Multi-loop multi-scale Feynman integrals for collider physics

Simone Zoia

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Swiss National Science Foundation

Universität Zürich UZH

Based on:

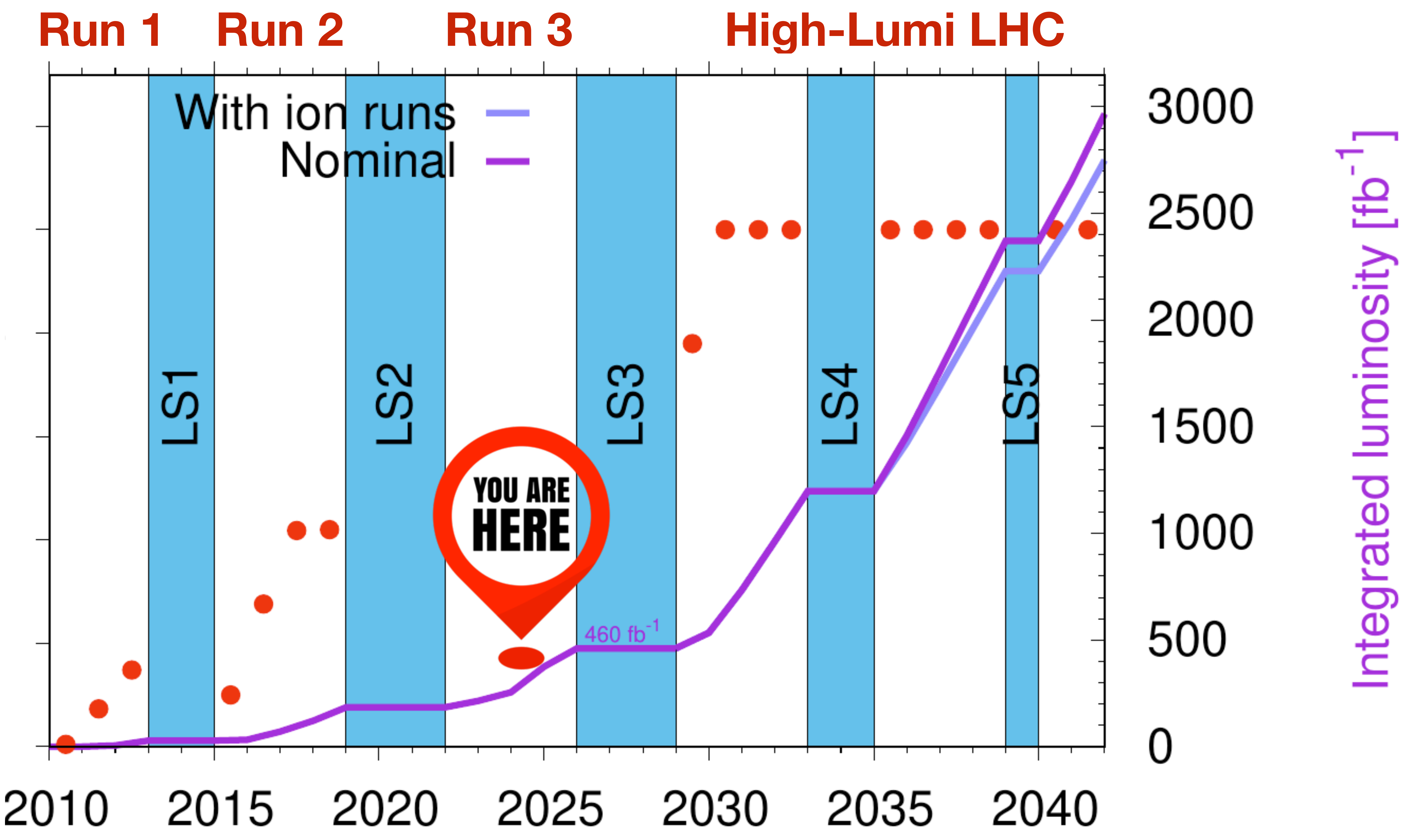
- with D. Chicherin, V. Sotnikov
  JHEP 01 (2022) 096, e-Print:2110.10111

- with S. Abreu, D. Chicherin, H. Ita, B. Page, V. Sotnikov, W. Tschernow
  Phys.Rev.Lett. 132 (2024) 14, 141601, e-Print:2306.15431

- with S. Badger, M. Becchetti, N. Giraudo
  JHEP 07 (2024) 073, e-Print:2404.12325

- with S. Badger, M. Becchetti, C. Brancaccio, H. B. Hartanto
  in preparation

We're only at the dawn of the LHC physics programme



We'll be able to test the Standard Model at (sub)-percent accuracy…

provided that the theoretical predictions can keep up!

Plot by R. T. Garcia, lhc-commissioning.web.cern.ch

A long way from theory to experiment

Scattering amplitudes

IR subtraction

PDFs

Parton showers

Resummation

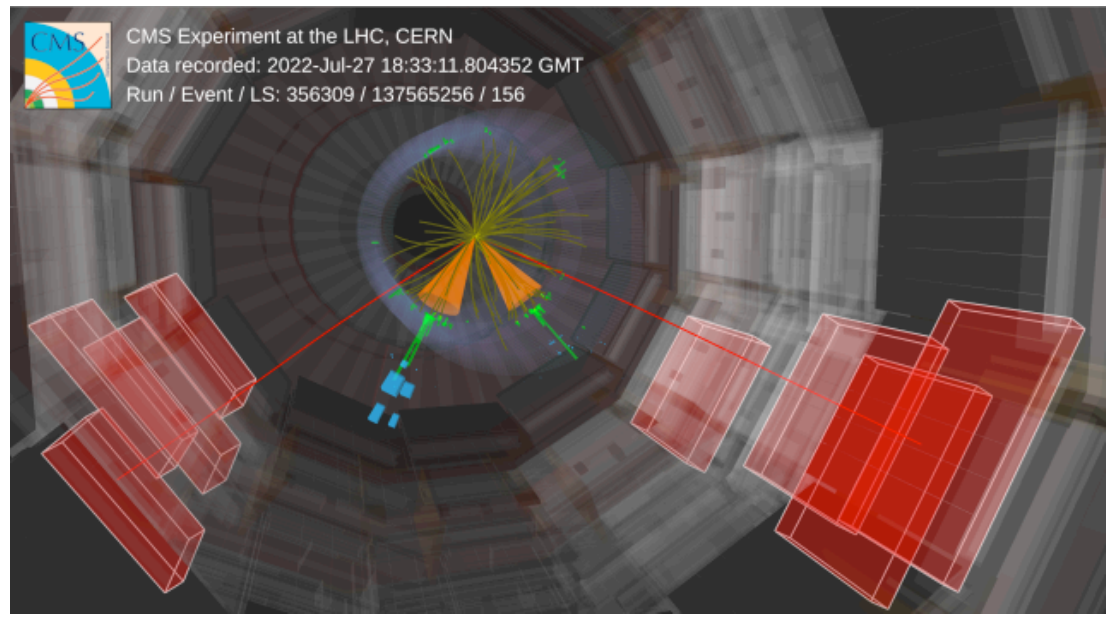
Hadronisation

…

Monte Carlo event generator

Detector simulation

Experimental data



It takes a village to make a prediction!

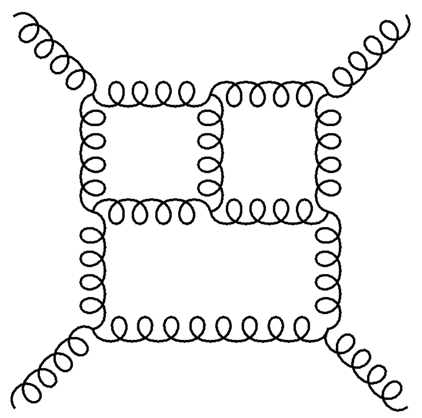
Precision at the LHC demands *at least* NNLO QCD

$$\mathrm{d}\sigma = \boxed{\mathrm{d}\sigma^{\mathrm{LO}}} + \boxed{\alpha_S\, \mathrm{d}\sigma^{\mathrm{NLO}}} + \boxed{\alpha_S^2\, \mathrm{d}\sigma^{\mathrm{NNLO}}} + \ldots$$

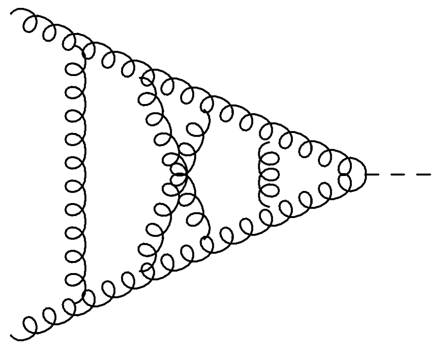
$\approx 10 - 30\,\%$ $\approx 1 - 10\,\%$

Loop frontier

N³LO $2 \to 2$



N⁴LO $2 \to 1$



\# loops

The difficulty grows with

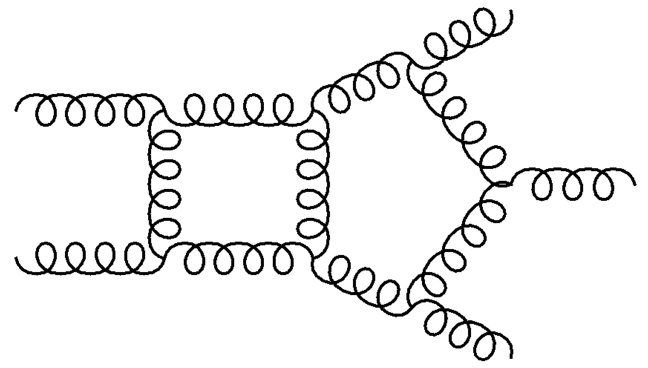
- order in the coupling (# loops)
- number of particles

\# particles

$2 \to n$

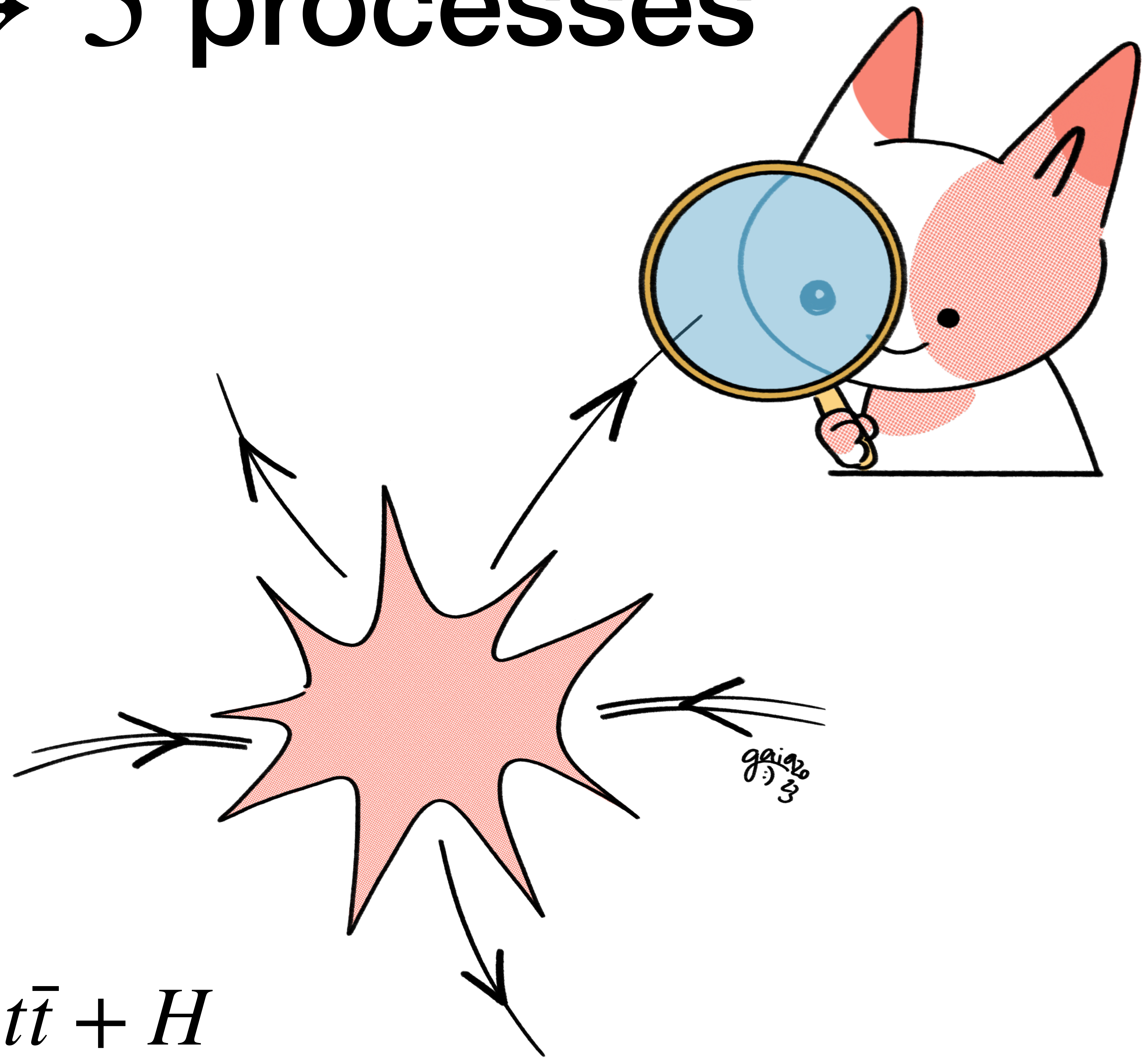
Multiplicity frontier

NNLO $2 \to 3$



This talk

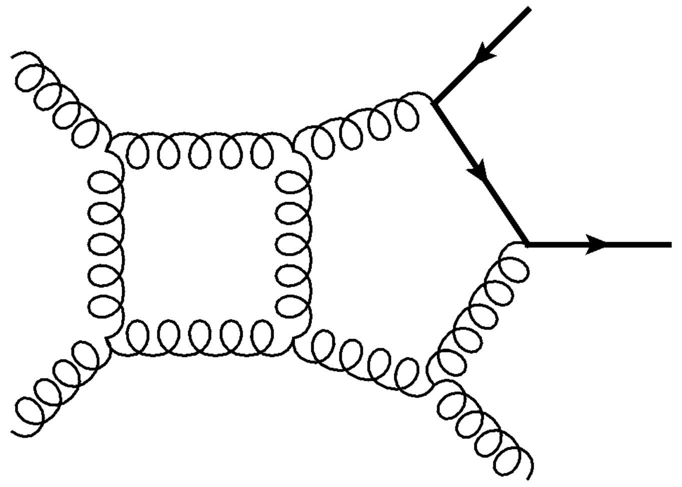
Rich phenomenology with $2 \to 3$ processes



From the Les Houches' “Precision wish-list”:

- $pp \to \gamma\gamma + j,\ pp \to \gamma\gamma\gamma$
- $pp \to 3j$
- $pp \to V + 2j,\ pp \to V + b\bar{b},\ pp \to VV' + j$
- $pp \to H + 2j$
- $pp \to t\bar{t} + j,\ pp \to t\bar{t} + \gamma,\ pp \to t\bar{t} + W,\ pp \to t\bar{t} + H$

Main bottleneck: two-loop five-particle scattering amplitudes



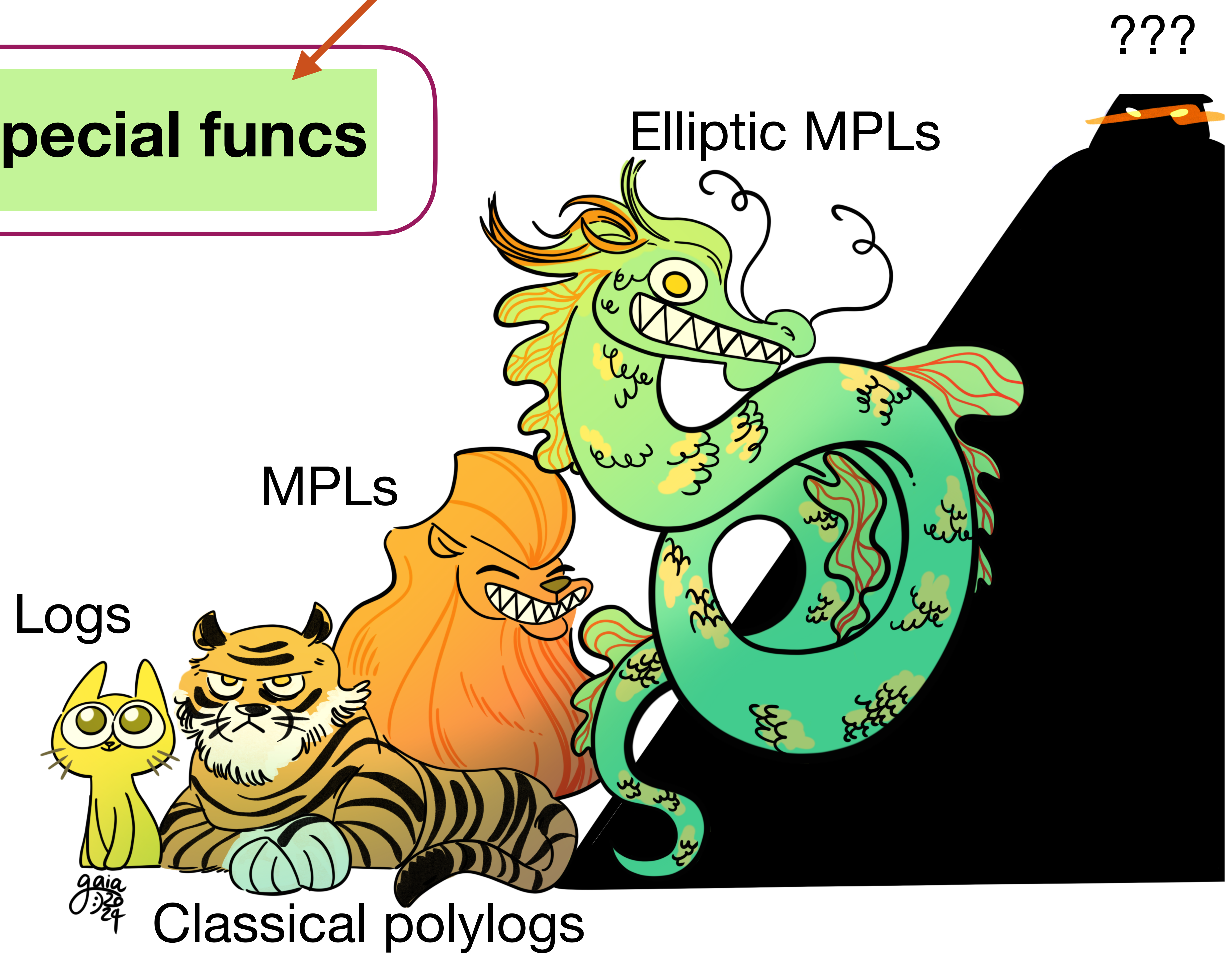
Unfortunately, phenomenology is very demanding

$$\textbf{amplitude} = \sum \textbf{Feynman diagrams}$$

?

Loop integration

$$\textbf{amplitude} = \sum \textbf{rational coeffs} \times \textbf{special funcs}$$

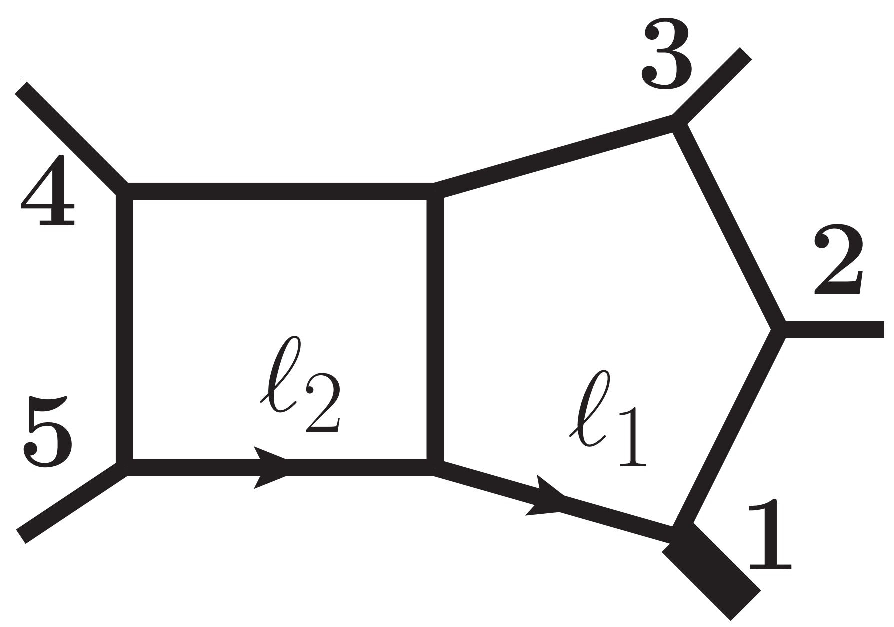


Wish list:

- Compact expressions
- Cancellation of the UV/IR poles
- "***Fast & furious***" evaluation in physical phase space

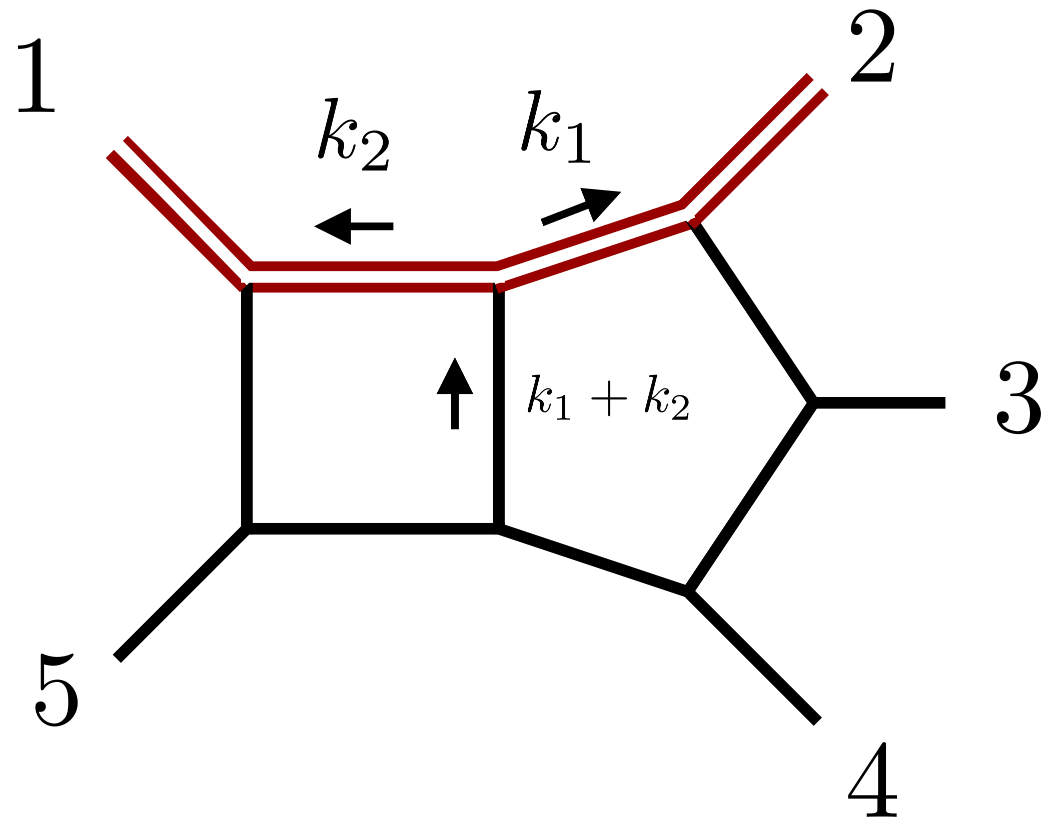
Outline

- The method of choice for multi-scale integrals: differential equations
- *Polylogarithmic* special function bases
- First steps into the "elliptic" world



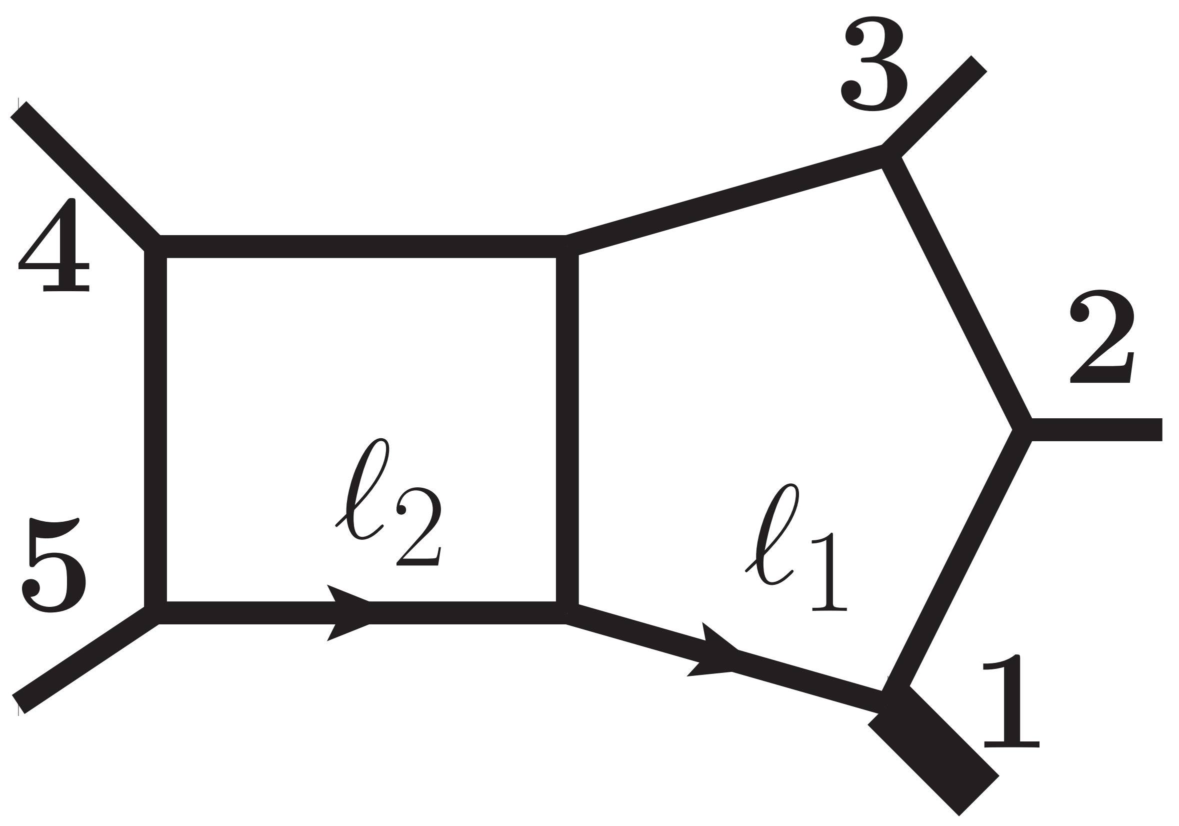
Running examples: two-loop five-particle Feynman integrals for

- H/V+2 jets production @ LHC
- $\mathrm{t\bar{t}}$+jet production @ LHC



Integral families

Scalar Feynman integrals with the same propagator structure = **integral family**



$$I_{\vec{a}}(X;\epsilon) = \int \frac{\mathrm{d}^D\ell_1}{\mathrm{i}\pi^{D/2}} \frac{\mathrm{d}^D\ell_2}{\mathrm{i}\pi^{D/2}} \frac{\rho_9^{-a_9}\rho_{10}^{-a_{10}}\rho_{11}^{-a_{11}}}{\rho_1^{a_1}\dots\rho_8^{a_8}}$$

$$\left\{ I_{\vec{a}}(X;\epsilon) \,|\, \forall\, \vec{a} \in \mathbb{Z}^{11} \right\}$$

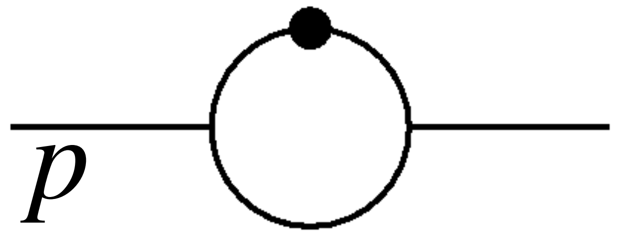
$$\rho_1 = \ell_1^2$$
$$\rho_2 = (\ell_1 - p_1)^2 - m_t^2$$
$$\rho_3 = (\ell_1 - p_1 - p_2)^2$$
$$\dots$$

Dimensional regularisation: $D = 4 - 2\epsilon$

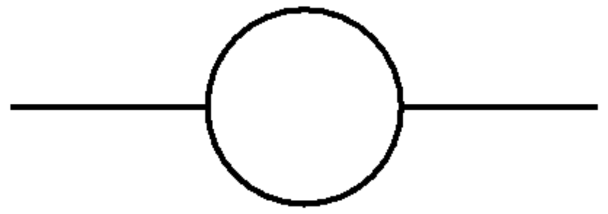
6 kinematic variables: $X = \left(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, p_1^2\right)$, with $s_{ij} = (p_i + p_j)^2$

Integral bases, alias master integrals

Identities among the $\mathrm{I}_{\vec{a}}$'s, e.g. Integration-By-Parts (IBP) relations *Tkachov '81; Chetyrkin, Tkachov '81; Laporta 2000*



$$= \frac{3-D}{p^2} \times$$



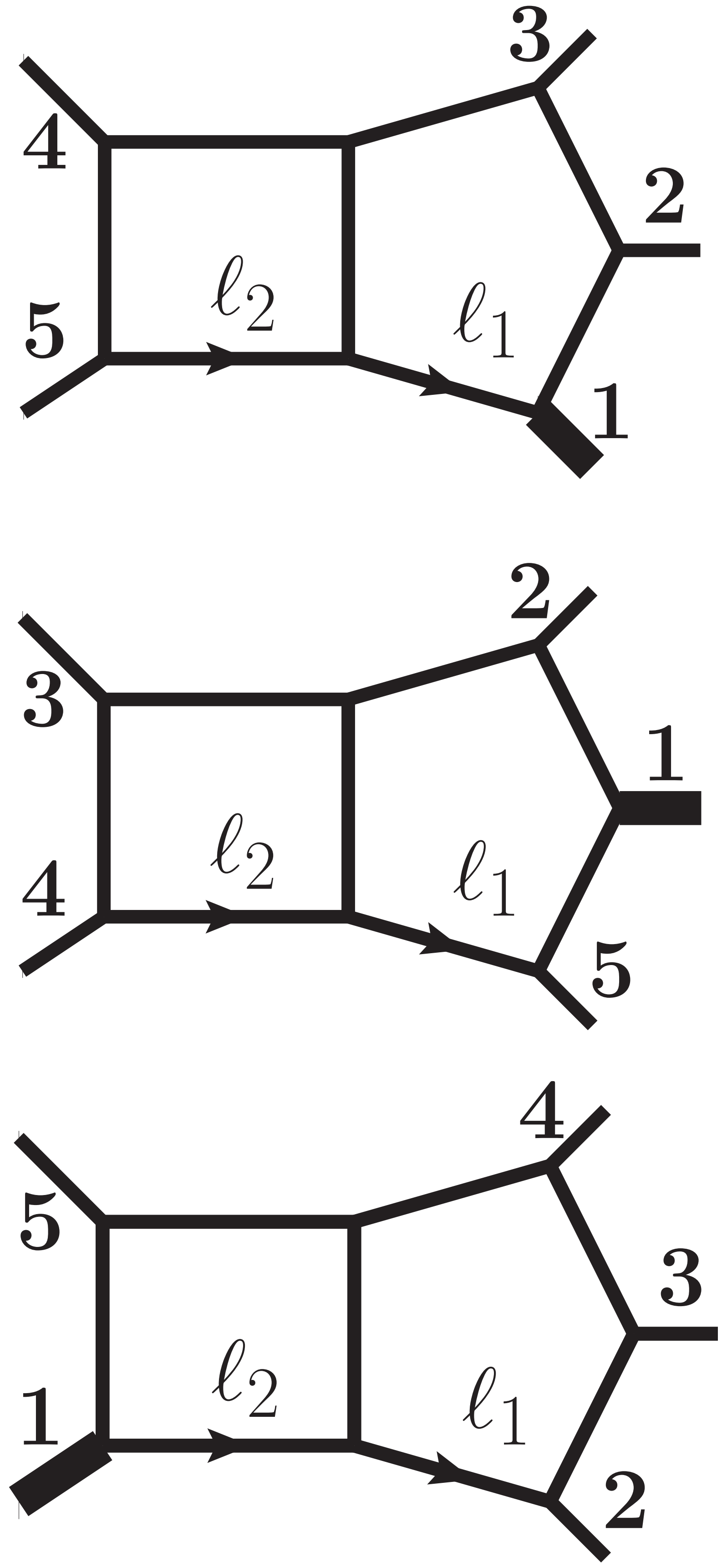
→ Finite-dimensional *basis*: $\overrightarrow{\mathrm{F}}(X;\epsilon)$

$$\mathrm{I}_{\vec{a}} = \sum_i c_i(\vec{a};X,\epsilon)\,F_i$$

IBP reduction = solution of **large** linear system of equations

Often a bottleneck, not for this talk — thanks to **FiniteFlow** *Peraro 2019*

2-loop 5-pt 1-mass master integrals

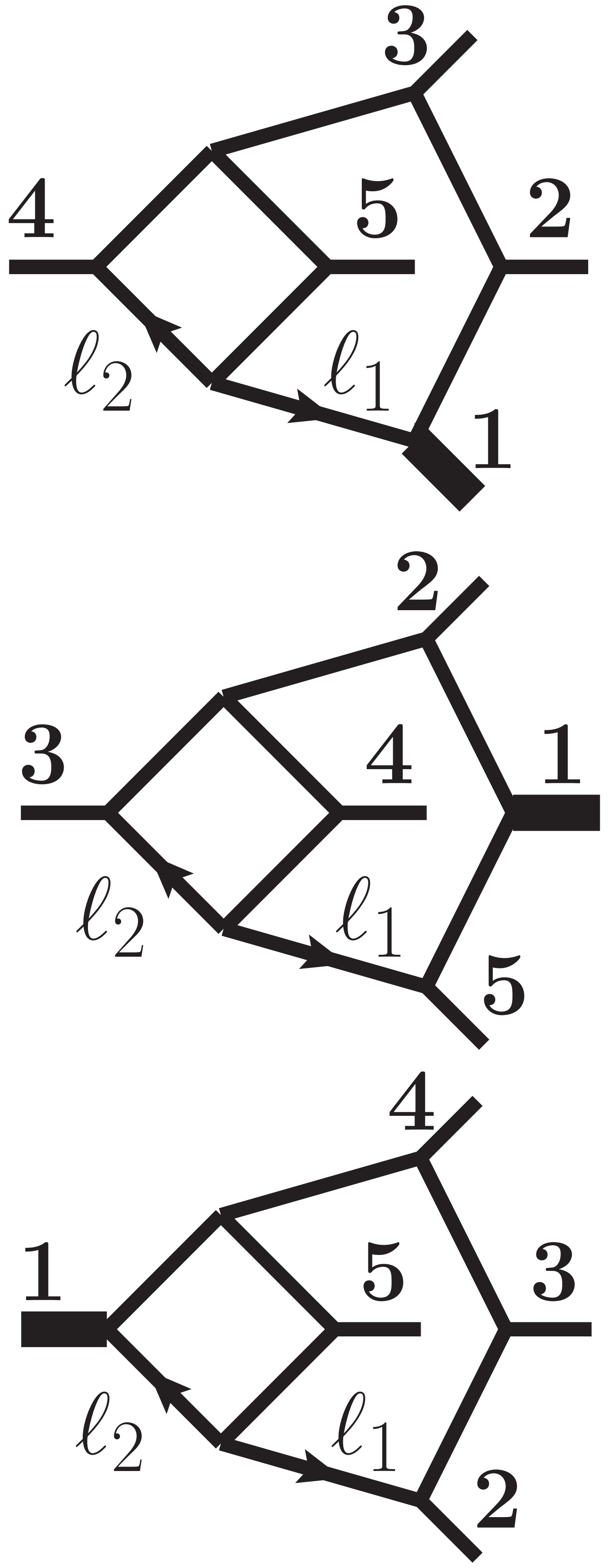


74

75

86

*Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020; Canko, Papadopoulos, Syrrakos 2020; Syrrakos 2020; Chicherin, Sotnikov, **SZ** 2021*

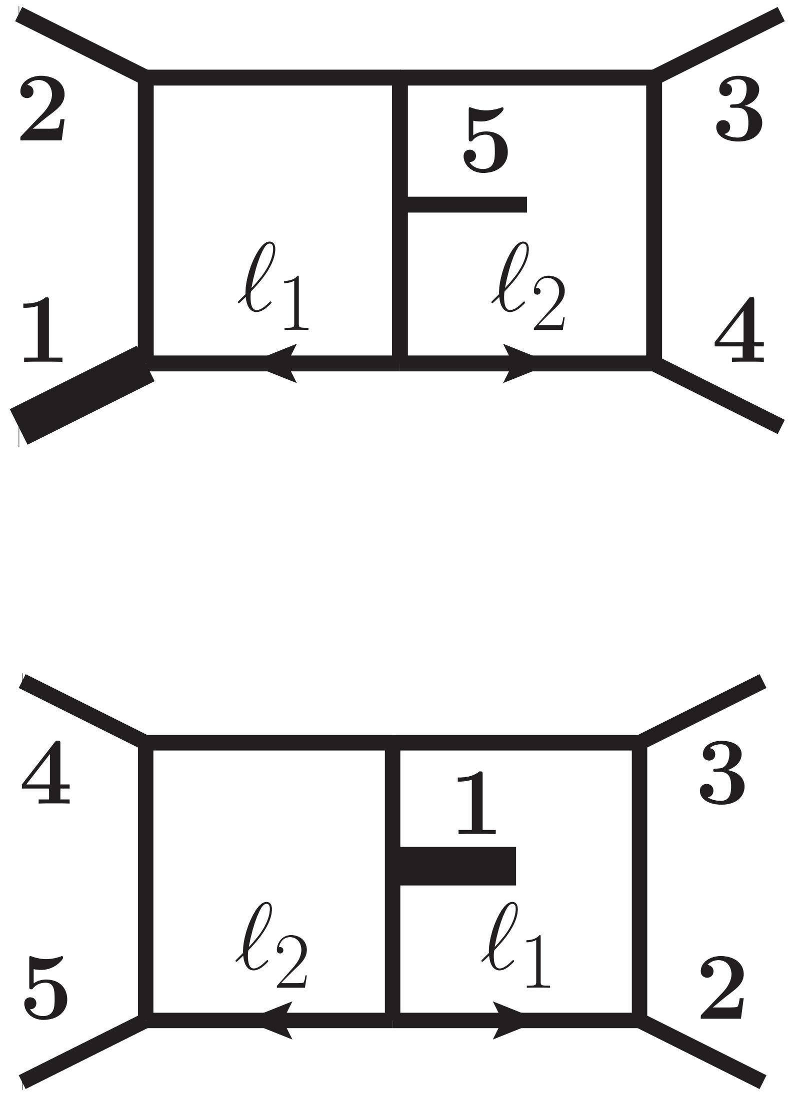


86

86

135

Abreu, Ita, Moriello, Page, Tschernow 2021; Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 2022



142

179

*Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, **SZ** 2023*

Integrating by differentiating

Barucchi, Ponzano '73; Kotikov '91; Bern, Dixon, Kosower '94; Gehrmann, Remiddi 2000

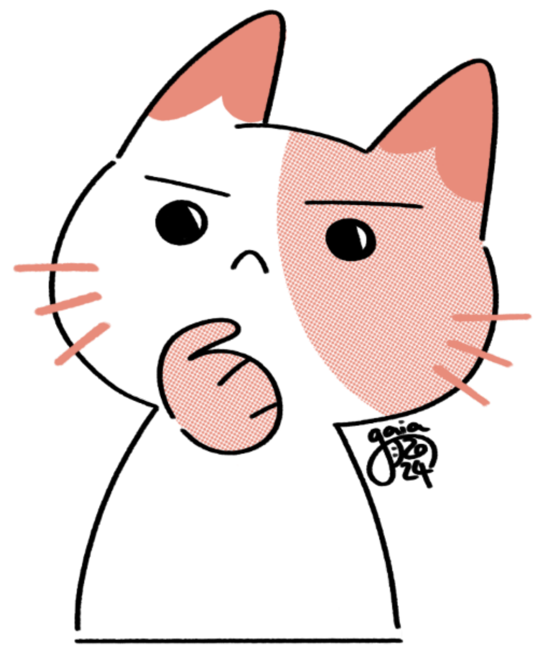
Integral families by construction closed under differentiation

$$\frac{\partial}{\partial s_{12}} \vec{\mathrm{F}}(X;\epsilon) = \sum_{\vec{a}} c_{\vec{a}}(X;\epsilon)\ \mathrm{I}_{\vec{a}}(X;\epsilon)$$

$$= A_{s_{12}}(X;\epsilon) \cdot \vec{\mathrm{F}}(X;\epsilon)$$

IBP reduction

System of 1st order linear PDEs for the MIs $\vec{\mathrm{F}}$



How do we solve it? Is there a "natural" basis?

Canonical form

Henn 2013

$$\mathrm{d}\vec{\mathrm{F}}(X;\epsilon) = \epsilon\, \mathrm{d}\tilde{A}(X) \cdot \vec{\mathrm{F}}(X;\epsilon)$$

$$\mathrm{d}\vec{\mathrm{F}} = \frac{\partial \vec{\mathrm{F}}}{\partial s_{12}} \mathrm{d}s_{12} + \ldots + \frac{\partial \vec{\mathrm{F}}}{\partial m_t^2} \mathrm{d}m_t^2$$

- Factorisation of ϵ makes ϵ-expansion of the solution easy

$$\vec{F}(X;\epsilon) = \sum_{k\geq 0} \epsilon^k F^{(k)}(X) \quad \Longrightarrow \quad \vec{\mathrm{F}}^{(k)}(X) = \int \mathrm{d}\tilde{A} \cdot \vec{F}^{(k-1)} + \text{const}.$$

Solution in terms of *iterated integrals*

- In the "***polylogarithmic***" cases, the connection matrix $\tilde{A}(X)$ takes the form

$$\tilde{A}(X) = \sum_i A_i \log W_i(X)$$

$W_i(X)$ = **letters** = singularities of the solution

1-mass pentagon alphabet: 204 letters

127 rational

$$W_1 = p_1^2\,,$$

$$\{W_2,\ldots,W_5\} = \{\sigma\,(s_{12}) : \sigma \in S_4/S_3[3,4,5]\}\,,$$

$$\{W_6,\ldots,W_{11}\} = \{\sigma\,(s_{23}) : \sigma \in S_4/(S_2[2,3]\times S_2[4,5])\}\,,$$

$$\{W_{12},\ldots,W_{15}\} = \{\sigma\,(2\,p_1\cdot p_2) : \sigma \in S_4/S_3[3,4,5]\}\,,$$

$$\{W_{16},\ldots,W_{27}\} = \{\sigma\big(2\,p_2\cdot(p_3+p_4)\big) : \sigma \in S_4/S_2[3,4]\}\,,$$

77 algebraic

$$\{W_{186},\ldots,W_{188}\} = \left\{\sigma\left(\frac{\Omega^{--}\Omega^{++}}{\Omega^{-+}\Omega^{+-}}\right) : \sigma \in S_4/(S_2[2,3]\times S_2[4,5]\times S_2[s_{23},s_{45}])\right\}\,,$$

$$\{W_{189},\ldots,W_{194}\} = \left\{\sigma\left(\frac{\tilde\Omega^{--}\tilde\Omega^{++}}{\tilde\Omega^{-+}\tilde\Omega^{+-}}\right) : \sigma \in S_4/(S_2[3,4]\times S_2[2,5])\right\}\,,$$

where

$$\Omega^{\pm\pm} = s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} \pm s_{34}\sqrt{\Delta_3^{(1)}} \pm \sqrt{\Delta_5}\,,$$

$$\tilde\Omega^{\pm\pm} = p_1^2 s_{34} \pm \sqrt{\Delta_5} \pm \sqrt{\Sigma_5^{(1)}}\,,$$

10 square roots:

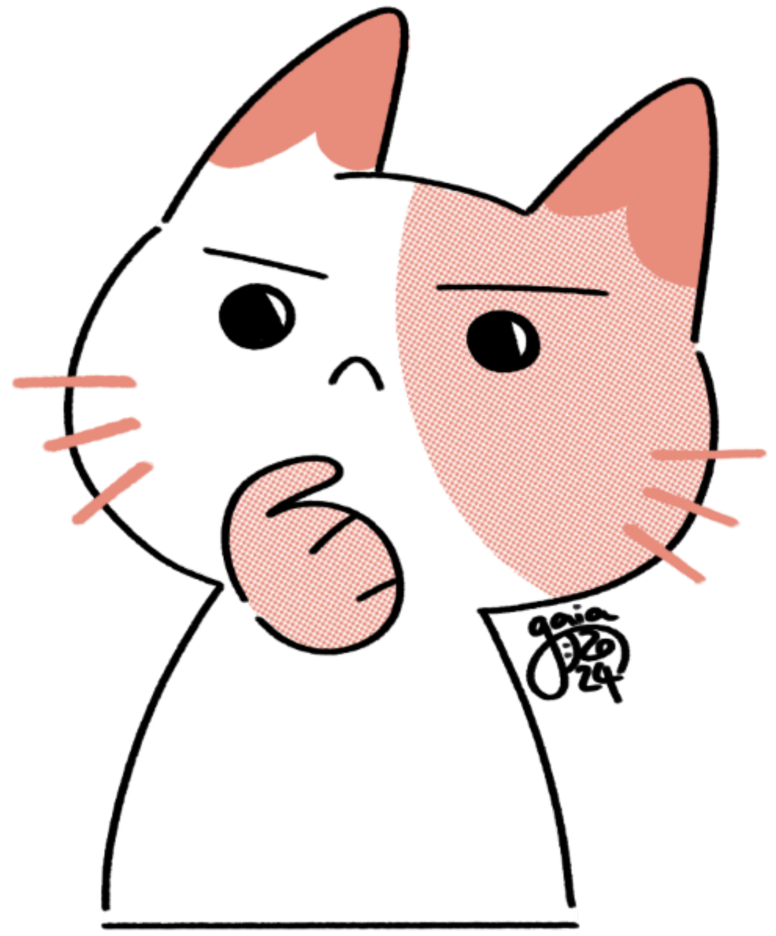
$$\Sigma_5 = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})$$

$$\begin{aligned}\Delta_5 &= \det G(p_1,p_2,p_3,p_4)\\ &= (s_{12}s_{15} - s_{12}s_{23} - p_1^2 s_{34} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2\\ &\quad - 4s_{23}s_{34}s_{45}(p_1^2 - s_{12} - s_{15} + s_{34})\,.\end{aligned}$$

Massless case: 31 letters, 1 square root → Escalation 😨

Crucial to have a systematic procedure to find/construct the letters!

How do we construct a canonical basis?



How to construct a canonical basis?

A lot of progress, but still no general algorithm

General approach: study **leading singularities** *Arkani-Hamed, Bourjaily, Cachazo, Trnka 2012*

Parameterise loop integrand and take residues (≡ partial fraction) until all integration variables are localised

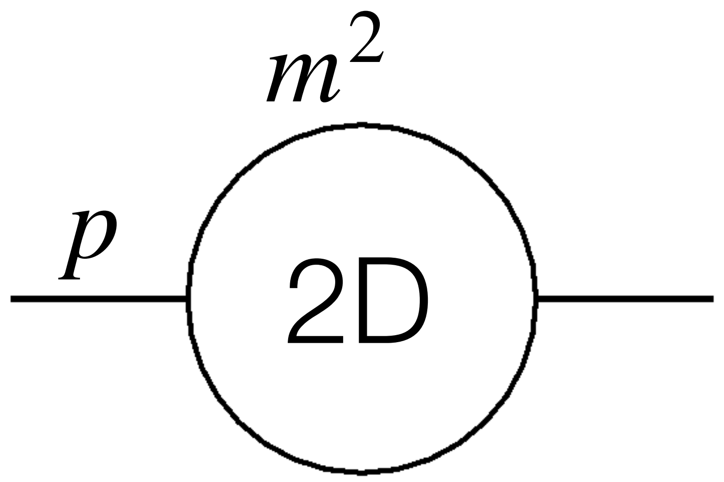
Construct integrands with **at most simple poles** + **constant leading singularities**

↑ *"dlog" integrand*

Analysis automated in *DlogBasis* *Wasser 2016; Henn, Mistlberger, Smirnov, Wasser 2020*

→ Practical limitations

Dlog integrands and leading singularities



$$\propto \int \frac{\mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_2}{p^2\,(\alpha_1\alpha_2 - x)\big[(1+\alpha_1)(1+\alpha_2) - x\big]}$$

$$= \int \frac{\mathrm{d}\alpha_1}{p^2(\alpha_1 + \alpha_1^2 + x)} \wedge \Big[\mathrm{d}\log(\alpha_1\alpha_2 - x) - \mathrm{d}\log(1 + \alpha_1 + \alpha_2 + \alpha_1\alpha_2 - x)\Big]$$

$$= \frac{1}{p^2\sqrt{1-4x}} \times \int \mathrm{d}\log(\cdots) \wedge \mathrm{d}\log(\cdots)$$

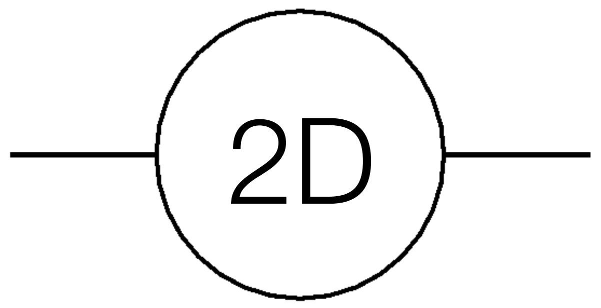
$$\mathrm{d}\log(z + c) = \frac{\mathrm{d}z}{z + c}$$

$$x = \frac{m^2}{p^2}$$

Leading singularity

“dlog” form

$$\longrightarrow \quad p^2\sqrt{1-4x} \times$$

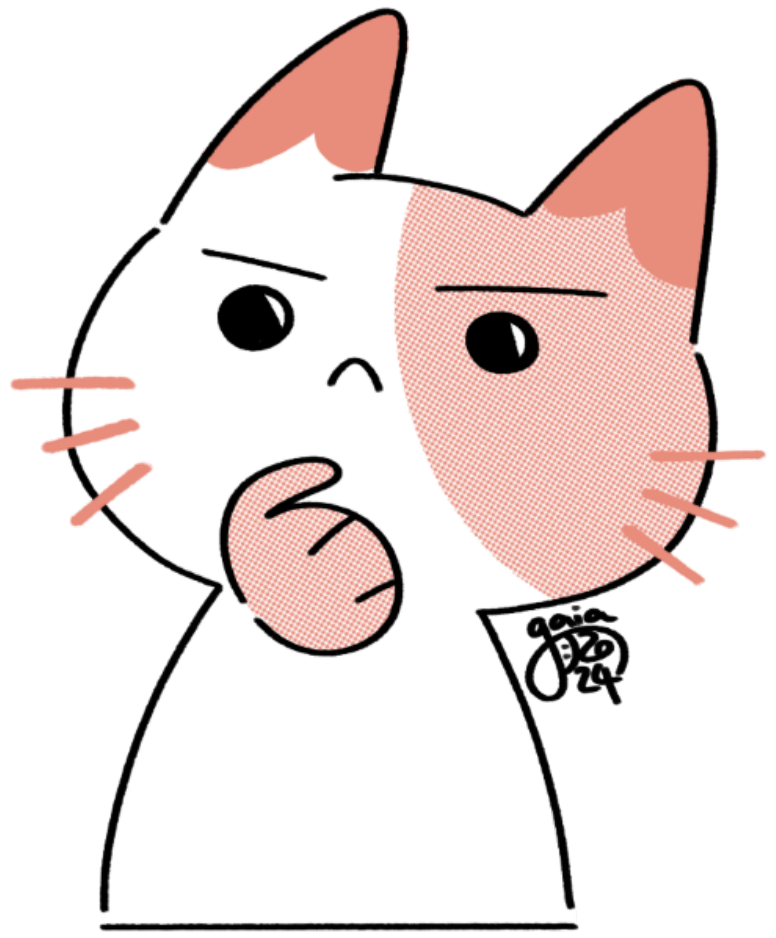


✔

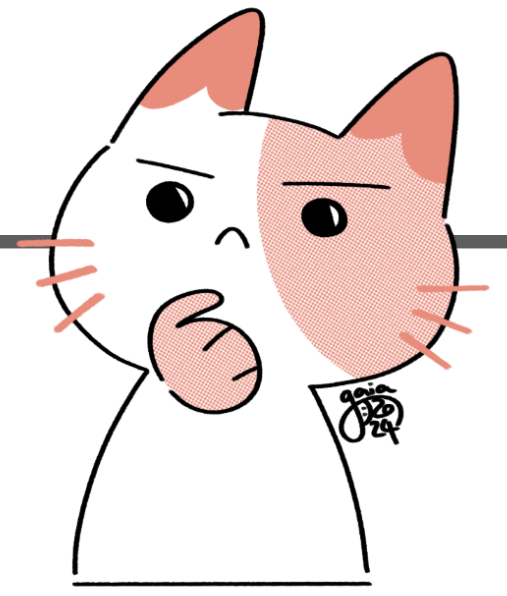
How do we solve the canonical DEs (efficiently)?

$$\mathrm{d}\,\vec{\mathrm{F}}(X;\epsilon) = \epsilon\,\mathrm{d}\tilde{A}(X)\cdot\vec{\mathrm{F}}(X;\epsilon)$$

$$\tilde{A}(X) = \sum_i A_i \log W_i(X)$$



How to solve the DEs?



1. Express MIs in terms of **multiple polylogarithms**

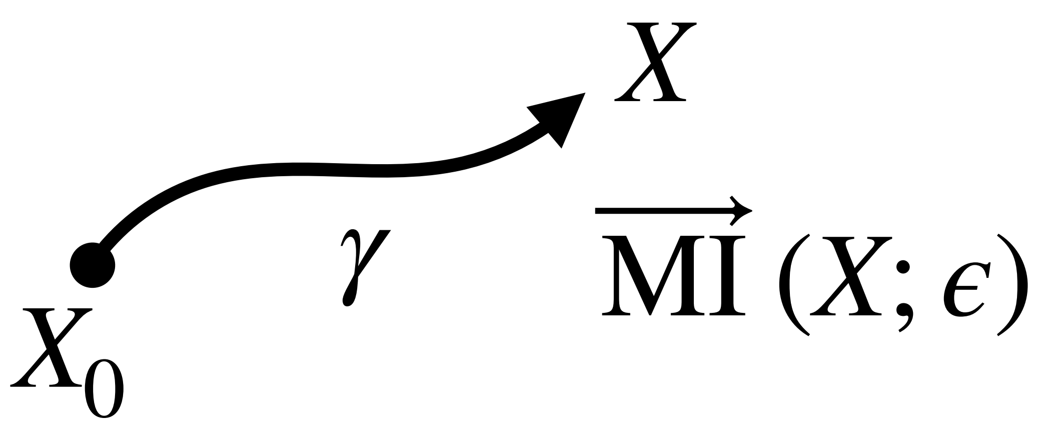
$$G\left(z_1,\ldots,z_n;x\right)=\int_0^x \frac{\mathrm{d}t_1}{t_1-z_1}\int_0^{t_1}\frac{\mathrm{d}t_2}{t_2-z_2}\cdots\int_0^{t_{n-1}}\frac{\mathrm{d}t_n}{t_n-z_n}$$

😀 Well understood functions, libraries for numerical evaluation

🙁 Difficult or impossible to obtain, functional relations

2. Integrate DEs numerically using **generalised series expansions** *Moriello 2019*

DiffExp *Hidding 2020*, SeaSyde *Armadillo et al. 2022*, AMFlow *Ma, Liu 2022*



😀 Very flexible and easy to set up

🙁 Time consuming, forced to evaluate the MIs

How to solve the DEs?

1. Express MIs in terms of **multiple polylogarithms**

$$G\left(z_1, \ldots, z_n; x\right) = \int_0^x \frac{\mathrm{d}t_1}{t_1 - z_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2 - z_2} \ldots \int_0^{t_{n-1}} \frac{\mathrm{d}t_n}{t_n - z_n}$$

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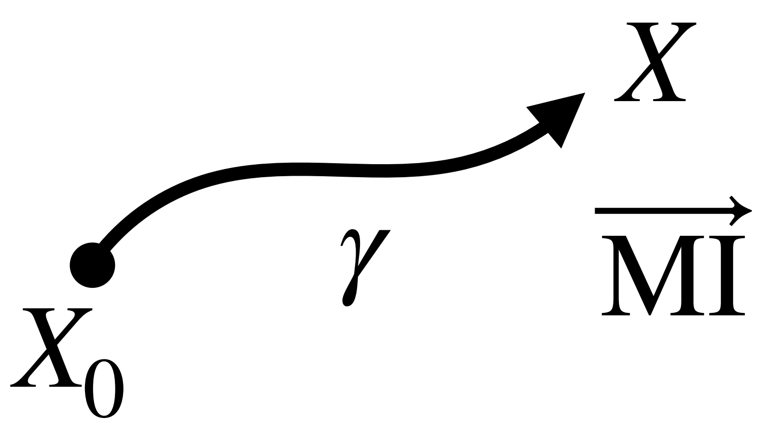
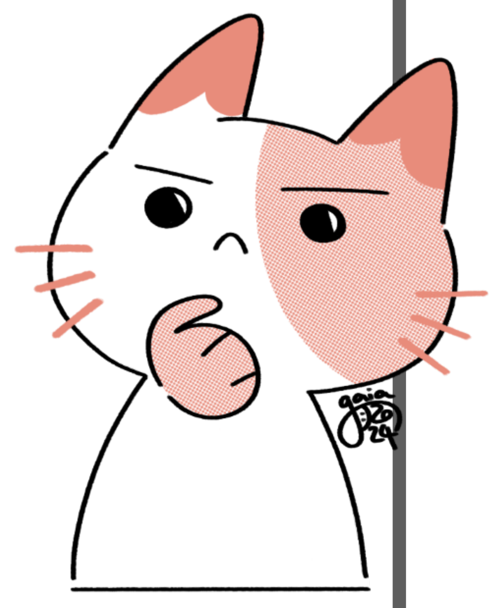


$\overrightarrow{\mathrm{MI}}\,(X;\epsilon)$

😀 Very flexible and easy to set up

🙁 Time consuming, forced to evaluate the MIs



The “pentagon functions” approach

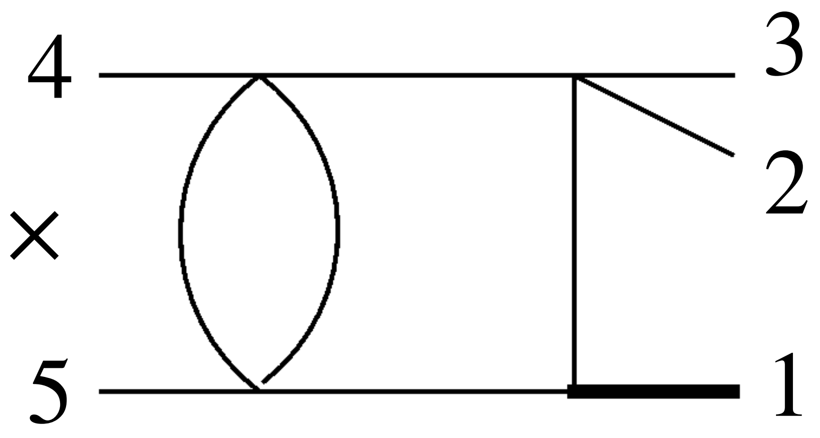
Particularly successful for 2-loop 5-point *Gehrmann, Henn, Lo Presti 2018; Chicherin, Sotnikov 2020; Chicherin, Sotnikov, **SZ** 2021; Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, **SZ** 2023*

The “philosophy” goes back to the *symbol map* Goncharov, Spradlin, Vergu, Volovich 2010

Very recent 3-loop developments made in UZH *Gehrmann, Henn, Jakubčík et al. 2024*

Construct a **basis** of algebraically independent special functions: $\vec{f} = \{f_i^{(w)}\}$

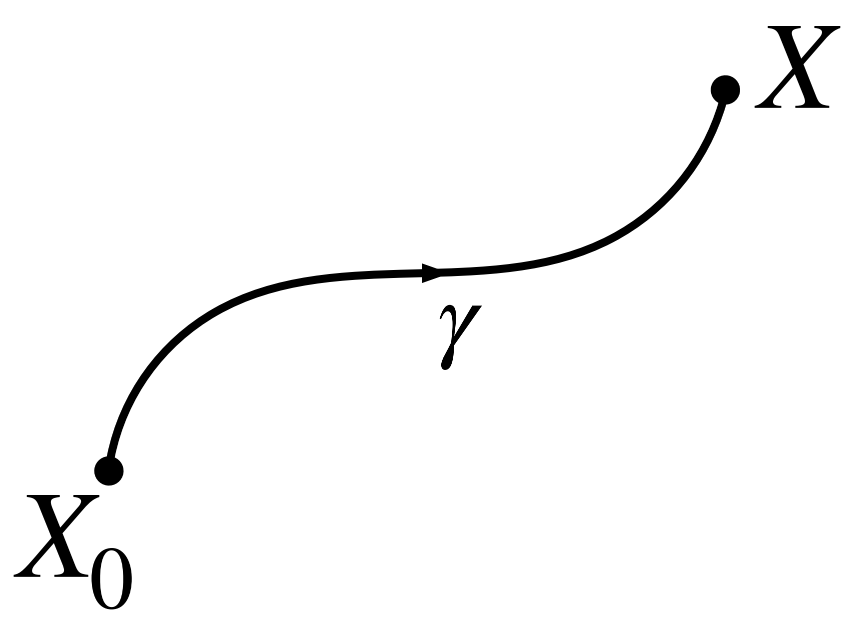
$\epsilon^3(1-2\epsilon)\sqrt{\Delta_3^{(1)}}\times$



$$= \epsilon^2 f_{23}^{(2)} + \epsilon^3\left[\frac{1}{4}\left(f_1^{(1)} - f_6^{(1)}\right) f_{23}^{(2)} + \frac{1}{2} f_3^{(3)} - \frac{1}{2} f_{29}^{(3)}\right] + \epsilon^4 f_{47}^{(4)} + \mathcal{O}\left(\epsilon^5\right)$$

Efficient numerical evaluation through one-fold integrals *Caron-Huot, Henn 2014*

Chen iterated integrals



$$\left[W_{i_1}, \ldots, W_{i_n}\right]_{X_0}(X) = \int_\gamma \mathrm{d}\log W_{i_n}(X') \left[W_{i_1}, \ldots, W_{i_{n-1}}\right]_{X_0}(X')$$

$n =$ transcendental weight

All functional relations become manifest in terms of iterated integrals

$$\mathrm{Li}_2(z) + \frac{1}{2}\log^2(-z) + \mathrm{Li}_2\left(\frac{1}{z}\right) + \frac{\pi^2}{6} = 0$$

$\mathbb{Q}$-linearly independent

$$\mathrm{Li}_2(z) = -\left[1-z, z\right]_{-1} - \log 2\left[z\right]_{-1} - \frac{\pi^2}{12}$$

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = \left[1-z, z\right]_{-1} - \left[z, z\right]_{-1} + \log 2\left[z\right]_{-1} - \frac{\pi^2}{12}$$

$$\frac{1}{2}\log^2(-z) = \left[z, z\right]_{-1}$$

Use $[W_1, \ldots, W_n]$ as basis of a vector space and view MIs/functions as vectors

$$\mathrm{Li}_2(z) = \Big([1-z, z], [z, z], [z], 1\Big) \cdot \begin{pmatrix} -1 \\ 0 \\ -\log(2) \\ -\frac{1}{12} \end{pmatrix}$$

Extract function basis from MI coefficients

$$\overrightarrow{\mathrm{MI}}(X;\epsilon) = \sum_{w\geq 0} \epsilon^w \, \overrightarrow{\mathrm{MI}}^{(w)}(X)$$

Solve DEs in terms of Chen iterated integrals up to required order (here, $w = 4$)

$$\left\{\mathrm{MI}_i^{(0)}\right\} \longrightarrow \{1\}$$

$$\left\{\mathrm{MI}_i^{(1)}\right\} \longrightarrow \left\{f_k^{(1)}\right\}$$

$$\left\{\mathrm{MI}_i^{(2)}\right\} \cup \left\{f_i^{(1)} \times f_j^{(1)}\right\} \longrightarrow \left\{f_k^{(2)}\right\}$$

$$\left\{\mathrm{MI}_i^{(3)}\right\} \cup \left\{f_i^{(2)} \times f_j^{(1)}\right\} \cup \left\{f_i^{(1)} \times f_j^{(1)} \times f_k^{(1)}\right\} \longrightarrow \left\{f_k^{(3)}\right\}$$

Gaussian elimination / Solution of a linear system of equations

$$\left\{f_i^{(w)}\right\}_{i,w=1,\ldots,4}$$

Algebraically independent

& irreducible

Tune ordering in Gaussian elimination to simplify the amplitudes

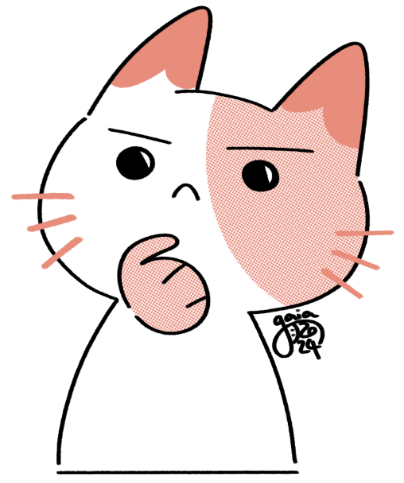
If $2\,\mathrm{MI}_1^{(1)}(X) + \mathrm{MI}_2^{(1)}(X) = 0$, do we choose

$$\mathrm{MI}_2^{(1)}(X) = -\,2\,\mathrm{MI}_1^{(1)}(X)$$

or

$$\mathrm{MI}_1^{(1)}(X) = -\,\frac{1}{2}\,\mathrm{MI}_2^{(1)}(X)$$

?



- Prefer products of lower-weight functions over higher weight functions

  ⇒ Pentagon functions are irreducible, high-weight functions are minimised

- Prefer 1-loop functions over 2-loop ones

  ⇒ Only these can appear in the UV/IR poles of 2-loop amplitudes

- Identify noteworthy analytic properties and isolate them in the minimal subsets of functions

  ⇒ E.g. cancellation of certain letters, divergences, integrable singularities…

Need to know relations among boundary values

We only know $\overrightarrow{\mathrm{MI}}^{(w)}(X_0)$ numerically

Previous approach: high-precision evaluation of MPLs + PSLQ algorithm

Ferguson, Bailey '92

$$\mathrm{MI}_1^{(2)}(X_0) = -1.644934067\ldots$$
$$\mathrm{MI}_2^{(2)}(X_0) = 0.4060916335\ldots$$
$$\mathrm{MI}_3^{(2)}(X_0) = 1.436746367\ldots$$

$\longrightarrow$

$$3\,\mathrm{MI}_1^{(2)}(X_0) + 4\,\mathrm{MI}_2^{(2)}(X_0) - 2\,\mathrm{MI}_3^{(2)}(X_0) = 0$$

- Very heavy from computational point of view (e.g. ~3000-digit precision in *Chicherin, Sotnikov, **SZ** 2021*) 😕
- Relies on MPL representation ☹️

The new algorithm

*Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, **SZ** 2023*

1. Select MI coefficients for the basis at **symbol** level: $\left\{f_i^{(w)}\right\}$

Goncharov, Spradlin, Vergu, Volovich 2010

Symbol = iterated integral stripped of boundary information

$$\mathrm{Li}_2(z) = -\left[1-z, z\right]_{-1} - \log 2\left[z\right]_{-1} - \frac{\pi^2}{12} \longrightarrow \mathcal{S}\left[\mathrm{Li}_2(z)\right] = -\left[1-z, z\right]$$

2. **Ansatz**: all MI coefficients are polynomials in $\left\{f_i^{(w)}\right\}$ + ζ_2 and ζ_3 (up to weight 4)

$$\mathrm{MI}^{(2)}(X) = \sum_i \alpha_i f_i^{(2)}(X) + \sum_{i\leq j} \beta_{ij} f_i^{(1)}(X) f_i^{(1)}(X) + \gamma\,\zeta_2 \qquad \alpha_i, \beta_{ij}, \gamma \in \mathbb{Q}$$

Fixed by symbol-level analysis

Fixed by evaluation at X_0 + rationalisation

Summary of the algorithm

Input:
- canonical DEs
- numerical boundary values

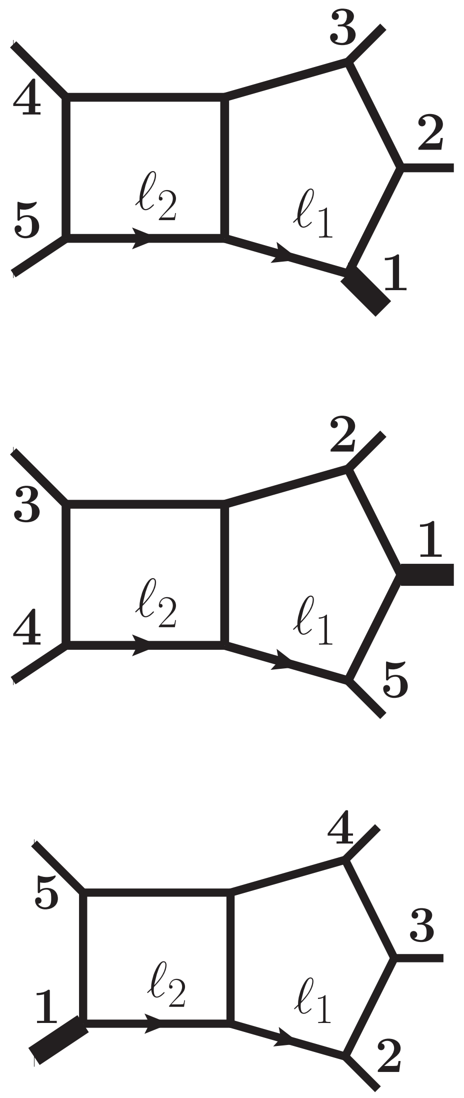
Only needed at the accuracy required for the evaluation (~70 digits)

Easy to obtain using **AMFlow** *Liu, Ma, Wang 2018; Liu, Ma 2022*

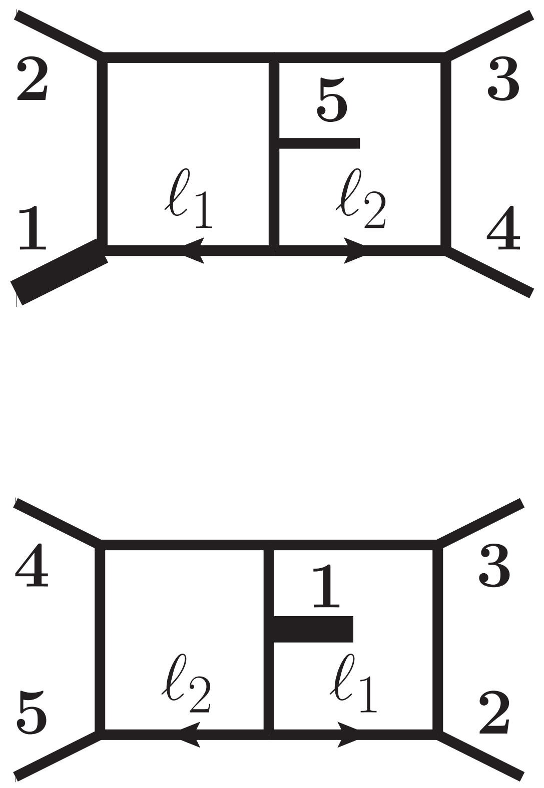
Output:
- function basis $\{f_i^{(w)}\}$ (written in terms of iterated integrals)
- relations among the boundary values
- expression of all MI coefficients as polynomials in $\{f_i^{(w)}\}$ and ζ values

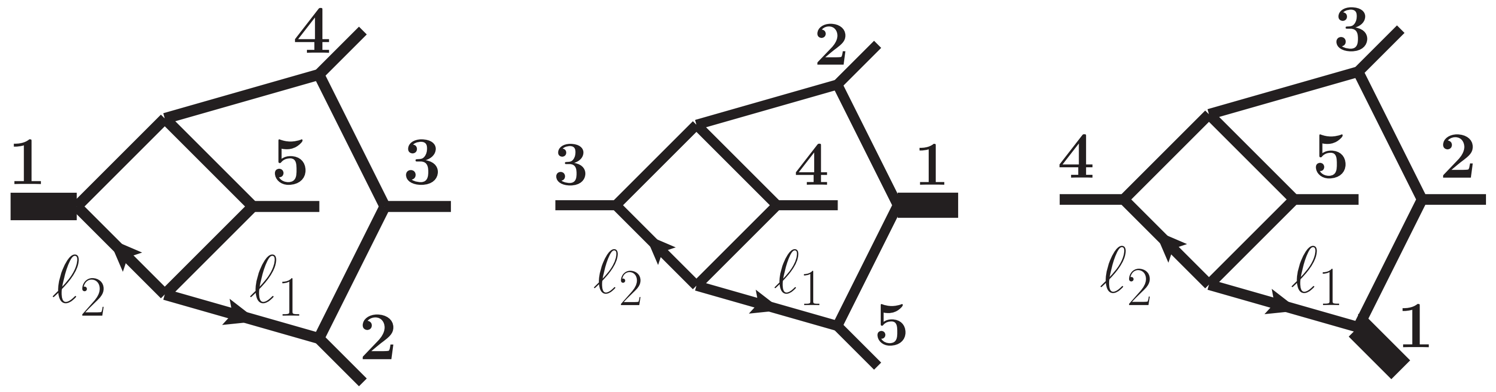
One-mass pentagon functions

*Chicherin, Sotnikov, **SZ** 2021; Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, **SZ** 2023*



weight	$P \cup PB$	HB	DP	Total
1	11	0	0	11
2	25	10	0	35
3	145	72	0	217
4	675	305	48	1028





All 4! permutations of external massless legs → all that is needed for any amplitude

All functions evaluated in ~1s (double precision)

Efficient numerical evaluation

Weight 1 & 2: explicit expressions using *symbol* technology *Duhr, Gangl, Rhodes 2011*

$$f_2^{(1)}(X) = \log\left(-s_{34}\right)$$
$$f_2^{(2)}(X) = \mathrm{Li}_2\left(\frac{s_{14}}{p_1^2}\right) + \log\left(-\frac{s_{14}}{p_1^2}\right)\log\left(1-\frac{s_{14}}{p_1^2}\right) + \mathrm{i}\pi\log\left(s_{15}-s_{23}+s_{45}\right) - \mathrm{i}\pi\log\left(p_1^2\right)$$

Weight 3 & 4: numerical integration of 1-fold integral representation *Caron-Huot, Henn 2014*

$$f^{(3)} \sim \int_0^1 \mathrm{d}t\, \frac{\mathrm{d}\log}{\mathrm{d}t} \times f^{(2)} \qquad f^{(4)} \sim \int_0^1 \mathrm{d}t\, \log \times \frac{\mathrm{d}\log}{\mathrm{d}t} \times f^{(2)}$$

Implemented in C++ library **PentagonFunctions++** *Sotnikov*

Efficient numerical evaluation

Weight 1 & 2: explicit expressions using *symbol* technology *Duhr, Gangl, Rhodes 2011*

$$f_2^{(1)}(X) = \log\left(-s_{34}\right)$$

$$f_2^{(2)}(X) = \mathrm{Li}_2\left(\frac{s_{14}}{p_1^2}\right) + \log\left(-\frac{s_{14}}{p_1^2}\right)\log\left(1-\frac{s_{14}}{p_1^2}\right) + i\pi\log\left(s_{15}-s_{23}+s_{45}\right) - i\pi\log\left(p_1^2\right)$$

Weight 3 & 4: numerical integration of 1-fold integral representation *Caron-Huot, Henn 2014*

$$f^{(3)} \sim \int^1 \mathrm{d}t\, \frac{\mathrm{d}\log}{} \times f^{(2)} \qquad f^{(4)} \sim \int^1 \mathrm{d}t\, \log \times \frac{\mathrm{d}\log}{} \times f^{(2)}$$

Implemented in C

Recently extended to weight-6 for 3-loop applications

Liu, Matijašić, Miczajka, Xu, Xu, Zhang 2024

Amplitudes ready for deployment in NNLO QCD phenomenology

$\mathrm{pp} \to 3\gamma$ *Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020*

$\mathrm{pp} \to 2\gamma + \mathrm{j}$ *Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021*

$\mathrm{pp} \to 3\mathrm{j}$ *Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022*

$\mathrm{pp} \to \gamma + 2\mathrm{j}$ *Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet,* ***SZ*** *2023*

$\mathrm{pp} \to \mathrm{W} + \mathrm{b\bar{b}}$ *Bayu Hartanto, Poncelet, Popescu,* ***SZ*** *2022; Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini 2023*

$\mathrm{pp} \to \mathrm{W} + \mathrm{t\bar{t}}$ *Buonocore, Devoto, Grazzini, Kallweit, Mazzitelli, Rottoli, Savoini 2023*

$\mathrm{pp} \to \mathrm{Z} + \mathrm{b\bar{b}}$ *Mazzitelli, Sotnikov, Wiesemann 2024*

$\mathrm{pp} \to \mathrm{H} + \mathrm{t\bar{t}}$ *Devoto, Grazzini, Kallweit, Mazzitelli, Savoini 2024*

Massification of $\mathrm{t/b}$

… and more to come!

Amplitudes ready for deployment in NNLO QCD phenomenology

$\mathrm{pp} \to 3\gamma$ *Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020*

$\mathrm{pp} \to 2\gamma + \mathrm{j}$ *Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021*

$\mathrm{pp} \to 3\mathrm{j}$

$\mathrm{pp} \to \gamma$

$\mathrm{pp} \to \mathrm{W}$ *...azzitelli, Rottoli, Sav...*

$\mathrm{pp} \to \mathrm{W}$ ...

$\mathrm{pp} \to \mathrm{Z} + \mathrm{b}\bar{\mathrm{b}}$ *Mazzitelli, Sotnikov, Wiesemann 2024*

$\mathrm{pp} \to \mathrm{H} + \mathrm{t}\bar{\mathrm{t}}$ *Devoto, Grazzini, Kallweit, Mazzitelli, Savoini 2024*

Massification of t/b

[Submitted on 9 Dec 2024]

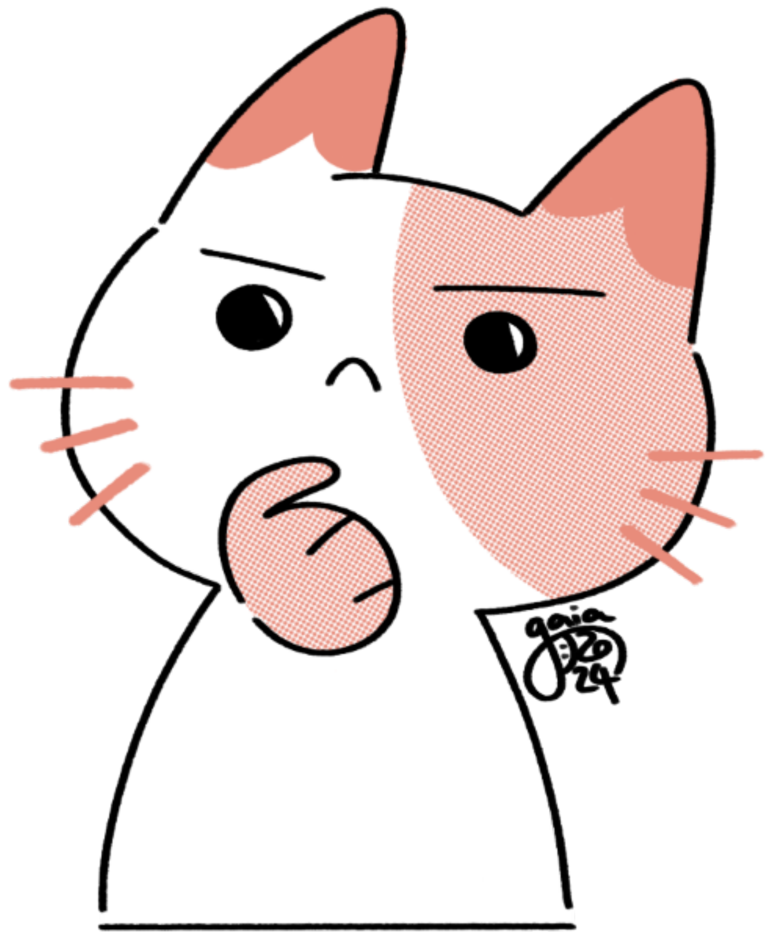
Full-colour double-virtual amplitudes for associated production of a Higgs boson with a bottom-quark pair at the LHC

Simon Badger, Heribertus Bayu Hartanto, Rene Poncelet, Zihao Wu, Yang Zhang, Simone Zoia

We present the double-virtual amplitudes contributing to the production of a Higgs boson in association with a $b\bar{b}$ pair at the Large Hadron Collider. We perform the computation within the five-flavour scheme, which employs massless bottom quarks and finite bottom-Yukawa coupling, taking into account all the colour structures. We derive the analytic form of the helicity amplitudes through finite-field reconstruction techniques. The analytic expressions have been implemented in a public C++ library, and we demonstrate that evaluations are sufficiently stable and efficient for use in phenomenological studies.

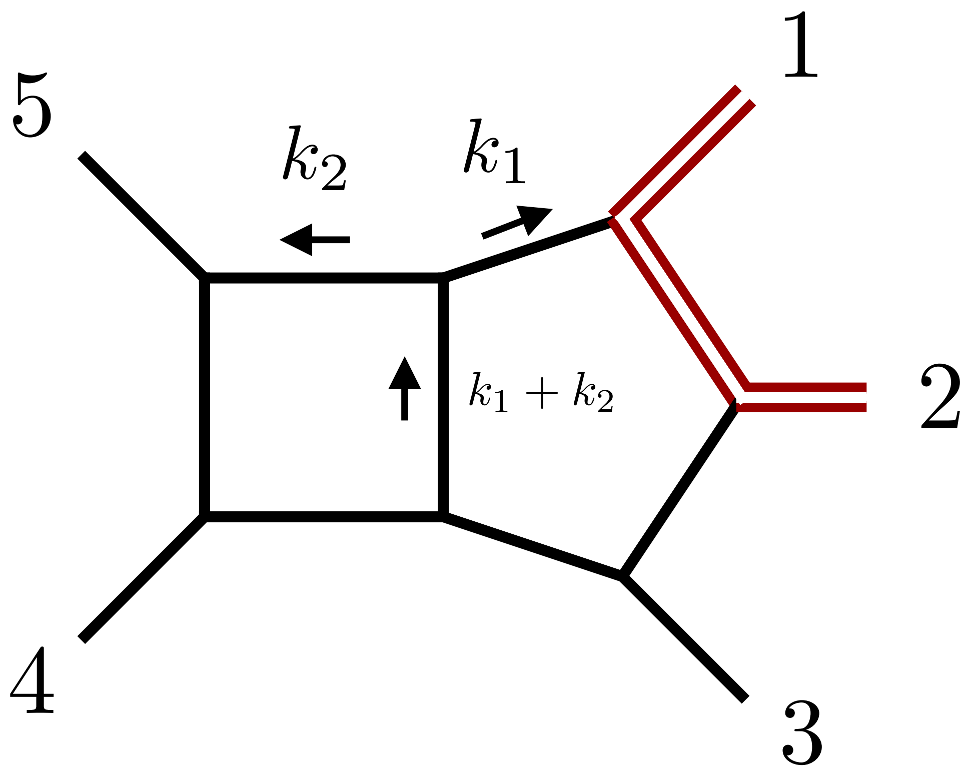
… and more to come!

What about non-polylogarithmic cases?



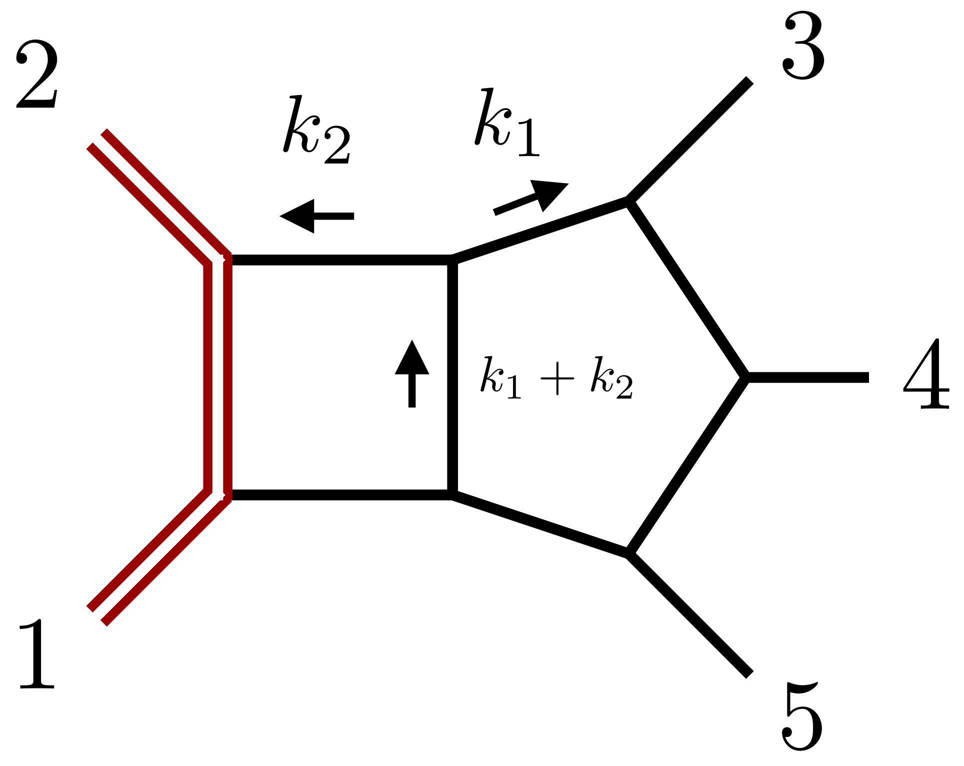
Two-loop families to $t\bar{t}j$ production @ leading colour

Badger, Becchetti, Giraudo, ***SZ*** *2024*



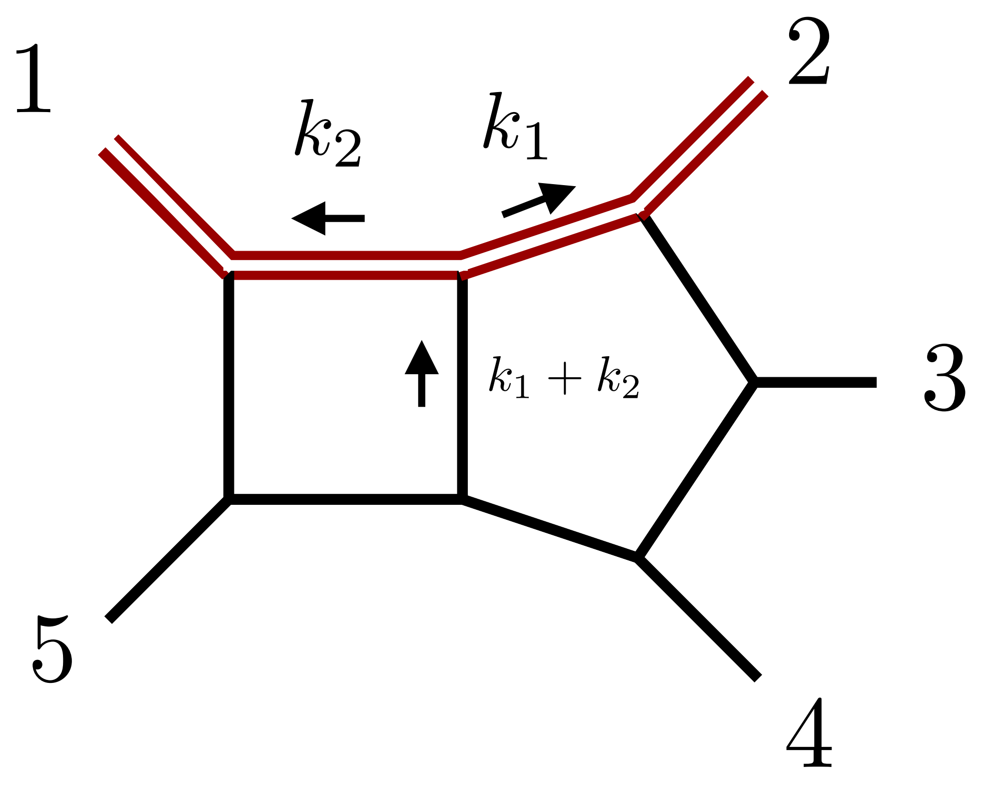
88 MIs

Badger, Becchetti,
Chaubey, Marzucca 2022



109 MIs

Polylogarithmic ✔

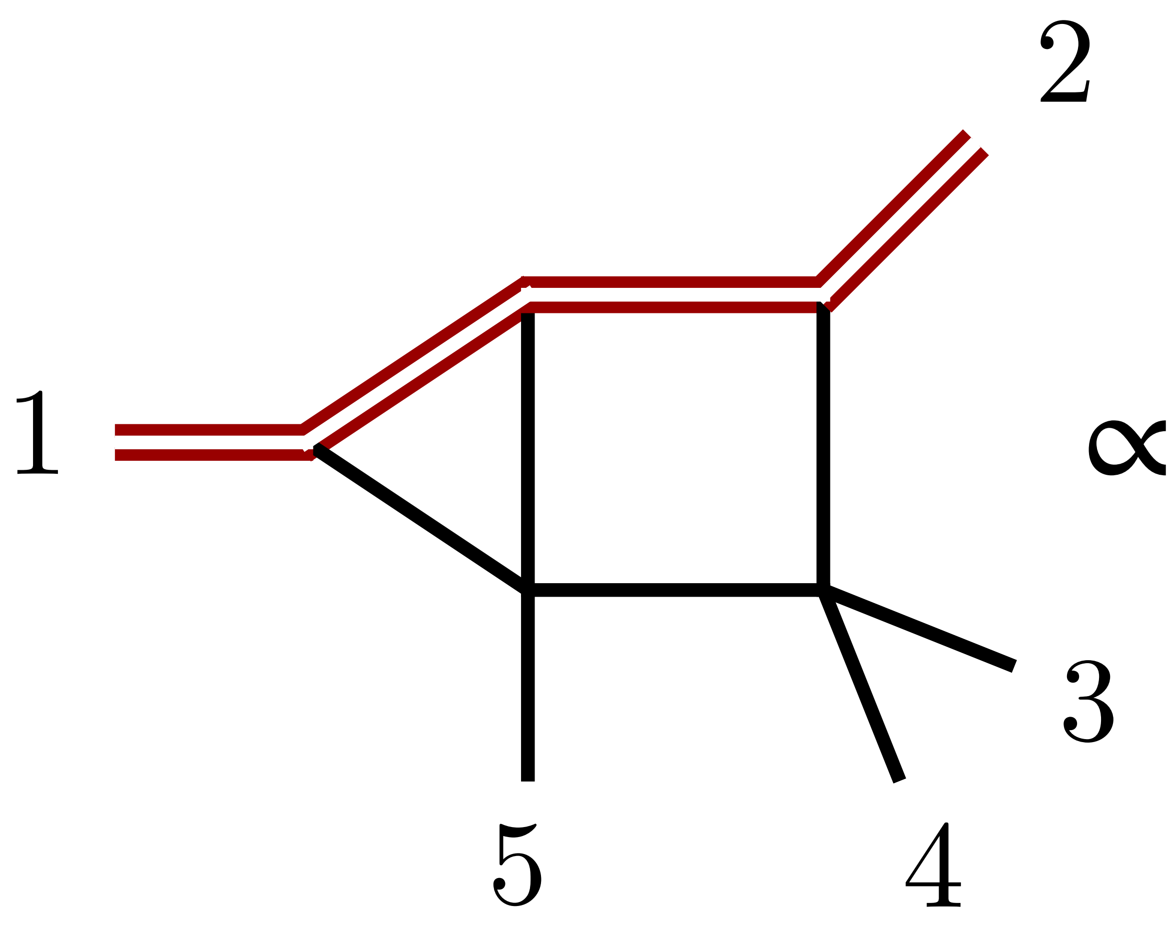


121 MIs

Elliptic curve 🆘

Nested square roots 🤔

Dlog analysis obstructed by an elliptic curve



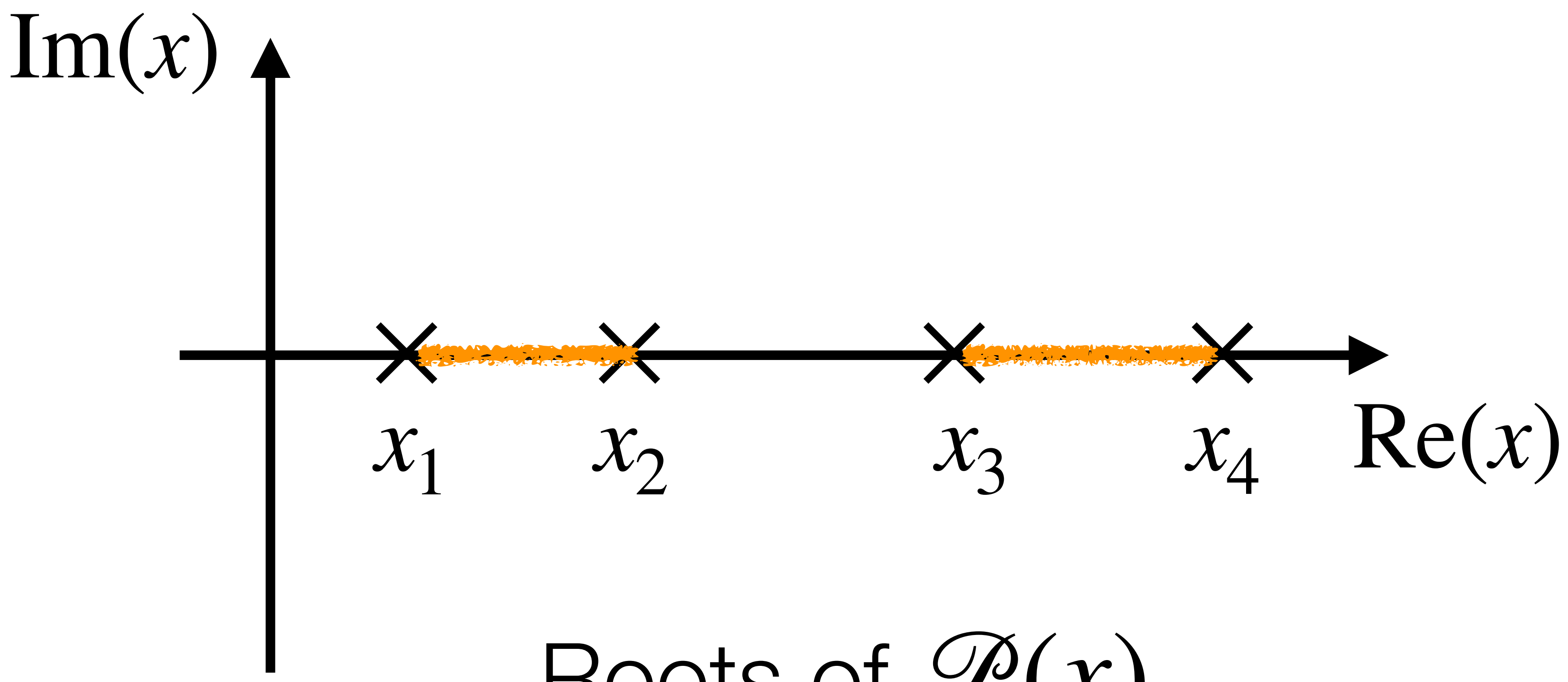
$$\propto \int \frac{\mathrm{d}x}{\sqrt{\mathscr{P}(x)}} \wedge \mathrm{dlogs}$$

3 MIs in this sector

Degree-4 polynomial with distinct roots

$$\mathscr{P}(x) = \left(x + m_t^2\right)\left(x - 3m_t^2\right)\left(\mathscr{P}_0 + \mathscr{P}_1 x + \mathscr{P}_2 x^2\right)$$

Elliptic curve: $y^2 = \mathscr{P}(x)$

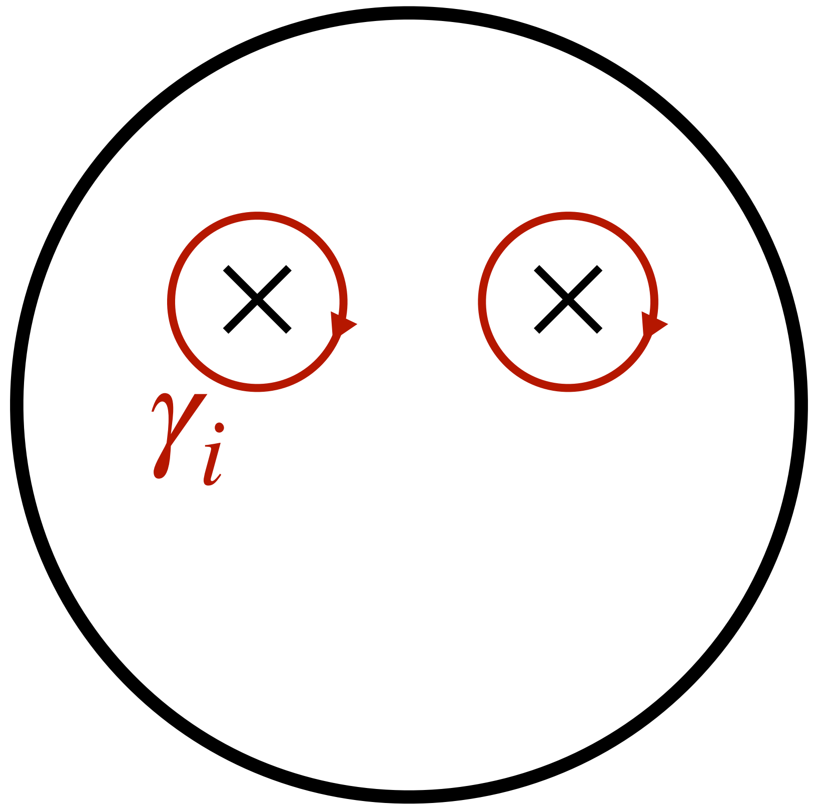


Roots of $\mathscr{P}(x)$

With some imagination, this is a donut!

Punctured Riemann sphere

$$\mathbb{C} \cup \{\infty\} \backslash \{z = c_i\}$$



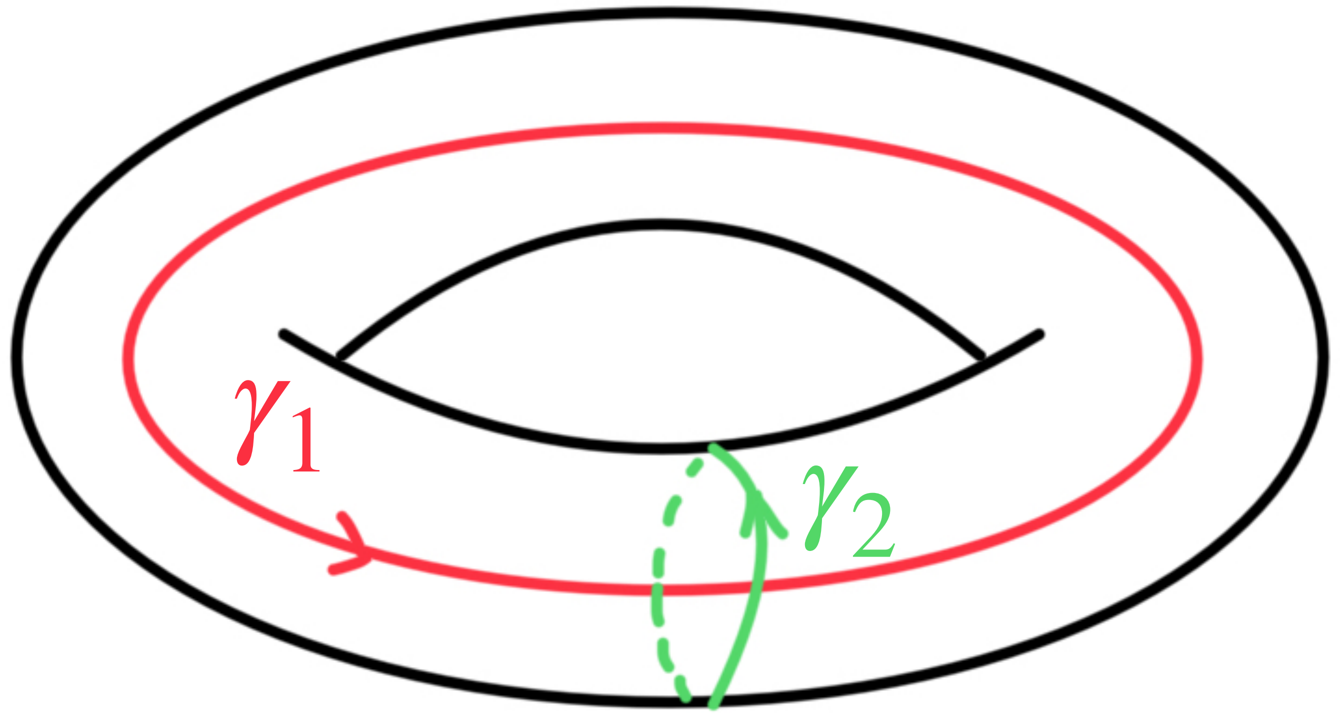
Differential forms: $\dfrac{\mathrm{d}x}{x - c_i}$

Leading singularities ≡ residues

$$\oint_{\gamma_i} \frac{\mathrm{d}x}{x - c}$$

Torus

$$\mathbb{C} \cup \{\infty\} \backslash \{\text{branch cuts}\}$$



Differential forms: $\dfrac{\mathrm{d}x}{\sqrt{\mathscr{P}(x)}}, \ldots$

Periods of the elliptic curve

$$\oint_{\gamma_1} \frac{\mathrm{d}x}{\sqrt{\mathscr{P}(x)}} \propto K\left(\frac{(x_3 - x_2)(x_4 - x_1)}{(x_3 - x_1)(x_4 - x_2)}\right)$$

Heroic effort into developing the maths we need

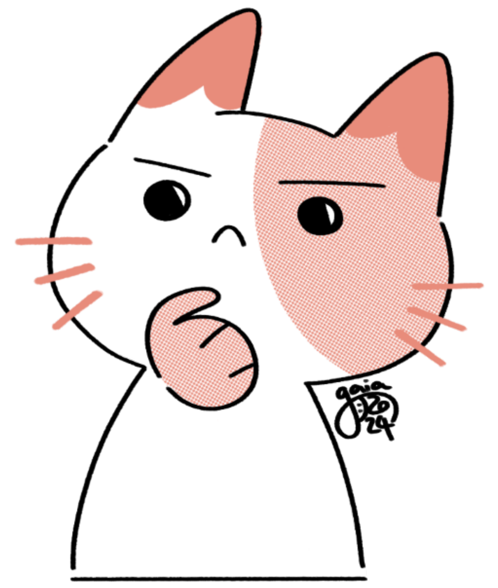
Review: arXiv:2203.07088 (*Bourjaily, Broedel, Chaubey, Duhr…, Marzucca…*)

Last week: arXiv:2412.02300 (*Duhr, Porkert, Stawinski*), arXiv:2411.19042 (*Forner, Nega, Tancredi*)

Still challenging to obtain “canonical” DEs & most applications have few variables

We have 5 ratios + elliptic curve is coupled to the nested root 😨

How far can we go without canonical DEs?

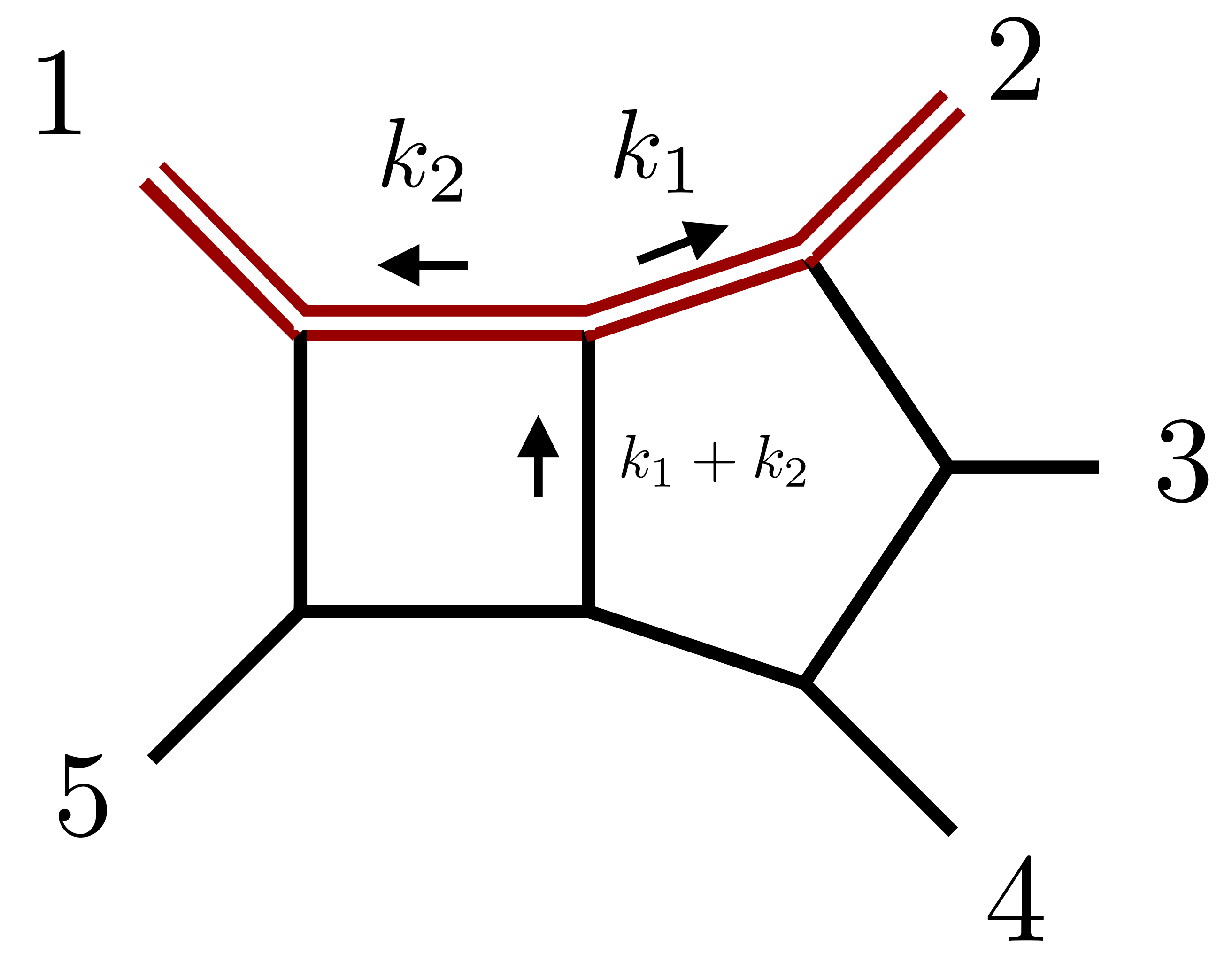


Get as close to canonical as possible

*Badger, Becchetti, Giraudo, **SZ** 2024*

$$\mathrm{d}\vec{g}(X;\epsilon) = \sum_{k=0}^{2} \epsilon^k\, \Omega^{(k)}(X) \cdot \vec{g}(X;\epsilon)$$

$$\Omega^{(k)}(X) = \sum_i A_i^{(k)}\, \mathrm{d}\log W_i(X) \; + \; \sum_j B_j^{(k)}\, \omega_j(X)$$



"Non-logarithmic" one-forms, e.g. $\omega(x,y) = \omega_x(x,y)\,\mathrm{d}x + \omega_y(x,y)\,\mathrm{d}y$

(Linearly independent)

Algebraic functions

Push the trouble into the finite part

Choose the "problematic" MIs so that they start at order ϵ^4

In principle, possible because the ϵ-poles of 2-loop amplitudes are determined by 1-loop amplitudes

In practice not obvious:

- arbitrary MIs have ϵ-poles
- the "non-problematic" MIs are coupled to the problematic ones through the DEs

New ingredient: numerical evaluations at random points to determine which $g_i^{(k)}(X) = 0$

The boundary values conspire… to help us!

$$\vec{g}(X;\epsilon) = \sum_{k\geq 0} \epsilon^k \, \vec{g}^{(k)}(X)$$

$$\mathrm{d}\vec{g}^{(k)}(X) = \Omega^{(0)}(X)\cdot\vec{g}^{(k)}(X) + \Omega^{(1)}(X)\cdot\vec{g}^{(k-1)}(X) + \Omega^{(2)}(X)\cdot\vec{g}^{(k-2)}(X)$$

$$\mathrm{d}\vec{g}^{(0)}(X) = 0$$

$$\mathrm{d}\vec{g}^{(1)}(X) \sim \mathrm{d}\log \times \, \vec{g}^{(0)}(X)$$

Drop all vanishing $g_i^{(k)}(X)$

Can be solved in terms of iterated integrals

⇒ pentagon functions ✓

$$\mathrm{d}g_{15}^{(2)}(X) = \frac{1}{24}\left[12\, g_{103}^{(1)}(X_0) + 8\, g_{110}^{(1)}(X_0) + 4\, g_{111}^{(1)}(X_0) + 3\, g_{118}^{(1)}(X_0) - 48\, g_{63}^{(1)}(X_0)\right]\omega_2(X) + \ldots$$

= 0

Almost everything is polylogarithmic!

Only 6 $g_i^{(4)}(X)$ cannot be expressed in terms of iterated integrals this way

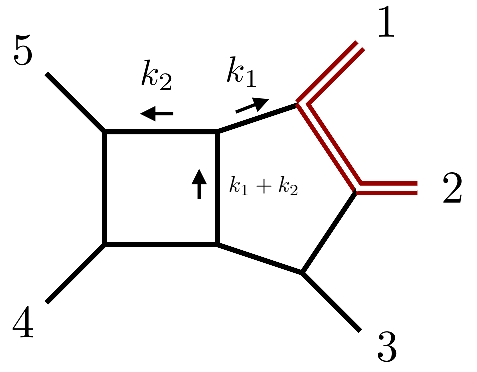
We just add them to our function basis as independent functions

$$
\begin{aligned}
g_{15}(X;\epsilon) = & \frac{1}{48} + \frac{\epsilon}{24}\left(F_1^{(1)} - 2\,F_2^{(1)} + 2\,F_4^{(1)} - 2\,F_6^{(1)}\right) \\
& - \frac{1}{48}\epsilon^2\left[7\,F_1^{(2)} + 7\,F_3^{(2)} + \frac{25}{4}\zeta_2 - \frac{15}{4}\left(F_1^{(1)}\right)^2 + 15\,F_1^{(1)}\,F_2^{(1)} - 8\left(F_2^{(1)}\right)^2 + \ldots\right] \\
& + \frac{1}{8}\epsilon^3\left[F_5^{(3)} + \frac{820}{9}\zeta_3 + \frac{7}{12}\zeta_2 F_2^{(1)} - \left(F_2^{(1)}\right)^3 + 5\left(F_3^{(1)}\right)^2 F_5^{(1)} + 6\,F_1^{(1)}F_2^{(1)}F_6^{(1)} + \ldots\right] \\
& + \epsilon^4\,F_1^{(4^*)} + \mathcal{O}\left(\epsilon^5\right)\,,
\end{aligned}
$$

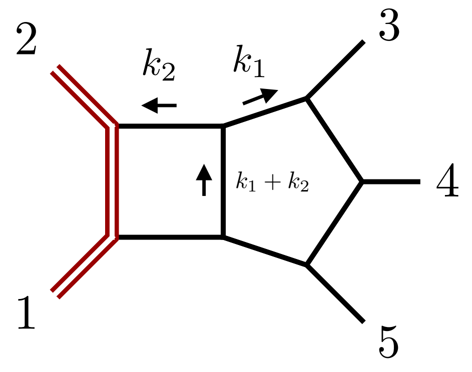
Polylogarithmic

Non-polylogarithmic

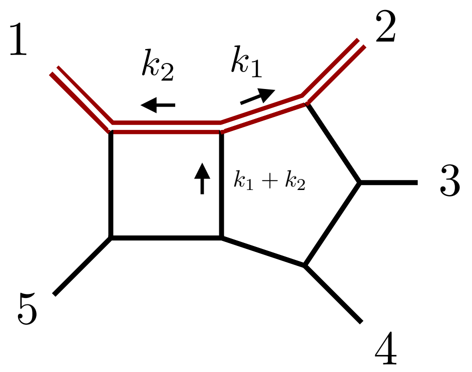
All special functions for the 2-loop amplitudes @ leading colour



2 permutations



1 permutation



2 permutations

transcendental weight	1	2	3	4	4*	all
# of functions	6	8	45	166	12	237

Polylogarithmic, can be evaluated efficiently with available technique!

Non-polylogarithmic

Non-polylog. funcs. defined via auxiliary DEs, much simpler than those for the MIs

$$\mathrm{d}\vec{G}(X) = M(X) \cdot \vec{G}(X)$$

$$\vec{G}(X) = \left(F_1^{(4*)}, \ldots, F_{12}^{(4*)}, \ldots, F_1^{(1)}, 1\right)^{\mathrm{T}}$$

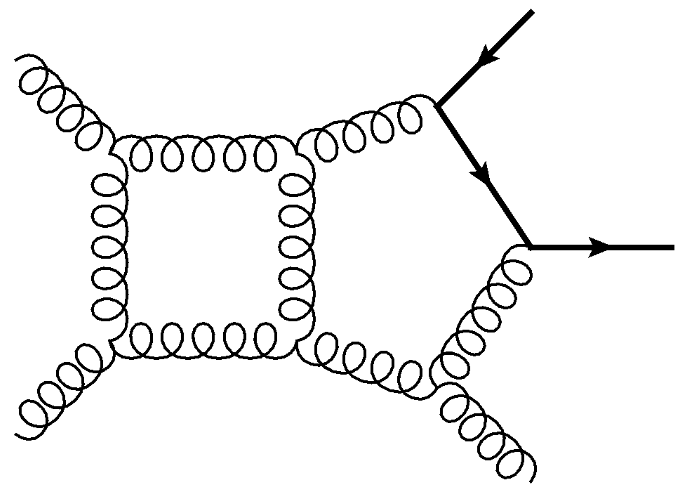
84 functions

Hidding 2020

Solved with **DiffExp** in ~1 min / point (~15 s / segment)

First results for the two-loop $gg \to gt\bar{t}$ amplitudes

*Badger, Becchetti, Brancaccio, Hartanto, **SZ**, in preparation*



Leading colour helicity finite remainders:

$$F^{(2)}(X) = \sum_i R_i(X)\ \mathrm{mon}_i\left(F_j^{(w)}\right)$$

Rational coefficients

Monomials of special functions

For now, evaluated numerically through finite-field routine

Thanks to our representation of the MIs:

- Analytic cancellation of UV/IR poles
- Dramatic simplification in the rational coefficients (O(100) drop in polynomial degrees)

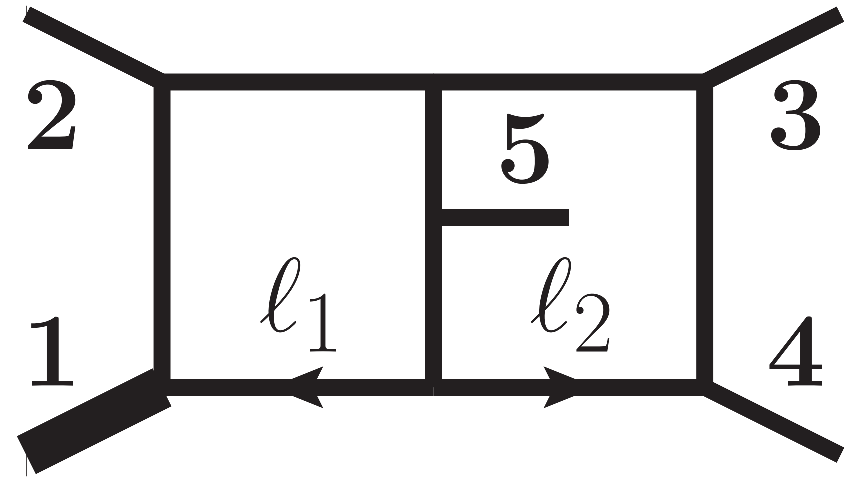
Ready for analytic computation!

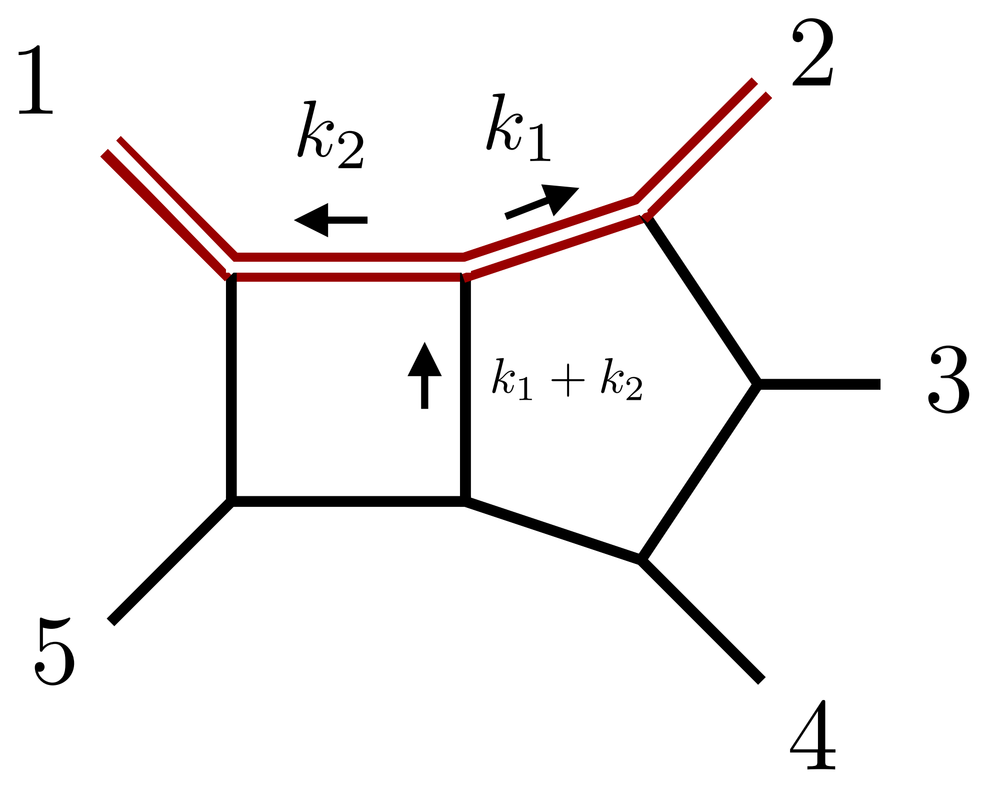
Conclusions

General algorithm for constructing a special-function basis to solve canonical DEs

- Numerical evaluation suitable for pheno.
- Substantial simplification of the scattering amplitudes

➡ All 2-loop 5-pt. integrals with 1 external massive leg





First steps towards the extension of this method to non-polylogarithmic integrals

➡ 2-loop 5-pt. integrals for $\mathrm{pp} \to \mathrm{t\bar{t}}$+jet @ leading colour

Thank you!