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Applications of General Relativity

Sheet 5

Spring 2021
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Website: `http://www.physik.uzh.ch/en/teaching/PHY519/`

Exercise 1 [Gravitational field of a moving particle]

Consider a particle of mass M moving with constant velocity $\boldsymbol{V}$. Calculate the gravitomagnetic potential and the equation of motion for a test particle to order $\mathcal{O}(v/c)$.

Hint: Start in the particles rest frame Σ' with the equations (4.9) and (4.10) for the metric perturbation in the script and then perform a coordinate transformation to the global frame Σ. Since we are working at $\mathcal{O}(h)$ the transformation is nothing than a LORENTZ boost.

Exercise 2 [Particles in the field of a gravitational wave]

Show that the curves $r = r(\varphi)$ described by

$$r^2(\varphi) = R^2 \begin{cases} 1 - 2h\cos(2\varphi)\cos(\omega t) \\ 1 - 2h\sin(2\varphi)\cos(\omega t) \end{cases} \tag{1}$$

for $h \ll 1$ are ellipses. How is the eccentricity e related to h?

Hint: Start from the following parametric form of an ellipse,

$$r(\varphi) = \frac{b}{\sqrt{1 - e^2\cos^2\varphi}}\,, \tag{2}$$

and assume $e^2 \ll 1$.

Exercise 3 [Gravitational Bremsstrahlung]

The gravitational wave analogue of Bremsstrahlung can be generated by a small mass m scattering off a large mass $M \gg m$ with impact parameter b. Assume that the large mass sits at $(0,0,0)$, that $E = 0$ (parabolic orbit) and that the orbit lies in the x-y-plane. Calculate the gravitational wave amplitude at a position on the z-axis.

Hint: For the LORENZ gauge trace reversed metric $\gamma_{\mu\nu} = h_{\mu\nu} - 1/2\ \eta_{\mu\nu} h$ and slowly moving sources the gravitational wave amplitude can be calculated as[1]

$$\gamma_{ij}(t, \boldsymbol{x}) = \frac{2G}{r}\frac{\mathrm{d}^2 I_{ij}(t_\mathrm{r})}{\mathrm{d}t^2}, \tag{3}$$

[1] See Caroll 2004 for derivation and restrictions of this formula.

where t_r is the retarded time and $I_{ij}(t) = \int T_{00}(t) y^i y^j \, \mathrm{d}y^3$ is the quadrupole tensor. The hyperbolic orbit of the small mass m is described by the parametric solution

$$r(\varphi) = \frac{2b}{1+\cos(\varphi)} \qquad \dot{\varphi} = \sqrt{\frac{M}{8b^3}}\,[1+\cos(\varphi)]^2\,. \tag{4}$$