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Applications of General Relativity

Solutions 7

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Exercise 1 [Gravitational waves from a Binary System]

a) Let us solve the problem of circular orbits in Newtonian gravity. We have that the two bodies m_1 and m_2 are following a circular orbit around a common center, with radius respectively r_1 and r_2, in the x-y plane. Their positions are:

$$x_1^i = r_1(\cos\omega t, \sin\omega t, 0), \quad x_2^i = -r_2(\cos\omega t, \sin\omega t, 0)$$

Newton's second law gives:

$$\frac{Gm_1m_2}{(r_1+r_2)^2} = m_1r_1\omega^2 = m_2r_2\omega^2$$

We find, with $R = r_1 + r_2$ and $M = m_1 + m_2$:

$$\frac{r_2}{m_1} = \frac{r_1}{m_2} = \frac{R}{M}$$
$$\omega^2 = \frac{GM}{R^3}$$

The density is:

$$\rho\left(t, x^i\right) = m_1\delta\left(x^i - x_1^i(t)\right) + m_2\delta\left(x^i - x_2^i(t)\right)$$

So that the quadrupole moment is:

$$I_{ij}(t) = \int \rho\,(t,y)\, y^i y^j d^3y = m_1x_1^i(t)x_1^j(t) + m_2x_2^i(t)x_2^j(t)$$
$$I_{xx} = m_1{r_1}^2\cos^2\omega t + m_2{r_2}^2\cos^2\omega t = \mu R^2\cos^2\omega t$$
$$I_{xy} = m_1{r_1}^2\cos\omega t\sin\omega t + m_2{r_2}^2\cos\omega t\sin\omega t = \mu R^2\cos\omega t\sin\omega t$$
$$I_{yy} = m_1{r_1}^2\sin^2\omega t + m_2{r_2}^2\sin^2\omega t = \mu R^2\sin^2\omega t$$

where $\mu = \frac{m_1m_2}{M}$.

The gravitational-wave tensor is thus, far away from the source and on the z-axis:

$$h_{ij}^{TT}(t,x^i) = \frac{2G}{|x|}\frac{d^2}{dt^2}I_{ij}(t-|x|)$$
$$h_{xx}^{TT}(t,x^i) = \frac{4G\mu R^2\omega^2}{z}\left(\sin^2\omega t_r - \cos^2\omega t_r\right) = -\frac{4G^2\mu M}{Rz}\cos 2\omega t_r = -h_{yy}^{TT}(t,x^i)$$
$$h_{xy}^{TT}(t,x^i) = -\frac{8G\mu R^2\omega^2}{z}\sin\omega t_r\cos\omega t_r = -\frac{4G^2\mu M}{Rz}\sin 2\omega t_r$$

where $t_r = t - z$ is the retarded time.

b) The energy loss is given by (see textbook or lecture notes)

$$P = -\frac{G}{5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}\right\rangle, \tag{1}$$

where

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl} I_{kl} \qquad \text{and} \qquad J^{ij} = \eta^{i\alpha}\eta^{j\beta} J_{\alpha\beta}. \tag{2}$$

The nonzero components of the third derivative of the reduced quadrupole moment tensor are

$$\frac{d^3 J^{xx}}{dt^3} = \zeta \sin(2\omega t) = -^{yy}, \frac{d^3 J^{xy}}{dt^3} = -\zeta\cos(2\omega t) =^{yx} \tag{3}$$

Substituting into (1), and averaging the trig functions over one oscillation gives

$$P = -\frac{32}{5} G\mu^2 R^4 \omega^6 \qquad \text{or} \qquad P = -\frac{32}{5} G^4 \frac{\mu^2 M^3}{R^5} \tag{4}$$

c) The Newtonian energy of the binary is the following:

$$E_N = E_{kin} + E_{pot} = \frac{1}{2}\left(m_1 {r_1}^2\omega^2 + m_2 {r_2}^2\omega^2\right) - \frac{Gm_1 m_2}{R} = -\frac{1}{2}\frac{G\mu M}{R}$$

We thus have:

$$\frac{dE_N}{dt} = \frac{1}{2}\frac{G\mu M}{R^2}\frac{dR}{dt} = -\frac{dE_{GW}}{dt}$$

From this, we can infer:

$$\frac{dR}{dt} = -\frac{64 G^3 \mu M^2}{5R^3}$$

d) We have ω in terms of R:

$$\omega = \sqrt{\frac{GM}{R^3}} \quad \Longrightarrow \quad \frac{d\omega}{dt} = -\frac{3}{2}\sqrt{\frac{GM}{R^5}}\frac{dR}{dt} = \frac{96}{5}\sqrt{\frac{G^7\mu^2 M^5}{R^{11}}} = \frac{96}{5} G^{5/3}\mu M^{2/3}\omega^{11/3}$$

We thus find that the derivative of the frequency depends on $\mathcal{M}^{5/3}$, where $\mathcal{M}$ is the chirp mass defined as: $\mathcal{M} = \mu^{3/5} M^{2/5}$.

e) We can integrate $\omega(t)$:

$$\omega^{-11/3}\frac{d\omega}{dt} = \frac{96}{5}(G\mathcal{M})^{5/3} \quad \Longrightarrow \quad \omega(t) = \left(\frac{256}{5}\right)^{-3/8} (G\mathcal{M})^{-5/8}(t_0 - t)^{-3/8}$$

where t_0 is a constant.

This diverges as $t \to t_0$, so the coalescence time is $T_{coal} = t_0$, with

$$t_0 = \frac{5}{2^8}\frac{1}{G^{5/3}\mathcal{M}^{5/3}\omega^{8/3}}. \tag{5}$$

f) In the case of a Sun-Earth-like system $P \sim 200$ W where we assumed $R \approx 1AU \sim 1.5 \times 10^{11}$ m, $m_1 \approx 2 \times 10^{30}$ kg ≈ 1.5 km, $m_2 = 6 \times 10^{24}$ kg ≈ 0.5 cm for $G = c = 1$. For two neutron stars at the same separation: $P \approx 2.4 \times 10^{14}$ W. The shrinking of the orbit is $dR/dt \sim 10^{-20}$ m/sec $\approx 3 \times 10^{-13}$ m/year and $dR/dt \sim 2 \times 10^{-14}$ m/sec $\approx 6 \times 10^{-7}$ m/yr, respectively.