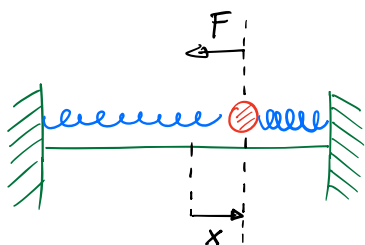


1.7 One-dimensional systems & the phase diagram

Before moving to the Lagrangian formalism, let's consider the particularly simple case of a conservative Newtonian system with a single effective degree of freedom.

Example: Elastic force over an horizontal plane [harmonic oscillator]
(gravity irrelevant, no friction, perfect elasticity)



$$F = -kx \quad p = m\dot{x}$$

$$U = \frac{1}{2} Kx^2 + U_0 \quad T = \frac{1}{2} m \dot{x}^2$$

Note that we can redefine variables (or units) such that the e.o.m. assume the form

$$\dot{p} = -q \quad \& \quad E = U + T = \frac{1}{2} p^2 + \frac{1}{2} q^2$$

More generally, for one-dimensional conservative systems we have

$$\dot{p} = f(q) = -\frac{\partial U(q)}{\partial q} \quad (1.46)$$

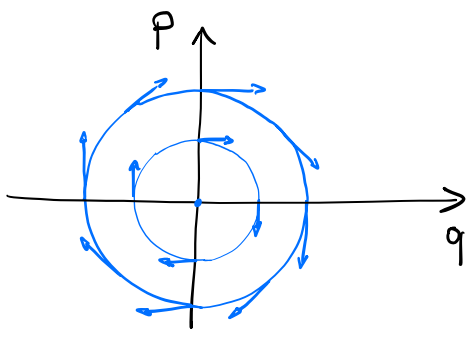
This equation is equivalent to the following set of eq.s:

$$\begin{cases} \dot{q} = p \\ \dot{p} = f(q) \end{cases} \quad \longleftrightarrow \quad \text{phase flow} \quad (1.47)$$

The $\{p, q\}$ plane is called the phase plane.

Eq. (1.47) defines a vector field in such plane, and the solutions to (1.46) corresponds to motions in such plane.

Going back to the example of the harmonic oscillator,
phase plane & flow assume the corresponding form:



Each curve in this plane corresponds to a well-defined energy level

The **phase velocity** vector in each point (q, p) of the plane has components $(p, -q)$, i.e. it has the same magnitude of the radius and it is always perpendicular.

The motion in this plane is therefore a uniform circular motion $q = r_0 \cos(\varphi_0 - t)$ $p = r_0 \sin(\varphi_0 - t)$ $r_0 = \sqrt{2E}$.

The origin is a special orbit corresponding to the **equilibrium position**

Let's consider now a less simple example, where the potential energy has the form shown in Fig 1

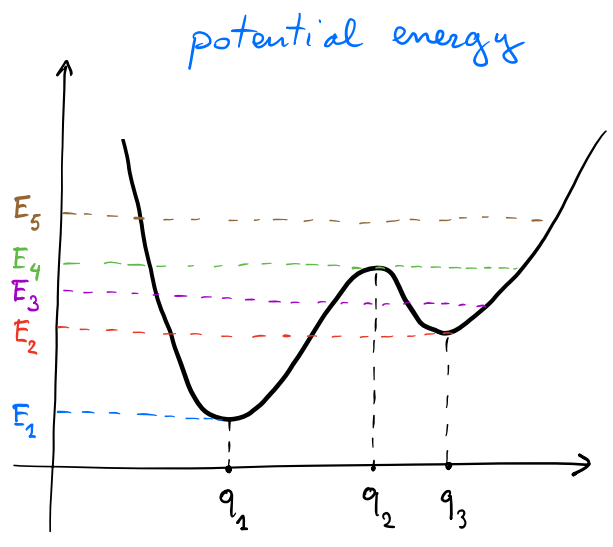


Fig 1

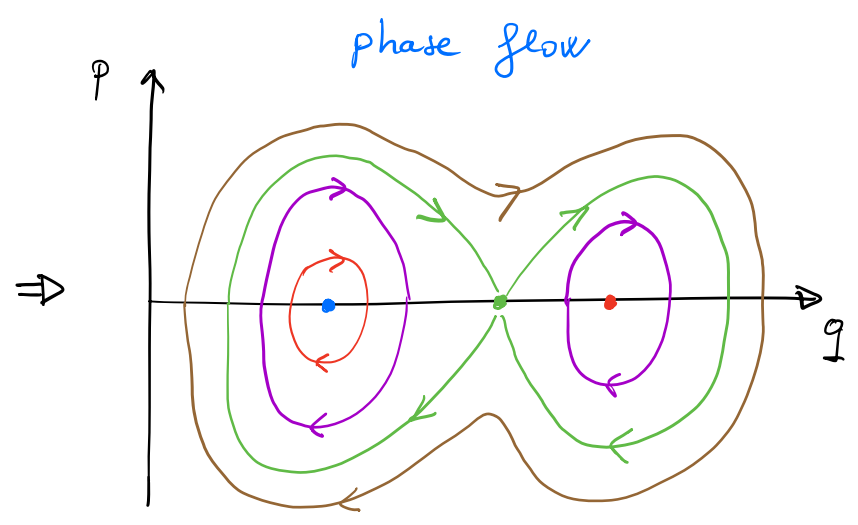


Fig 2

The structure of the phase flow, shown in Fig. 2, can be understood qualitatively without solving completely the e.o.m.

Key features:

- 1) The phase curve corresponding to the energy level E_1 is a point ($\leftrightarrow$ equilibrium in q_1)
- 2) The phase curves corresponding to E_2 & E_3 have two separate parts
- 3) The phase curve corresponding to E_4 has three separate parts.

The E_4 curve has 3 parts since the duration of the motion along any of the non point-like branches is infinite.

In general, the time difference to go from a point q_i to a point q_f along a trajectory is:

$$t_f - t_i = \int_{q_i}^{q_f} \frac{dq}{\sqrt{2(E - U(q))}} \quad (1.48)$$

Proof:

$$E = \frac{1}{2} \dot{q}^2 + U(q) \rightarrow \dot{q} = \frac{dq}{dt} = \sqrt{2(E - U(q))} \rightarrow (1.48)$$

Using this general result and expanding $U(q)$ around the relative maximum in q_2 :

$$U(q) = E_4 + \left. \frac{d^2 U}{dq^2} \right|_{q=q_2} \cdot (q - q_2)^2 + O[(q - q_2)^3]$$

it follows that the time to reach q_2 from any of the points along the trajectory with $E = E_4$ is infinite.