MMP I
Tutorial 7

HS 2017
Prof. M. Grazzini

S. Devoto, J. Yook, J. Mazzitelli
`http://www.physik.uzh.ch/en/teaching/PHY312/HS2017.html`

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Exercise 1: Calculus of variation (6 Pts.)
An airplane is to travel from point $A = (0, 0)$ to point $B = (d, 0)$. In this problem we assume that the surface of the Earth is actually a plane and the trajectory $y(x)$ will be a curve in the $x - y$ plane. An airplane costs more money to fly at a lower altitude than at a higher one. We wish to minimize the cost of a trajectory between the points A and B. The cost of travelling a distance $\mathrm{d}s$ at an altitude y is given by $e^{-y/h}\mathrm{d}s$, with h a given constant.

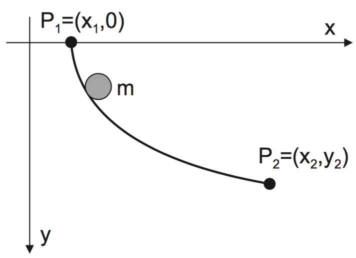
a) Give an expression for the cost of the voyage between the points A and B and express the problem of minimizing this cost as a variational problem. Show that the associated Euler-Lagrange equation can be written as:
$hy''(x) + (y'(x))^2 + 1 = 0$.

b) Find the general solution to this Euler-Lagrange equation, as well as the solution satisfying the boundary conditions for the trip from A to B.

Exercise 2: Brachistochrone problem (5 Pts.)
A bead with mass m and zero initial velocity moves on a frictionless wire placed in a (homogeneous) gravitational field from $P_1 = (x_1, 0)$ to $P_2 = (x_2, y_2)$. Find the curve $y(x)$ which minimizes the time T, i.e. for which the integral

$$T[y] = \int_{x_1}^{x_2} \mathrm{d}x \; F[y(x), y'(x), x] \tag{2.1}$$

is extremal. What is the shape of the wire?



Hint: Express T as a path-integral over the inverse velocity which can be found by looking at the total energy of the wire (which is conserved). Use the substitution $y = \kappa \sin^2(\phi/2)$ to solve the integral (κ is not a new constant but should be chosen conveniently as a function of the constants already introduced).

– please turn over –

Exercise 3: Minimizing a functional (3 Pts.)
Find the function $y(x)$ subject to the condition $y(0) = 0$ and $y(1) = 1$ that minimises the functional:

$$F[y] = \int_0^1 \mathrm{d}x((y')^2 - 4y^2). \tag{3.1}$$