SYSTEMS OF FIRST ORDER ODE AND ODE OF HIGHER ORDER

We consider a set of n continuous functions $f_i : A \to \mathbb{R}$ $\quad A \subset \mathbb{R} \times \mathbb{R}^n$

$$(x, y_1 \dots y_n) \to f_i(x, y_1 \dots y_n)$$

x independent variable $\qquad y_i$ unknown functions

$$\begin{cases} y_1' = f_1(x, y_1 \dots y_n) \\ y_2' = f_2(x, y_1 \dots y_n) \\ \vdots \\ y_n' = f_n(x, y_1 \dots y_n) \end{cases}$$

system of first order ODE

We can also formally write it as $\underline{y}' = \underline{f}(x, \underline{y})$ $\qquad \underline{y}(x_0) = \underline{y_0}$

set of initial conditions

We now consider an n-order ODE

$$y^{(n)} = f(x, y, y' \dots y^{(n-1)})$$

This equation is equivalent to a system of first-order ODEs

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = f(x, y_1 \dots y_n) \end{cases}$$

If $y_1 \dots y_n$ is a solution of the system, by replacing $y_1 \to y$ $\quad y_2 \to y' \dots$ $\quad y_n \to y^{(n-1)}$ we have a solution of the differential equation.

Conversely, if $y(x)$ is a solution of the n-order ODE, then the vector

$\underline{y} = (y_1 \dots y_n) = (y, y' \dots y^{(n-1)})$ is a solution of the linear system.

LINEAR SYSTEMS OF 1-ST ORDER ODEs

We now consider the special case in which the functions $f_i(x, y_1 \dots y_m)$ are linear in $y_1 \dots y_m$

$$\begin{cases} y_1' = a_{11}(x) y_1 + \dots \; a_{1m}(x) y_m + b_1(x) \\ \vdots \\ y_m' = a_{m1}(x) y_1 + \dots \; a_{mm}(x) y_m + b_m(x) \end{cases}$$

$b_i : I \subset \mathbb{R} \to \mathbb{R}$

$a_{ij} : I \subset \mathbb{R} \to \mathbb{R}$

continuous functions

In matrix notation we can write $\underline{y}' = A(x)\underline{y} + \underline{b}(x)$

THEOREM (without proof)

If the functions $a_{ij}(x)$ and $b_i(x)$ are continuous, given $x_0 \in I$, $\underline{y}_0 \in \mathbb{R}^m$ there exists a unique solution of the system of class $C^1(I)$ such that $\underline{y}(x_0) = \underline{y}_0$

THE CASE $\underline{b} = 0$

In the case $\underline{b} = 0$ the system is said to be homogeneous. In this case the solutions of the system describe a m-dimensional space.

A set of solutions of the homogeneous system is said to be linearly independent if $\lambda_1 \underline{\phi}_1 + \dots \lambda_2 \underline{\phi}_2 = 0 \quad \forall x \in I$ only if $\lambda_1 = \lambda_2 = \dots \lambda_2 = 0$

Due to the linearity, if $\underline{\phi}_1$ and $\underline{\phi}_2$ are solutions $\Rightarrow$ $\alpha \underline{\phi}_1 + \beta \underline{\phi}_2$ is also solution

$\left(\underline{\phi}_1' = A \underline{\phi}_1 , \quad \underline{\phi}_2' = A \underline{\phi}_2 \qquad \alpha \underline{\phi}_1' + \beta \underline{\phi}_2' = A(\alpha \underline{\phi}_1 + \beta \underline{\phi}_2) \right)$

More generally, every linear combination of solutions is a solution, and there exist m linearly independent solutions.

Such solutions are said to provide a FUNDAMENTAL SYSTEM.

Considering n independent solutions

$$\underline{\phi}_j = \begin{pmatrix} \phi_{1j}(x) \\ \vdots \\ \phi_{nj}(x) \end{pmatrix}$$

we can construct the matrix

$$Y(x) = \begin{pmatrix} \phi_{11}(x) & \cdots & \phi_{1n}(x) \\ \vdots & & \\ \phi_{n1}(x) & \cdots & \phi_{nn}(x) \end{pmatrix}$$

$$\begin{matrix} \uparrow & & \uparrow \\ \underline{\phi}_1 & & \underline{\phi}_n \end{matrix}$$

$\Rightarrow$ WRONSKIAN MATRIX

THEOREM (Liouville)

A necessary and sufficient condition such that the vectors $\underline{\phi}_1 \ldots \underline{\phi}_n$ are linearly independent is that

(i) $\quad W(x) = \det Y(x) \neq 0 \quad \forall x \in I$

or equivalently that

(ii) $\quad W(x_0) \neq 0$ for at least a value $x_0 \in I$

Proof

Obviously (i) implies (ii). Suppose now that $W(x_0) \neq 0$ and that $\exists\, x_1 \in I$ such that $W(x_1) = 0$. This means that there exist $\alpha_1 \ldots \alpha_n$ not all vanishing such that $\underline{y}(x_1) = \sum_j \alpha_j \underline{\phi}_j(x_1) = 0$. This means that $\underline{y}$ is solution of the problem

$$\begin{cases} \dot{\underline{y}}(x) = A(x)\,\underline{y} & x \in I \\ \underline{y}(x_1) = 0 \end{cases}$$

But the $Z(x) \equiv 0$ null solution is also solution of the same problem ($Z' = AZ$ and $Z(x_1) = 0$) $\Rightarrow$ since the solution is unique we have $\underline{Y} = 0$ and $W(x_0) = 0$ ∎

We can now apply the above results to the case of the n-order ODE

$$y^{(n)} = \alpha_1(x)\, y^{(n-1)} + \dots \alpha_n(x)\, y$$

by rewriting it as a system we can define the Wronskian as

$$W(x) = \begin{vmatrix} \varphi_1(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & & \vdots \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & & \varphi_n^{(n-1)}(x) \end{vmatrix}$$

where $\phi_1(x) \dots \phi_n(x)$ are n independent solutions.

EXAMPLE

$$y'' = \frac{2}{x^2}\, y - \frac{2}{x}\, y' \qquad x > 0$$

this is an example of the EULER DIFFERENTIAL EQUATION (see later)

we can solve it through an ansatz

$$y = x^\alpha \qquad y' = \alpha x^{\alpha-1} \qquad y'' = \alpha(\alpha-1)\, x^{\alpha-2}$$

$$\alpha(\alpha-1)\cancel{x^{\alpha-2}} = \frac{2}{\cancel{x^2}}\, x^\alpha - \frac{2}{\cancel{x}}\, \alpha x^{\alpha-1}$$

$\Rightarrow$ we get an algebraic equation for α

$$\alpha(\alpha-1) = 2 - 2\alpha$$

$$\alpha^2 + \alpha - 2 = 0 \quad \Rightarrow \quad \alpha_1 = 1, \quad \alpha_2 = -2$$

$\Rightarrow$ we have two solutions of the DE

$$\phi_1(x) = x$$

$$\phi_2(x) = \frac{1}{x^2}$$

The Wronskian is

$$W(x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x^2} \\ 1 & -\frac{2}{x^3} \end{vmatrix} = -\frac{3}{x^2} \neq 0$$

$\Rightarrow$ ϕ_1 and ϕ_2 provide a fundamental system $\Rightarrow$ $\phi(x) = c_1 x + \dfrac{c_2}{x^2}$ general solution

EXAMPLE

$y'' + y = 0$ this is the equation of the harmonic oscillator $\Rightarrow$ $y_1 = \sin x$, $y_2 = \cos x$ are linearly indep. solutions

$$W(x) = \det\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = 1$$

EXAMPLE

$y'' - 4y' - 5y = 0$ this is a second-order linear equation with constant coefficients

ansatz $y = e^{\lambda x}$ $\Rightarrow$ $\lambda^2 - 4\lambda - 5 = 0$ $\quad \lambda_1 = -1$, $\lambda_2 = 5$

$$\lambda = \frac{4 \pm \sqrt{16+20}}{2}$$

$$W(x) = \begin{vmatrix} e^{-x} & e^{5x} \\ -e^{-x} & 5e^{5x} \end{vmatrix}$$

$$= 6e^{4x} \neq 0$$

EXAMPLE

$y'' - 6y' + 9y = 0$ ansatz $y = e^{\lambda x}$ $\Rightarrow$ $\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$

$y_1 = e^{3x}$ is a solution

how can we find a second solution?

$y_2 = xe^{3x}$ is also a solution!

$$y_2' = e^{3x} + 3xe^{3x}$$

$$y_2'' = 3e^{3x} + 3e^{3x} + 9xe^{3x}$$

$$y_2'' - 6y_2' + 9 = \cancel{3e^{3x}} + \cancel{3e^{3x}} + \cancel{9xe^{3x}} - 6(\cancel{e^{3x}} + \cancel{3xe^{3x}}) + \cancel{9xe^{3x}} = 0$$

How can we make this systematic?

HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

$$\begin{cases} y_1' = a_{11} y_1 + \dots a_{1n} y_n \\ \vdots \\ y_n' = a_{n1} y_1 + \dots a_{nn} y_n \end{cases}$$

$a_{ij} \in \mathbb{R}$

in compact form $\underline{y}' = A\underline{y}$

We try the ansatz $\underline{y} = e^{\lambda x}\underline{c}$ $\Rightarrow$ $\underline{y}' = \lambda e^{\lambda x}\underline{c} = A e^{\lambda x}\underline{c}$

$\Rightarrow$ $A\underline{c} = \lambda \underline{c}$ λ must be an eigenvalue of A

We have thus to study the characteristic polynomial $P(\lambda) = \det(A - \lambda I)$

The roots of $P(\lambda) = 0$ are called eigenvalues of A and the problem we have is the one of the diagonalization of A. Let us recall a few results in linear algebra. For each eigenvalue λ_i we define:

- $m_a(\lambda_i)$ algebraic multiplicity power of $(\lambda - \lambda_i)$ in $P(\lambda)$
- $m_g(\lambda_i)$ geometrical multiplicity $\dim(\ker(A - \lambda_i I))$

We have $m_g(\lambda_i) \leq m_a(\lambda_i)$. The matrix A is diagonalizable if

- $\sum_i m_a(\lambda_i) = n$
- $m_g(\lambda_i) = m_a(\lambda_i)$

If A is not diagonalizable it can be written in a form called JORDAN FORM

$$J = \begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix} \qquad \text{where} \qquad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

The dimension of each Jordan block is given by the algebraic multiplicity $m_a(\lambda_i)$.

We can now go back to our problem of solving the linear system.

- The simplest case is the one in which there are <u>n distinct real eigenvalues</u>.

  In this case there are n independent solutions $e^{\lambda_i x}\underline{c}_i$ where $\underline{c}_i$ is the eigenvector of A corresponding to the eigenvalue λ_i.

<u>EXAMPLE</u>

$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 4y_1 + y_2 \end{cases} \qquad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

$$P(\lambda) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = 3$$

$\underline{c}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\quad \underline{c}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and the general solution is

$$\underline{y} = \alpha e^{-x}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta e^{3x}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- The next case is the one in which, besides some single-multiplicity real eigenvalue, there are <u>complex</u> eigenvalues. Since A is real, if λ is a complex eigenvalue also λ^* is an eigenvalue

$$\left(\det(A - \lambda I) = 0 \quad \Rightarrow \quad \det(A - \lambda^* I) = 0 \right)$$

The solution associated to λ is

$$\underline{y} = e^{\lambda x}\underline{c}_1 = e^{(\mu + i\nu)x}(\underline{a} + i\underline{b}) = e^{\mu x}(\cos\nu x + i\sin\nu x)(\underline{a} + i\underline{b})$$

$$= e^{\mu x}\left(\cos\nu x\,\underline{a} - \sin\nu x\,\underline{b} + i(\sin\nu x\,\underline{a} + \cos\nu x\,\underline{b})\right)$$

The solution associated to λ^* is

$$\underline{y} = e^{\lambda^* x} \underline{c}^* = e^{(\mu - i\nu)x} (\underline{a} - i\underline{b}) = e^{\mu x}\Big(\cos \nu x\, \underline{a} - \sin \nu x\, \underline{b} - i\big(\sin \nu x\, \underline{a} + \cos \nu x\, \underline{b}\big)\Big)$$

We are looking for real solutions. We know that a linear combination of these solutions will be a solution $\Rightarrow$ we can simply take the real and imaginary parts of these solutions

$$\underline{y}_1(x) = e^{\mu x}\left(\cos \nu x\, \underline{a} - \sin \nu x\, \underline{b}\right)$$

$$\underline{y}_2(x) = e^{\mu x}\left(\sin \nu x\, \underline{a} + \cos \nu x\, \underline{b}\right)$$

EXAMPLE

$$\begin{cases} y_1' = y_1 - 2y_2 \\ y_2' = 2y_1 - y_3 \\ y_3' = 4y_1 - 2y_2 - y_3 \end{cases} \qquad A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & -1 \\ 4 & -2 & -1 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & -\lambda & -1 \\ 4 & -2 & -1-\lambda \end{vmatrix} = (1-\lambda)\left(\lambda(1+\lambda) - 2\right) + 2\left(-2(1+\lambda) + 4\right)$$

$$= (1-\lambda)\left[\lambda^2 + \lambda - 2 + 4\right]$$

$$= (1-\lambda)\left[\lambda^2 + \lambda + 2\right]$$

$$\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$$

$$\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{7}}{2}$$

$$\lambda_3 = 1$$

$$A - \lambda_1 I = \begin{pmatrix} \frac{3}{2} - i\frac{\sqrt{7}}{2} & -2 & 0 \\ 2 & \frac{1}{2} - i\frac{\sqrt{7}}{2} & -1 \\ 4 & -2 & -\frac{1}{2} - i\frac{\sqrt{7}}{2} \end{pmatrix}$$

$$\left(\frac{3}{2} - i\frac{\sqrt{7}}{2}\right)x - 2y = 0 \quad \Rightarrow \quad x = \frac{3}{2} + i\frac{\sqrt{7}}{2}$$

$$y = 2$$

$$z = 4$$

$$\underline{c}_1 = \begin{pmatrix} \frac{3}{2} + i\frac{\sqrt{7}}{2} \\ 2 \\ 4 \end{pmatrix}$$

Analogously we get $\underline{c}_2 = \begin{pmatrix} \frac{3}{2} - i\frac{\sqrt{7}}{2} \\ 2 \\ 4 \end{pmatrix} = \underline{c}_1^*$ $\qquad \underline{c}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

The solutions are thus

$$\underline{y}_1(x) = e^{-\frac{x}{2}} \left[\cos\frac{\sqrt{7}}{2}x \begin{pmatrix} 3/2 \\ 2 \\ 4 \end{pmatrix} - \sin\frac{\sqrt{7}}{2}x \begin{pmatrix} \sqrt{7}/2 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$\underline{y}_2(x) = e^{-\frac{x}{2}} \left[\sin\frac{\sqrt{7}}{2}x \begin{pmatrix} 3/2 \\ 2 \\ 4 \end{pmatrix} + \cos\frac{\sqrt{7}}{2}x \begin{pmatrix} \sqrt{7}/2 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$\underline{y}_3(x) = e^{x} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$\dashv$

We finally consider the case in which the matrix A has eigenvalues with multiplicity larger than 1. In this case one has to rely on the fact that the matrix can be written in JORDAN form.

$$A \sim \begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix}$$

Let us analyze the structure of the system corresponding to the r Jordan block.

$$J_r = \begin{pmatrix} \lambda_r & 1 & & \\ & \lambda_r & 1 & \\ & & \ddots & 1 \\ & & & \lambda_r \end{pmatrix}$$

$$\begin{cases} y_1' = \lambda_r y_1 + y_2 \\ \vdots \\ y_{m_r-1}' = \lambda_r y_{m_r-1} + y_{m_r} \\ y_{m_r}' = \lambda_r y_{m_r} \end{cases}$$

for example : $m_r = 2$

$$\begin{cases} y_1' = y_1\lambda + y_2 \\ y_2' = \lambda y_2 \end{cases}$$

$y_2 = e^{\lambda x} \Rightarrow y_1 = xe^{\lambda x} \Rightarrow \begin{pmatrix} xe^{\lambda x} \\ e^{\lambda x} \end{pmatrix}$ solution

$y_2 = 0 \Rightarrow y_1 = e^{\lambda x}$ $\qquad \begin{pmatrix} e^{\lambda x} \\ 0 \end{pmatrix}$ solution

In the general case the solution corresponding to the block is

$$\underline{y} = \begin{pmatrix} e^{\lambda x} & x e^{\lambda x} & \frac{1}{2}x^2 e^{\lambda x} & \cdots & \frac{1}{(m_2-1)!} x^{m_2-1} e^{\lambda x} \\ & e^{\lambda x} & x e^{\lambda x} & \cdots & \\ & & \ddots & & \\ & & & & e^{\lambda x} \end{pmatrix}$$

<u>EXAMPLE</u>

$$\begin{cases} x' = -2x + y + z \\ y' = x - 2y + z \\ z' = x + y - 2z \end{cases} \qquad A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

symmetric

$\Rightarrow$ diagonalizable

$$\begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = -(2+\lambda)\Big[\overbrace{(2+\lambda)^2}^{\lambda^2+4\lambda+3}-1\Big] - \big[-2-\lambda-1\big] + \big[1+2+\lambda\big]$$

$$= -\lambda^3 - 4\lambda^2 - 3\lambda - 2\lambda^2 - 8\lambda - 6 + 3 + \lambda + 3 + \lambda$$

$$= -\lambda(\lambda+3)^2$$

$\lambda_1 = \lambda_2 = -3$

$\lambda_3 = 0$

The eigenvalue $\lambda = -3$ has algebraic multiplicity $m_a(-3) = 2$

$\Rightarrow$ since the matrix is diagonalizable we expect it to also have $m_g(-3) = 2$

$$A + 3I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

this is indeed the case! the ker $(A+3I)$ has dimension 2

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ the general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\underline{v}_3} + \beta e^{-3t} \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{\underline{v}_1} + \gamma e^{-3t} \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\underline{v}_2}$$

EXAMPLE

$$\begin{cases} x' = x + z \\ y' = x \\ z' = x - y \end{cases} \qquad A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

A is not symmetric

$\Rightarrow$ let us see if it can be diagonalized

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 1 & -1 & -\lambda \end{vmatrix} = (1-\lambda)\lambda^2 + (-1+\lambda) = -(1+\lambda)(1-\lambda)^2$$

$\Rightarrow \quad \lambda_1 = -1 \qquad m_a(-1) = 1$

$\lambda_2 = 1 \qquad m_a(1) = 2$

Let us check the geometrical multiplicity

$$A - I = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \qquad (A - I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} z = 0 \\ x - y = 0 \\ x - y - z = 0 \end{cases}$$

$m_g(1) = 1 \qquad \underline{v_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow \quad m_g(1) \neq m_a(1)$

The matrix is not diagonalizable $\quad J_A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$A + I = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad (A + I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + z \\ x + y \\ x - y + z \end{pmatrix} = 0 \qquad \underline{v_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$\Rightarrow$ Two independent solutions are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t \qquad \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} e^{-t}$$

a third solution corresponding to $\lambda = 1$ can be found in the form

$$\begin{pmatrix} \alpha + \beta t \\ \gamma + \delta t \\ \rho + \vartheta t \end{pmatrix} e^t \quad \Rightarrow \quad \text{we find} \quad \begin{pmatrix} 1 + t \\ t \\ 1 \end{pmatrix} e^t$$

HOMOGENEOUS LINEAR M-ORDER DIFF. EQ.

WITH CONSTANT COEFFICIENTS

We want to apply what we have seen to the case of the equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots a_0 y = 0$$

$$\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-1}' = -a_0 y_0 \dots - a_{n-1} y_{n-1} \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & \dots & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & & \dots & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{pmatrix}$$

$$\det(A - \lambda I) = (-1)^n \left(\lambda^n + a_{n-1} \lambda^{n-1} + \dots a_0 \right)$$

this polynomial is what we get from the ansatz $y = e^{\lambda t}$

THEOREM

$\lambda_1 \dots \lambda_r$ real eigenvalues

$\mu_1 \dots \mu_s$ complex eigenvalues

$$\sum_{j=1}^{r} m_j + 2 \sum_{j=1}^{s} m_j = n$$

algebraic multiplicity

$\mu_j = \alpha_j + \beta_j i$

The solutions are

$$e^{\lambda_j x}, \ x e^{\lambda_j x} \ \dots \ x^{m_j - 1} e^{\lambda_j x} \qquad 1 \le j \le r$$

$$e^{\alpha_j x} \cos \beta_j x, \ x e^{\alpha_j x} \cos \beta_j x \ \dots \ x^{m_j - 1} e^{\alpha_j x} \cos \beta_j x \qquad 1 \le j \le s.$$

$$e^{\alpha_j x} \sin \beta_j x \ \dots \ x^{m_j - 1} e^{\alpha_j x} \sin \beta_j x \qquad 1 \le j \le s$$

The total number of solutions is n.

We discuss explicitly the case $n=2$

$n=2$

$$\begin{cases} y_0' = y_1 \\ y_1' = -a_0 y_0 - a_1 y_1 \end{cases} \qquad A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \qquad P(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -a_0 & -a_1 - \lambda \end{vmatrix} = \lambda(a_1 + \lambda) + a_0$$

$\lambda^2 + a_1\lambda + a_0 = 0 \quad \Rightarrow \quad \lambda = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$

- $a_1^2 - 4a_0 > 0 \quad \Rightarrow$ two distinct real solutions λ_1, λ_2

  $e^{\lambda_1 x}, e^{\lambda_2 x}$ independent solutions

- $a_1^2 - 4a_0 < 0 \quad \Rightarrow$ two complex solutions $\mu = -\dfrac{a_1}{2} \pm \dfrac{i}{2}\sqrt{4a_0 - a_1^2}$

  $= \alpha \pm i\beta$

  $e^{\alpha x}\cos\beta x \;, \; e^{\alpha x}\sin\beta x$

- $a_1^2 - 4a_0 = 0 \quad \Rightarrow$ one real solution with double multiplicity (assume $a_0 > 0$)

  $a_1 = 2\sqrt{a_0}$

  $A = \begin{pmatrix} 0 & 1 \\ -a_0 & -2\sqrt{a_0} \end{pmatrix} \qquad \lambda = -\dfrac{a_1}{2} = -\sqrt{a_0}$

  $A - \lambda I = A + \sqrt{a_0} = \begin{pmatrix} \sqrt{a_0} & 1 \\ -a_0 & -\sqrt{a_0} \end{pmatrix} \qquad m_g(\lambda) = 1$

  $\Rightarrow \quad A \sim \begin{pmatrix} -\sqrt{a_0} & 1 \\ & -\sqrt{a_0} \end{pmatrix}$

A fundamental system is $\begin{pmatrix} xe^{\lambda x} & e^{\lambda x} \\ e^{\lambda x} & 0 \end{pmatrix}$

$\Rightarrow \quad xe^{\lambda x}, \; e^{\lambda x}$ two independent solutions

EXAMPLE

$y^{(5)} + 4y^{(4)} + 2y^{(3)} - 4y'' + 8y' + 16y = 0$

$P(\lambda) = \lambda^5 + 4\lambda^4 + 2\lambda^3 - 4\lambda^2 + 8\lambda + 16 = (\lambda+2)^3(\lambda - 1 + i)(\lambda - 1 - i)$

$\Rightarrow$ the solutions are $e^{-2x}, \; xe^{-2x}, \; x^2e^{-2x}, \; e^x\cos x, \; e^x\sin x$

Example

$y'' - 2y' + y = 0$

$P(\lambda) = (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$

$y_1(x) = e^x \qquad y_2(x) = e^x x$

check: $y_2' = e^x + xe^x$

$y_2'' = e^x + e^x + xe^x$

$y_2'' - 2y_2' + y_2 = \cancel{2e^x} + \cancel{xe^x} - 2(\cancel{e^x} + \cancel{xe^x}) + \cancel{xe^x} = 0$

$y = \alpha y_1 + \beta y_2$

general solution

Euler Equation

$x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots a_1 x y' + a_0 y = 0$

two possible approaches:

- $x = e^t \Rightarrow$ we get a linear equation for $y(t)$ with constant coefficients
- ansatz $y = c x^m \Rightarrow$ look for algebraic solution for m

Example

$2x^2 y'' + 3xy' - 15y = 0$

$y = cx^m \Rightarrow 2m(m-1) + 3m - 15 = 0$

$m = \frac{-1 \pm \sqrt{1 + 4 \cdot 30}}{4} = \frac{5}{2},\ -3$

$\Rightarrow y_1 = x^{-3} \qquad y_2 = x^{5/2}$

$y = \alpha x^{-3} + \beta x^{5/2}$ general solution

INHOMOGENEOUS LINEAR SYSTEMS

We now consider the linear system of the form

$$\underline{y}' = A(x)\,\underline{y} + \underline{b}(x)$$

Assuming we know a fundamental system $Y(x)$ of solutions of the homogeneous system we can find the solution of the inhomogeneous problem by <u>variation of constants</u> (analogously to what done for linear equations)

$Y(x)$ fundamental system $\Rightarrow$ $\underline{y}(x) = Y(x)\,\underline{v}$ arbitrary solution of the homogeneous system.

$\Rightarrow$ $\underline{y}(x) = Y(x)\,\underline{v}(x)$ ansatz

$$\underline{y}'(x) = Y'\underline{v} + Y\underline{v}' = \cancel{AY\underline{v}} + Y\underline{v}' = \cancel{AY\underline{v}} + \underline{b}$$

$\Rightarrow$ $Y(x)\,\underline{v}'(x) = \underline{b}(x)$ $\quad \underline{v}'(x) = Y^{-1}(x)\,\underline{b}(x)$ $\quad$ (since Y is a fundamental system the matrix has an inverse

$\Rightarrow$ $\underline{v}(x) = \underline{v}_0 + \int_{x_0}^{x} Y^{-1}(s)\,\underline{b}(s)\,ds$ $\quad \underline{v}(x_0) = \underline{v}_0$ initial condition

The solution of the system can thus be written as

$$\underline{y}(x) = Y(x)\,\underline{v}_0 + \int_{x_0}^{x} Y(x)\,Y^{-1}(s)\,\underline{b}(s)\,ds$$

EXAMPLE

Solve the Cauchy problem

$$\begin{cases} y_1' = y_1 - y_2 + e^x & y_1(0) = 0 \\ y_2' = y_1 + y_2 + 1 & y_2(0) = 1 \end{cases} \qquad \underline{b}(x) = \begin{pmatrix} e^x \\ 1 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\qquad (1-\lambda)^2 + 1 = 0 \qquad \lambda^2 - 2\lambda + 2 = 0$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$\Rightarrow$ a fundamental system is $\quad Y(x) = e^x \begin{pmatrix} \cos x & \sin x \\ \sin x & -\cos x \end{pmatrix} \qquad Y(0) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$Y^{-1}(x) = e^{-x} \begin{pmatrix} \cos x & \sin x \\ \sin x & -\cos x \end{pmatrix} \qquad \underline{v_0} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

we have to compute

$$\int_0^x e^{-s} \begin{pmatrix} \cos s & \sin s \\ \sin s & -\cos s \end{pmatrix} \begin{pmatrix} e^s \\ 1 \end{pmatrix} ds = \int_0^x \begin{pmatrix} \cos s + e^{-s} \sin s \\ \sin s - e^{-s} \cos s \end{pmatrix} ds$$

$$= \begin{pmatrix} -\frac{1}{2} e^{-x} (\sin x + \cos x) + \sin x + \frac{1}{2} \\ -\frac{1}{2} e^{-x} (\sin x - \cos x) - \cos x + \frac{1}{2} \end{pmatrix}$$

The solution of the problem is thus

$$\underline{y}(x) = \frac{1}{2} \begin{pmatrix} e^x \sin x + e^x \cos x - 1 \\ 2e^x + e^x \sin x - e^x \cos x - 1 \end{pmatrix} + \begin{pmatrix} -e^x \sin x \\ e^x \cos x \end{pmatrix}$$

$$Y(x) \int_{x_0}^x Y^{-1}(s)\, \underline{b}(s)\, ds \qquad\qquad Y(x)\, \underline{v_0}$$

$$= \begin{pmatrix} \frac{1}{2} e^x \cos x - \frac{1}{2} e^x \sin x - \frac{1}{2} \\ e^x + \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x - \frac{1}{2} \end{pmatrix}$$