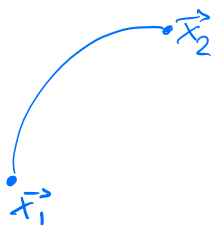


1.3 Work, energy & conservative forces

(7)

We define the work done by a force along a given trajectory as

$$W_{12} = \int_{\text{path } \vec{x}_1 \rightarrow \vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x} = \int_{t_1}^{t_2} \vec{F}(\vec{x}(t)) \frac{d\vec{x}}{dt} dt \quad (1.6)$$



The work is the integral of the force along the path defined by the trajectory

For $m=0$ (as always assumed from now on, unless otherwise stated)

$$\begin{aligned} W_{12} &= m \int_{t_1}^{t_2} \frac{d}{dt} [\dot{\vec{x}}(t)] \dot{\vec{x}}(t) dt = \frac{m}{2} \int_{t_1}^{t_2} \frac{d}{dt} [\dot{\vec{x}}(t)^2] dt \\ &= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \equiv T_2 - T_1 \end{aligned} \quad (1.7)$$

where we have defined the scalar quantity

$$T = \frac{1}{2} m \dot{\vec{x}} \cdot \dot{\vec{x}} \equiv \frac{1}{2} m v^2 \quad (1.8)$$

that we denote as the kinetic energy of the particle.

A force is said to be conservative if the work is the same for any physically possible path between the same initial and final points. This implies null work on closed path

$$\oint \vec{F} d\vec{x} = 0 \quad \left(\begin{array}{l} \text{possible alternative} \\ \text{def. of conservative force} \end{array} \right) \quad (1.9)$$

(8)

A necessary and sufficient condition for $\vec{F}$ to be conservative is

$$\boxed{\vec{F} = -\vec{\nabla} V(\vec{x})} \quad (1.10)$$

where V is a scalar function depending only on the position that we denote as the potential (or the potential energy)

$$W_{12} = \int \vec{F} d\vec{x} = \left| \begin{array}{c} \text{cons.} \\ \text{force} \end{array} \right. - \int \vec{\nabla} V(\vec{x}) d\vec{x} = - \int_{\vec{x}_1}^{\vec{x}_2} dV(\vec{x}) = V_1 - V_2 \quad (1.11)$$

Given (1.7) this implies $T_1 + V_1 = T_2 + V_2$



Theorem of Energy Conservation:

If the forces acting on a particle are conservative, then the total energy

(1.12)

$$\boxed{E(\vec{x}, \dot{\vec{x}}) = T(\dot{\vec{x}}) + V(\vec{x})}$$

is conserved

Another way to check this is by computing explicitly $\frac{dE}{dt}$

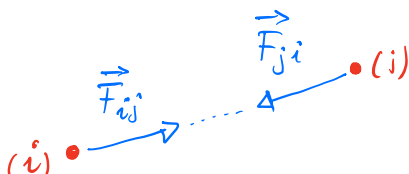
$$\begin{aligned} \frac{d}{dt} E(\vec{x}, \dot{\vec{x}}) &= \frac{d}{dt} T(\dot{\vec{x}}) + \frac{d}{dt} V(\vec{x}) = \frac{d}{dt} \left(\frac{1}{2} m \dot{\vec{x}}^2 \right) + \vec{\nabla} V \cdot \dot{\vec{x}} \\ &= \dot{\vec{x}} \left[m \ddot{\vec{x}} - \vec{F} \right] = 0 \end{aligned}$$

1.4 Systems of point particles

(9)

We now proceed generalizing the concepts so far introduced to a system of n point-like particles.

The system is closed, if the force $\vec{F}_i$ acting on the i -th particle can be decomposed as

$$\vec{F}_i = \sum_{j=1 \neq i}^n \vec{F}_{ij} \quad (1.13)$$


The diagram illustrates the forces between two particles, labeled (i) and (j). Particle (i) is on the left and particle (j) is on the right. A blue arrow labeled $\vec{F}_{ij}$ points from particle (i) towards particle (j). Another blue arrow labeled $\vec{F}_{ji}$ points from particle (j) towards particle (i). A dashed line connects the two particles, and the arrows are positioned along this line, indicating they are internal forces within the system.

where $\vec{F}_{ij} = -\vec{F}_{ji}$ (3rd Newton's law, or principle of action & reaction)
[$\Rightarrow$ prove this follows from Galileo's relativity for 2-particle system].

For open systems, we decompose $\vec{F}_i$ as

$$\boxed{\vec{F}_i = \sum_{j=1 \neq i}^n \vec{F}_{ij} + \vec{F}_i^{(ext)}} \quad \boxed{\vec{F}_{ij} = -\vec{F}_{ji}} \quad (1.14)$$

Let's define the total momentum of the system

$$\vec{P} = \sum_i m_i \dot{\vec{x}}_i = \sum_i \vec{p}_i \quad (1.15)$$

The center of mass ($\vec{X}$) and the total mass (M)

$$\vec{X} = \frac{\sum_i m_i \vec{x}_i}{\sum_i m_i} \quad M = \sum_i m_i \quad (1.16)$$

such that

$$\vec{P} = M \dot{\vec{X}} \quad (1.17)$$

It is easy to prove that

Theorem: the rate of change of total momentum of a system is equal to the sum of all the external forces acting on the system

Proof:

$$\frac{d\vec{P}}{dt} = \sum_i \dot{\vec{P}}_i = \sum_{\substack{i,j=1 \\ i \neq j}}^n \vec{F}_{ij} + \sum_i \vec{F}_i^{(ext)} = \sum_i \vec{F}_i^{(ext)}$$

$\searrow$ given $\vec{F}_{ij} = -\vec{F}_{ji}$

From this theorem it follows that

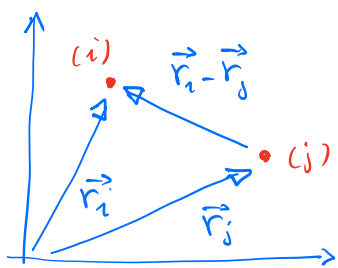
- ⊙ The momentum of a closed system is conserved
- ⊙ The c.o.m. of a closed system moves uniformly

Let's consider now the angular momentum of a system of particles with respect to a given point. The total angular momentum is

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \quad (1.18)$$

and its time variation reads

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \sum_i \frac{d}{dt} (\vec{r}_i \times \vec{p}_i) = \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \vec{F}_i^{(ext)} + \sum_{\substack{i,j=1 \\ i \neq j}}^n \vec{r}_i \times \vec{F}_{ij} \\ &= \sum_i \vec{r}_i \times \vec{F}_i^{(ext)} + \underbrace{\sum_{i>j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}}_{\downarrow} \end{aligned} \quad (1.19)$$



$\emptyset$ if $\vec{F}_{ij}$ is parallel to $\vec{r}_i - \vec{r}_j$
 which we assume to hold for all
 internal forces
 ("strong" law of action and reaction)



Theorem: the rate of change of total angular momentum of a system is equal to the sum of all the external torques acting on the system

(11)

Let's finally look at the energy of the system.

The work of all forces when considering the evolution of the system from time t_1 to time t_2 is

$$W_{12} = \sum_i \int_{\{x_i(t_1)\}}^{\{x_i(t_2)\}} \vec{F}_i d\vec{x}_i = \sum_i \int_{t_1}^{t_2} \frac{1}{2} m_i (\dot{\vec{x}}_i)^2 = T_2^{\text{tot}} - T_1^{\text{tot}} \quad (1.20)$$

$$T^{\text{tot}} = \frac{1}{2} \sum_i m_i v_i^2 = \text{total kinetic energy} \quad (1.21)$$

It maybe convenient to change variables and separate the motion of the c.o.m. of the system

$$\vec{x}_i = \vec{r}_i + \vec{X} \quad \left\{ \begin{array}{l} \vec{X} = \text{c.o.m. coordinate} \\ \vec{r}_i = \text{particle coord in the c.o.m. frame} \end{array} \right.$$

Then the total kin. energy becomes

$$T^{\text{tot}} = \frac{1}{2} \sum_i m_i (\dot{\vec{r}}_i + \dot{\vec{X}})(\dot{\vec{r}}_i + \dot{\vec{X}}) = \underbrace{\left[\frac{1}{2} M V^2 \right]}_{\text{kin. energy of the c.o.m.}} + \underbrace{\left[\frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^2 \right]}_{\text{"internal" kinetic energy}}$$

where we have used the

fact that $\sum_i m_i \dot{\vec{r}}_i = 0$

(c.o.m. momentum in the c.o.m. frame)

Let's now consider the case of conservative forces.

In order to identify the different contributions to the potential energy, we decompose the total work as

$$W_{12} = W_{12}^{\text{ext}} + W_{12}^{\text{int}} \quad (1.22)$$

$$W_{12}^{\text{ext}} = \sum_i \int_{\{x_i(t_1)\}}^{\{x_i(t_2)\}} \vec{F}_i^{(\text{ext})} d\vec{x}_i = \sum_i V_i^{\text{ext}} \Big|_{t_1} - \sum_i V_i^{\text{ext}} \Big|_{t_2} = V_1^{\text{ext}} - V_2^{\text{ext}} \quad (1.23)$$

potential associated to the external forces

$$W_{12}^{\text{int}} = \sum_{i \neq j} \int_{\{x_i(t_1)\}}^{\{x_i(t_2)\}} \vec{F}_{ij} \cdot d\vec{x}_i$$

Focusing on internal forces, the hypothesis of conservative forces implies there exists a scalar function $V_{ij}(\vec{r}_i, \vec{r}_j)$ such that

$$\begin{cases} \vec{F}_{ij} = -\vec{\nabla}_{\vec{x}_i} V_{ij}(\vec{x}_i, \vec{x}_j) \equiv -\vec{\nabla}_i V_{ij}(\vec{x}_i, \vec{x}_j) \\ \vec{F}_{ji} = -\vec{\nabla}_{\vec{x}_j} V_{ij}(\vec{x}_i, \vec{x}_j) \equiv -\vec{\nabla}_j V_{ij}(\vec{x}_i, \vec{x}_j) \end{cases}$$

In order to satisfy the strong law of action & reaction we must have

$$\boxed{V_{ij} = \varphi(|\vec{r}_i - \vec{r}_j|)} \quad |\vec{r}_i - \vec{r}_j| \equiv |\vec{r}_{ij}| \equiv \sqrt{\vec{r}_{ij} \cdot \vec{r}_{ij}} \quad (1.24)$$

where $\varphi(s)$ is a generic scalar function.

We find indeed:

$$\vec{\nabla}_i \varphi(|\vec{r}_i - \vec{r}_j|) = \frac{d}{ds} \varphi(s) \frac{1}{s} \Big|_{s=|\vec{r}_{ij}|} \cdot (\vec{r}_i - \vec{r}_j) \equiv \varphi'(|\vec{r}_{ij}|) \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_{ij}|} \quad (1.25)$$

hence

$$\vec{F}_{ij} = -\vec{\nabla}_i V_{ij} \equiv -\vec{\nabla}_{ij} V_{ij} = +\vec{\nabla}_j V_{ij} = -\vec{F}_{ji}$$

where $\vec{\nabla}_{ij}$ denotes the gradient with respect to the variable $\vec{r}_{ij}$

We are now ready to compute the internal work.

For each pair of particles we have

$$\vec{F}_{ij} d\vec{x}_i + \vec{F}_{ji} d\vec{x}_j = -\vec{\nabla}_i V_{ij} d\vec{r}_i - \vec{\nabla}_j V_{ij} d\vec{r}_j = -(\vec{\nabla}_{ij} V_{ij}) \cdot d\vec{r}_{ij}$$

That implies

$$W_{12}^{\text{int}} = - \sum_{i>j} \int \vec{\nabla}_{ij} V_{ij} d\vec{r}_{ij} = +\frac{1}{2} \sum_{i \neq j} V_{ij} \Big|_{t_1} - \frac{1}{2} \sum_{i \neq j} V_{ij} \Big|_{t_2} \quad (1.26)$$

↑
potential associated to internal forces

(the $\frac{1}{2}$ factor is there because each ij pairs appears twice)

We can thus define a total potential energy

$$V^{\text{tot}} = \boxed{\sum_i V_i^{\text{ext}}} + \boxed{\frac{1}{2} \sum_{i \neq j} V_{ij}} \quad (1.27)$$

and $E^{\text{tot}} = V^{\text{tot}} + T^{\text{tot}}$ is conserved.

In general, the internal potential energy is non-zero and, most important, can vary with time. Only in specific systems it is separately conserved, these are what we call rigid bodies.

Formally a rigid body can be defined as a system of particles where the $\vec{r}_{ij}$ are fixed (cannot vary with time) $\rightarrow$ the internal forces do not work.