MMP I
Solution Sheet 5

HS 21
Prof. Ph. Jetzer

L. Buonocore, M. Loechner, X. Liu, M. Ebersold
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Exercise 1 [Heat Conduction in a Ball (6 points)]

$$K = \{\vec{x} = (x, y, z) \in \mathbb{R}^3 | \vec{x}^2 \leq 1\}$$
$$\partial K = \{\vec{x} \in \mathbb{R}^2 \ | \vec{x}^2 = 1\}$$

Temperature distribution: $u(t, \vec{x})$ with $t > 0, x \in K$

$u(t, \vec{x})$ satisfies the **heat conduction differential equation**

$$\frac{\partial u}{\partial t} = \Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u \tag{1}$$

with the boundary contidions

1. $u(t, \vec{x}) = 0$ for $\vec{x} \in \partial K$
2. at $t = 0$: $u(0, \vec{x}) \equiv u(0, r) = T(r) \in C^2(\mathbb{R}_+)$, $r = |\vec{x}|$.

   We have only positive temperatures and the temperature distribution $T(1) = 0$

Since the initial temperature distribution $T(r)$ is spherically symmetric and also the driving forces arising from 1 do not favour any particular direction, the problem can be solved with the spherically symmetric ansatz

$u(t, \vec{x}) = u(t, r)$ with $r = \sqrt{(x^2 + y^2 + z^2)}$.

a) Goal: write Δ in terms of spherical coordinates

   coordinate dependence: $u(t, r) = u(t, \sqrt{x^2 + y^2 + z^2})$

"Jacobian": $\left[\frac{\partial r}{\partial x^i} = \frac{\partial}{\partial x^i}\sqrt{x^2+y^2+z^2} = \frac{(2x^i)}{2\sqrt{(x^2+y^2+z^2)}} = \frac{x^i}{r}\right]$

"Hessian": $\left[\frac{\partial^2 r}{\partial (x^i)^2} = \frac{\partial}{\partial x^i}\left(\frac{x^i}{r}\right) = \frac{r - x^i\frac{\partial r}{\partial x^i}}{r^3} = \frac{r^2-(x^i)^2}{r^3} = \frac{(x^j)^2+(x^k)^2}{r^3}, j,k \neq i\right]$

$$\Rightarrow \frac{\partial u}{\partial x^i} = \frac{\partial r}{\partial x^i}\frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{\partial^2 u}{\partial (x^i)^2} = \frac{\partial^2 r}{\partial (x^i)^2}\frac{\partial u}{\partial r} + \frac{\partial r}{\partial x^i}\frac{\partial}{\partial x^i}\frac{\partial u}{\partial r}$$

$$\frac{\partial}{\partial x^i}\frac{\partial u}{\partial r} = \frac{\partial r}{\partial x^i}\frac{\partial}{\partial r}\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 u}{\partial (x^i)^2} = \frac{\partial^2 r}{\partial (x^i)^2}\frac{\partial u}{\partial r} + \left(\frac{\partial r}{\partial x^i}\right)^2\frac{\partial^2 u}{\partial r^2}$$

$$\Rightarrow \Delta u = \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] =$$

$$= \left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]\frac{\partial u}{\partial r} + \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right]\frac{\partial^2 u}{\partial r^2} =$$

$$= \left[\frac{(y^2+z^2)+(x^2+z^2)+(x^2+y^2)}{r^3}\right]\frac{\partial u}{\partial r} + \left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right]\frac{\partial^2 u}{\partial r^2} =$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}.$$

b) Ansatz: $u(t,r) = f(t)\cdot g(r)$, $g(r) = \frac{h(r)}{r}$

$$\Rightarrow g(r)\frac{\partial f}{\partial t} = f(t)\Delta g$$

$$\Rightarrow \frac{\dot{f}(t)}{f(t)} = \frac{\Delta g(r)}{g(r)} = const =: -C \in \mathbb{R}$$

The equation is valid because the two terms are equal $\forall r,t$. We choose the constant $-C$ to simplify the notation.

$$\Delta g = \Delta\frac{h}{r} = \frac{\partial^2}{\partial r^2}\frac{h}{r} + \frac{2}{r}\frac{\partial}{\partial r}\frac{h}{r} =$$

$$= \frac{\partial}{\partial r}\left[\frac{h'r-h}{r^2}\right] + \frac{2}{r}\left[\frac{h'r-h}{r^2}\right] =$$

$$= \frac{(h''r+h'-h')r^2-(h'r-h)2r}{r^4} + \frac{2h'r-2h}{r^3} =$$

$$= \frac{h''r^2-2h'r+2h+2h'r-2h}{r^3} = \frac{h''}{r}$$

$$\Rightarrow \frac{h''/r}{h/r} = \frac{h''}{h} = -C$$

$$\Rightarrow h'' + hC = 0$$

$\Rightarrow$ **Ansatz**: $h(r) = A\cos(kr) + B\sin(kr)$

$\Rightarrow h''(r) = -k^2 h(r) \Rightarrow k = \pm\sqrt{C}$

Behaviour at $r = 0$**:** $\lim_{r\to 0} g(r) = \lim_{r\to 0} \frac{A\cos(kr)+B\sin(kr)}{r} = \lim_{r\to 0} A\frac{1}{r} + B\frac{kr}{r}$

$\Rightarrow$ The cosine term is not a physical solution!

$\Rightarrow g(r) = B\frac{\sin(kr)}{r} = B\frac{\sin(\sqrt{C}r)}{r}$

$\frac{\dot{f}}{f} = C \Rightarrow f(t) = Ke^{-Ct}$

$f(0) = K \Rightarrow f(t) = f(0)e^{-Ct}$

c)

$$\begin{aligned} u(t,1) = 0 &\Leftrightarrow f(t)g(1) = f(t)B\frac{\sin(k)}{1} = 0 \\ &\Leftrightarrow \sin k = 0 \Leftrightarrow k = n\pi, n \in \mathbb{Z} (\Leftrightarrow C = n^2\pi^2) \\ &\Rightarrow u(t,r) = f(t)\cdot B\cdot\frac{\sin(n\pi r)}{r} \\ &\to f(t) = f(0)e^{-n^2\pi^2 t} \end{aligned}$$

d) **Superposition** (of already found solutions)

Since we are dealing with a linear differential equation, the superposition principle can be applied:

$$u(t,r) = \sum_n b_n f_n(t) g_n(t) = \frac{f(0)}{r}\sum_n b_n e^{-n^2\pi^2 t}\sin(n\pi r)$$

e) **Fourier coefficients:** $u(0,r) = T(r) = \frac{f(0)}{r}\sum_n b_n \sin(n\pi r)$

$$\begin{aligned} \Rightarrow \frac{r\cdot T(r)}{f(0)} &= \sum b_n \sin(n\pi r) \\ \Rightarrow_{x=\pi r} \frac{xT(\frac{x}{\pi})}{\pi f(0)} &= \sum b_n \sin(nx) \qquad \text{Fourier Series!} \\ \Rightarrow b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{xT(\frac{x}{\pi})}{\pi f(0)}\sin(nx)dx \\ &= \frac{1}{\pi}\int_{-1}^{1}\frac{rT(r)}{f(0)}\sin(n\pi r)\pi dr \\ &= \int_{-1}^{1}\frac{rT(r)}{f(0)}\sin(n\pi r)dr \end{aligned}$$

If $T(r)$ is odd, this integral vanishes! $\to$ not good $\to T(r) = T(-r)$ even $\to T(r) = T(|r|)$

Exercise 2 [Growth of Bacteria (4 points)]

a)

$$\underline{\frac{dN}{dt} = \alpha\sqrt{N}} \quad \Rightarrow \quad \frac{dN}{\sqrt{N}} = \alpha dt$$
$$\Rightarrow \quad \int_{N_0}^{N} \frac{d\tilde{N}}{\sqrt{\tilde{N}}} = \int_0^t \alpha d\tilde{t}$$
$$\Rightarrow \quad 2\sqrt{N} - 2\sqrt{N_0} = \alpha t \Rightarrow \sqrt{N} = \frac{\alpha t}{2} + \sqrt{N_0}$$
$$\Rightarrow \quad \underline{N(t) = \left(\frac{\alpha t}{2} + \sqrt{N_0}\right)^2}$$

b)

$$\underline{\frac{dN}{dt} = \alpha N} \quad \Rightarrow \quad \frac{dN}{N} = \alpha dt$$
$$\Rightarrow \quad \int_{N_0}^{N} \frac{d\tilde{N}}{\tilde{N}} = \int_0^t \alpha d\tilde{t}$$
$$\Rightarrow \quad \ln(N) - \ln(N_0) = \ln(\frac{N}{N_0}) = \alpha t$$
$$\Rightarrow \quad \frac{N}{N_0} = e^{\alpha t} \quad \Rightarrow \quad \underline{N(t) = N_0 e^{\alpha t}}$$

c) Removal rate: $\dot{k}$ with $k = \text{const}$ (between t_1 and t_2, $\int_{t_1}^{t_2} \dot{k}\, dt$ bacteria are removed)

$$\underline{\frac{dN}{dt} = \alpha N - \dot{k}} \quad \Rightarrow \quad \frac{dN}{\alpha N - \dot{k}} = dt \quad \Rightarrow \quad \int_{N_0}^{N} \frac{d\tilde{N}}{\alpha\tilde{N} - \dot{k}} = \int_0^{\tilde{t}} dt$$
$$\Rightarrow \quad \frac{1}{\alpha} \ln\left(\frac{\alpha N - \dot{k}}{\alpha N_0 - \dot{k}}\right) = t$$
$$\Rightarrow \quad \alpha N - \dot{k} = (\alpha N_0 - \dot{k}) e^{\alpha t}$$
$$\Rightarrow \quad \underline{N(t) = \frac{1}{\alpha}\left((\alpha N_0 - \dot{k}) e^{\alpha t} + \dot{k}\right)}$$

d)

$$\underline{\frac{dN}{dt} = \alpha\left(1 - \frac{N}{\beta}\right) N} \quad \Rightarrow \quad \frac{dN}{N(1 - \frac{N}{\beta})} = \frac{\beta dN}{N(\beta - N)} = \frac{dN}{N} + \frac{dN}{\beta - N} = \alpha dt$$
$$\Rightarrow \quad \int \frac{dN}{N} + \int \frac{dN}{\beta - N} = \alpha \int dt$$
$$\Rightarrow \quad \ln|N| - \ln|\beta - N| = -\alpha t + c$$
$$\Rightarrow \quad \ln\left|\frac{\beta - N}{N}\right| = -\alpha t - c$$

$$\Rightarrow \quad \left|\frac{\beta - N}{N}\right| = e^{-\alpha t - c}$$

$$\Rightarrow \quad \frac{\beta - N}{N} = Ce^{-\alpha t} \qquad (C = \pm e^{-c})$$

Solving for N and determining C using $N(0) = N_0$ we find

$$\underline{N(t) = \frac{\beta N_0}{N_0 + (\beta - N_0)e^{-\alpha t}}}$$

Solutions for $N_0 = 0$:

a) $N(t) = \frac{\alpha^2}{2}t^2$

b) $N(t) = 0 \;\; \forall t$

c) $N(t) = \frac{1}{\alpha}\left(-\dot{k}e^{\alpha t} + \dot{k}\right) = \frac{\dot{k}}{\alpha}\left(1 - e^{\alpha t}\right)$

d) $N(t) = 0 \quad \forall t$

We see that **b)** and **d)** provide reasonable solutions for $N_0 = 0$. The solutions for **a)** and **c)** are mathematically correct, but have a problem because they demand that bacteria come out of nothing since $N(t > 0) > 0$ but $N_0 = 0$.