MMP I
Tutorial 10

HS 2019
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Exercise 1: Hilbert space $H = \mathbb{C}^n$ (8 Pts.)

Let $\{|\chi\rangle\}_i$, $1 \leq i \leq n$ be an orthonormal basis in $H = \mathbb{C}^n$, $T \in L(H, H)$ a linear operator (an endomorphism) and U a unitary linear operator, i.e. $U^\dagger U = 1$.

a) Show that $\{|\phi\rangle\}_i$ with $|\phi_i\rangle \equiv U\,|\chi_i\rangle$ also form an orthonormal basis.

b) Show that if λ is an eigenvalue of U we have $|\lambda| = 1$. What is $||U||$?

c) Assume T has an eigenvalue λ. Show that the eigenvectors to this eigenvalue form a Hilbert space. Show that λ^* is an eigenvalue of $T^\dagger$.

d) Let $l \leq n$. Show $P \equiv \sum_{j=1}^{l} |\chi_j\rangle\,\langle\chi_j|$ is a projector, i.e. $P^2 = P$ and $P^\dagger = P$. What is $||P||$?

e) Let R and S be two self-adjoint operators. Show that there exists a common basis where both of them are diagonal if and only if $[R, S] = 0$. To simplify the proof you may assume that no eigenvalue is degenerate.

Exercise 2: Norm of linear operators (4 Pts.)
The norm $||T||$ of a bounded linear operator T is defined as:

$$||T|| = \sup_{x\neq 0} \frac{||Tx||}{||x||} = \sup_{||x||=1} ||Tx||. \tag{2.1}$$

The aim of the exercise is to show that for a self-adjoint bounded operator $S = S^\dagger$ we can also write the norm $||S||$ as

$$\sup_{x\neq 0} \frac{|\langle x|Sx\rangle|}{||x||^2} = \sup_{||x||=1} |\langle x|Sx\rangle| \equiv \sigma. \tag{2.2}$$

a) For $S = S^\dagger$ show $||S|| \geq \sigma$.

b) For $k \in \mathbb{R}$ define

$$|v_\pm\rangle \equiv k|x\rangle \pm \frac{1}{k} S|x\rangle. \tag{2.3}$$

and prove the following chain of (in)equalities

$$||Sx||^2 = \frac{1}{4}(\langle Sv_+|v_+\rangle - \langle Sv_-|v_-\rangle) \leq \frac{1}{4}\sigma(||v_+||^2 + ||v_-||^2) = \frac{1}{2}\sigma(k^2||x||^2 + k^{-2}||Sx||^2). \tag{2.4}$$

Minimise the right hand side w.r.t. k^2 to show $||S|| \leq \sigma$.

Exercise 3: Orthonormal basis of a Hilbert space (4 Pts.)

Consider $H = L^2[0, \pi]$ and the operator given by $T(f) = gf$ with $g(x) = x$ for $0 \leq x \leq 1$; 1 for $1 \leq x \leq \pi$.

a) Let $u(x) = 1$ in $L^2[0, \pi]$ and consider the functions $\phi_n(x) = T^n(u)$, $n = 0, 1 \cdots$. Are they a complete set in $L^2[0, \pi]$?

b) Find eigenvalues and eigenvectors of T and discuss their degeneracy.