# Applications of General Relativity
# Sheet 4



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**Exercise 1** [Metric of a static star] (8 points)

The exterior spacetime of extended spherical objects can be approximated by the Schwarzschild metric. In this exercise we will derive the interior metric modeling the star by a perfect fluid with energy-momentum tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$. As a first step we will consider the general static, spherically symmetric metric

$$\mathrm{d}s^2 = \exp[2\alpha(r)]\mathrm{d}t^2 - \exp[2\beta(r)]\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1}$$

The corresponding non-vanishing Christoffel symbols are (up to permutations of the two lower indices)

$$\begin{aligned}
&\Gamma^t_{t\,r} = \partial_r\alpha \\
&\Gamma^r_{t\,t} = \partial_r\alpha\exp[2(\alpha-\beta)] \quad \Gamma^r_{r\,r} = \partial_r\beta \quad \Gamma^r_{\theta\,\theta} = -r\exp[-2\beta] \quad \Gamma^r_{\phi\,\phi} = -r\sin^2(\theta)\exp[-2\beta] \\
&\Gamma^\theta_{\theta\,r} = r^{-1} \quad \Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta) \\
&\Gamma^\phi_{\phi\,r} = r^{-1} \quad \Gamma^\phi_{\phi\,\theta} = \cot(\theta),
\end{aligned}$$

while the Ricci tensor takes the form

$$R_{tt} = \exp[2(\alpha-\beta)]\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right) \tag{2}$$

$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \tag{3}$$

$$R_{\theta\theta} = 1 - \exp[-2\beta] + (\partial_r\beta - \partial_r\alpha)\,r\exp[-2\beta] \tag{4}$$

$$R_{\phi\phi} = \sin^2(\theta)R_{\theta\theta}. \tag{5}$$

(i) In the fluid rest frame the velocity is pointing in the timelike direction. Using this and the normalization $u_\mu u^\mu = 1$ show that the energy momentum tensor can be written as

$$T_{\mu\nu} = \mathrm{diag}\left\{\exp[2\alpha]\,\rho, \exp[2\beta]\,p, r^2\,p, r^2\sin^2(\theta)\;p\right\}. \tag{6}$$

(ii) Write down the Einstein equations inside the star.

It will be convenient to define

$$m(r) = \frac{1}{2G}\big(r - r\exp[-2\beta]\big) \tag{7}$$

and to rewrite the Einstein equations in terms of $m(r)$.

(iii) Use the energy-momentum conservation $\nabla_\mu T^{\mu\nu}$ to show

$$(\rho + p)\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}. \tag{8}$$

Use the above equation together with the $rr$-component of the Einstein equations to derive the Tolman-Oppenheimer-Volkoff equation of a star

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)\left[Gm(r) + 4\pi G r^3 p\right]}{r\left[r - 2Gm(r)\right]}. \tag{9}$$

This is the relativistic version of the equation of hydrostatic equilibrium for a star.

(iv) Assuming an incompressible fluid with constant density $\rho_*$ out to the surface of the star, solve for the pressure $p(r)$ and write down the metric. What is the maximum mass a star of a given radius $R$ can have?

**Exercise 2** [Out of plane precession of S2 orbit] (5 points)
The center of our galaxy harbours a rotating black hole. While the exact Kerr solution for rotating black holes will be discussed at the end of the semester we will consider an interesting effect using the weak field results derived in the lecture.

Many of the stars within 1 pc from the central black hole have orbits that lie in a plane. This suggests that these stars formed from gas that is arranged in a disk-like structure. However, the orbit of the star S2 with $R_0 = 4.6 \times 10^{-3}$ pc and eccentricity $e = 0.87$ is inclined by 75° with respect to this plane. One possible origin for this inclination could be the precession of the orbital plane due to the black hole's spin. Using the age estimate of $t_{\mathrm{S2}} \approx 3 \times 10^6$ yr, calculate a lower bound on the black hole spin using the black hole mass $M_{\mathrm{BH}} = 3 \times 10^6\ M_\odot$.

*Hint:* As shown in the lecture, the "gravitomagnetic" equation of motion can be written as

$$\frac{d\boldsymbol{v}}{dt} = -\nabla\phi + 2\boldsymbol{\Omega} \wedge \boldsymbol{v}, \tag{10}$$

where

$$\boldsymbol{\Omega} = \frac{G}{c^2}\left[\frac{3\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{S})}{r^5} - \frac{\boldsymbol{S}}{r^3}\right]. \tag{11}$$

Here, $\boldsymbol{S}$ is the spin of the black hole. Since you are searching a lower bound for the spin, it is sufficient to estimate the precession rate for the configuration where the precession is maximal, i.e., when $\boldsymbol{S}$ lies on the orbital plane. Furthermore, you are allowed to simplify the system assuming circular orbits. Note that the black hole spin is conveniently parametrized in terms of the spin parameter

$$a = \frac{Sc}{GM^2}, \tag{12}$$

where $a < 1$. A black hole with $a = 1$ is said to be maximally spinning.