Exercise 1. ***Long-distance scattering***

Starting from the form of the wave function for $r \to \infty$

$$\psi(\vec{r}) = \mathrm{e}^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{\mathrm{e}^{ikr}}{r}, \tag{1}$$

(a) Compute the probability density of the outgoing spherical plane wave

$$\vec{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) .$$

(b) Show that the asymptotic solution in (1) solves the Schroedinger equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(\vec{r}) = E\psi(\vec{r})$$

for $r \to \infty$ as long as the scattering potential satisfies $\lim_{r\to\infty} r\,V(r) \to 0$.

Exercise 2. ***Elastic-Scattering from a central potential***

Consider the elastic scattering off a central potential

$$V(r) = \frac{\epsilon}{r^2}, \qquad \text{with} \qquad \epsilon \ll \frac{\hbar^2}{2m}\,,$$

where the associated time-independent Schroedinger equation reads

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{\epsilon}{r^2}\right)\psi(\vec{r}) = E\psi(\vec{r})\,o.$$

Using the usual partial-wave decomposition

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} R_l(r)P_l(\cos\theta)\,,$$

compute the phase shifts $\delta_l(k)$ and the scattering amplitude $f(\theta)$.

Hint: Recall that:

$$\sum_{l=0}^{\infty} P_l(\cos\theta) = \frac{1}{2\sin\theta/2}.$$

Exercise 3. ***1D scattering from a localized potential bounded by an infinite well***

A particle of mass m and energy E is incident from the left on the potential

$$V(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a \le x \le 0 \\ \infty & x > 0 \end{cases}$$

(a) Given the incoming wave $\psi_i(x) = Ae^{ikx}$ (where $k = \sqrt{2mE}/\hbar$), show that the reflected wave is $\psi_r(x) = Be^{-ikx}$, where B is given by

$$B = Ae^{-2ika}\left[\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}\right] ,$$

and $k' = \sqrt{2m(E + V_0)}/\hbar$.

(b) Check that the reflected wave has the same amplitude as the incident wave.

(c) Recall that the total wave function for $x < -a$ has the form

$$\psi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right) .$$

Find the phase shift δ for a very deep well ($E \ll V_0$).