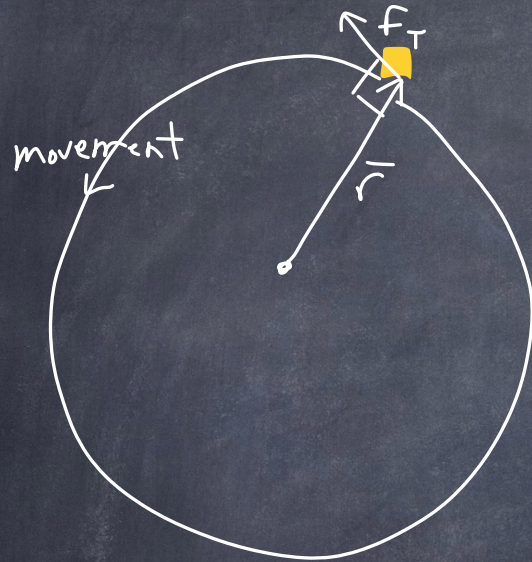


Ladder:
what if we add a person? 

More or less stable
for the cases below?



Consider a tangential force on an object, mass m , constrained to move in a circle.



The force is tangential, F_T

The force is unbalanced, creates a .

$$F_T = ma_T = m(r\alpha)$$

we multiply both sides by r :

$$rF_T = \underbrace{mr^2}_{\tau} \alpha$$

$$\tau = I\alpha$$

$$a_T = r\alpha \text{ for circular motion.}$$

$$\tau = I\alpha$$

$I \equiv mr^2$: "moment of inertia": this is for a particle of mass m at a radius r from the center of rotation.

Note: the parallels between linear + rotational motion:

linear motion

$$F = ma$$

rotational motion

$$\tau = I \alpha$$

I is
like mass
($I = mr^2$)

↑
angular
acceleration

Newton's
2nd law

$$\Sigma F = ma$$

Newton's second law
of rotation

$$\Sigma \tau = I \alpha$$