Applications of General Relativity
Solutions 5

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Exercise 1 [Gravitational field of a moving particle]

We are considering a particle of mass M moving with constant velocity $\boldsymbol{V}$. In the particles restframe Σ' we have

$$h'_{\mu\mu}(\boldsymbol{r}) = -\frac{2G}{c^2}\int \mathrm{d}^3r' \frac{\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{2GM}{c^2}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_M|} \tag{1}$$

and $h'_{0i} = 0$. Here $\boldsymbol{r}_M$ is the position of the mass M. If we now transfer to a general frame Σ, the metric changes according to $g_{\mu\nu} = \alpha^\kappa_\mu \alpha^\rho_\nu g'_{\kappa\rho}$. Since we are working at linear order in h and the mass M is moving on a straight line, the transformation is nothing but a Lorentz transformation $\alpha^\mu_\nu = \Lambda^\mu_\nu + \mathcal{O}(h)$. For the transformation matrix we have up

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & V^i/c \\ V^i/c & \mathbb{1} \end{pmatrix} \tag{2}$$

where we neglected terms $\mathcal{O}(V^2/c^2)$, i.e. set $\gamma = 1$. Note that we only need to transform the perturbation since $\Lambda^\kappa_\mu \Lambda^\rho_\nu \eta_{\kappa\rho} = \eta_{\mu\nu}$ and are thus left with

$$h_{\mu\nu} = \Lambda^\kappa_\mu \Lambda^\rho_\nu h'_{\kappa\rho}, \tag{3}$$

Evaluating the above expression we obtain for the metric perturbation in Σ

$$h_{\mu\mu} = h'_{\mu\mu} + \mathcal{O}\left(V^2/c^2\right) \qquad\qquad h_{0i} = 2\frac{V_i}{c}h'_{00} + \mathcal{O}\left(V^2/c^2\right) \tag{4}$$

The mass M is moving on a straight line in Σ, thus its position as a function of time can be written as

$$\boldsymbol{r}_M(t) = \boldsymbol{r}_{M,0} + \boldsymbol{V}t. \tag{5}$$

The gravitomagnetic potential is

$$\boldsymbol{h}(\boldsymbol{r}) = h_{0i}(\boldsymbol{r}) = -\frac{4GM}{c^3}\frac{\boldsymbol{V}}{|\boldsymbol{r}-\boldsymbol{r}_M(t)|}, \tag{6}$$

for the gravitomagnetic field we thus have

$$\boldsymbol{\Omega}(\boldsymbol{r}) = -\frac{2}{c}\boldsymbol{\nabla}_{\boldsymbol{r}} \times \boldsymbol{h}(\boldsymbol{r}) = \frac{2GM}{c^2}\frac{\boldsymbol{V}\times(\boldsymbol{r}-\boldsymbol{r}_M(t))}{|\boldsymbol{r}-\boldsymbol{r}_M(t)|^3}. \tag{7}$$

The equation of motion is calculated in analogy to the deriavtion presented in the lecture. The velocity of the test mass m will be symbolised by a lowercase $\boldsymbol{v}$. Evaluating the geodesic equation we obtain

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\gamma^i_{\alpha\beta}u^\alpha u^\beta = -\Gamma^i_{00}u^0u^0 - 2\Gamma^i_{0j}u^0u^j \tag{8}$$

$$= -c^2\Gamma^i_{00} - 2\Gamma^i_{0j}cv^j_m \tag{9}$$

where we have neglected terms $\mathcal{O}(v^2/c^2)$. For the Christoffel symbols we have

$$\Gamma^i_{00} = \frac{1}{2}\eta^{ij}\left(\partial_0 h_{j0} + \partial_0 h_{0j} - \partial_j h_{00}\right) = \partial_0 h^i_0 - \frac{1}{2}\partial^i h_{00} \tag{10}$$

$$\Gamma^i_{0j} = \frac{1}{2}\eta^{ik}\left(\partial_0 h_{jk} + \partial_j h_{0k} - \partial_k h_{0j}\right) = \frac{1}{2}\partial_0 h^i_j + \frac{1}{2}\eta^{ik}\left(\partial_j h_{0k} - \partial_k h_{0j}\right) \tag{11}$$

In contrast to the lecture, we now have a time dependent metric perturbation. The time dependence arises via $\boldsymbol{r}_M(t)$, such that the time derivative can be rewritten as

$$\frac{\partial h_{\mu\nu}}{\partial t} = -\boldsymbol{V}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} h_{\mu\nu}. \tag{12}$$

Hence we have for the equation of motion of the test mass m

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -c^2\partial_0 h_{i0} + \frac{c^2}{2}\partial_i h_{00} - \frac{c}{2}v^j\partial_0 h_{ij} - \frac{c}{2}v^j\left(\partial_j h_{0i} - \partial_i h_{0j}\right) \tag{13}$$

$$= -c^2\partial_0 h_{i0} + \frac{c^2}{2}\partial_i h_{00} - \frac{c}{2}v^j\partial_0 h_{ij} + \epsilon_{ikj}\Omega^k v^j \tag{14}$$

Neglecting $\partial_0 h_{i0} = \mathcal{O}(V^2/c^2)$ we finally have

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -GM\frac{(\boldsymbol{r}-\boldsymbol{r}_M)\cdot\boldsymbol{V}}{|\boldsymbol{r}-\boldsymbol{r}_M|^3}\boldsymbol{v} - GM\frac{\boldsymbol{r}-\boldsymbol{r}_M}{|\boldsymbol{r}-\boldsymbol{r}_M|^3} + \boldsymbol{\Omega}\times\boldsymbol{v} \tag{15}$$

We see that the gravitomagnetic forces due to moving masses are of order $\mathcal{O}(vV/c^2)$.

Exercise 2 [Particles in the field of a gravitational wave]
We start from the parametric form of an ellipse centered on the origin

$$r(\varphi) = \frac{b}{\sqrt{1-\epsilon^2\cos^2(\varphi)}}. \tag{16}$$

Thus we have for $\epsilon^2 \ll 1$

$$r^2(\varphi) = b^2\left(1 + \frac{\epsilon^2}{2}\big(1+\cos(2\varphi)\big)\right). \tag{17}$$

Our goal is to bring this in the form of the distortion of a circle due to an incoming gravitational wave

$$r^2(\varphi) = R^2\big(1 - 2h\cos(\omega t)\cos(2\varphi)\big) \tag{18}$$

which can be achieved by setting

$$R^2 = b^2\left(1+\frac{\epsilon^2}{2}\right) = \text{const.}, \qquad \epsilon^2 = -4h\cos(\omega t). \tag{19}$$

Using $b^2 = R^2(1-\epsilon^2/2)$ and $a^2 - b^2 = a^2\epsilon^2$ we obtain

$$a = R\big(1 - h\cos(\omega t)\big) \qquad b = R\big(1 + h\cos(\omega t)\big) \tag{20}$$

Equations 19 and 20 describe the time dependence of the ellipticity and the semi-minor and -major axis. It might seem troublesome that for $-\pi/2 \leq \omega t \leq \pi/2$ we have $b > a$ and $\epsilon^2 < 0$. But this can be intepreted as a phase shift by $\pi/2$ which means that the ellipse is rotated by $90°$ with respect to its standard orientation.

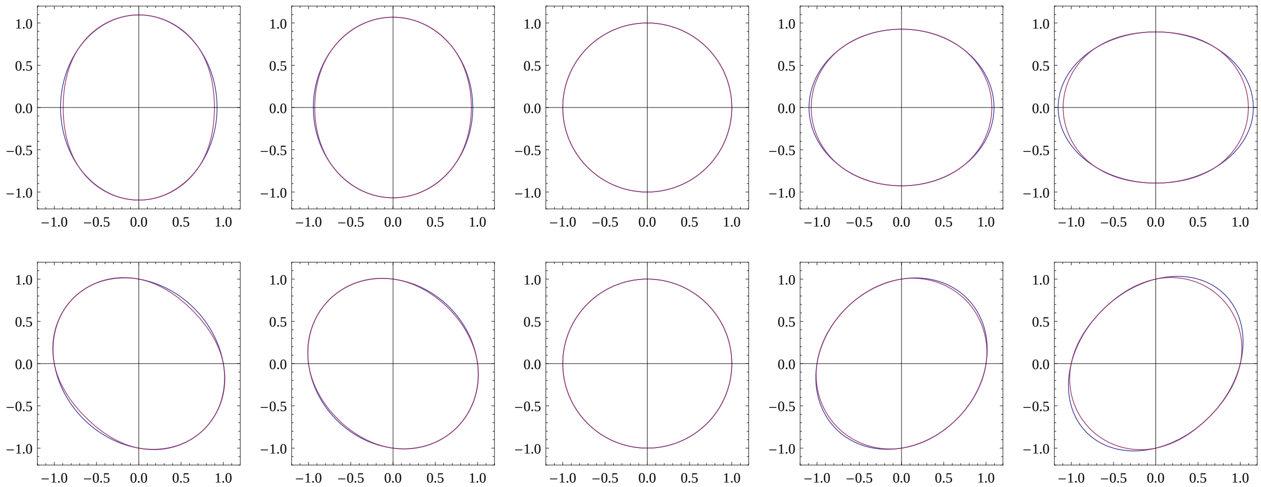

Figure 1: Time dependence of the elliptical distortion for $\omega t = 0, \pi/4, \pi/2, 3\pi/4, \pi$. *Top row:* h_+ polarization *Bottom row:* $h_\times$ polarization

For the second polarization we have to add a phase factor of $-\pi/4$ to the ellipse Eq. (16).

Exercise 3 [Gravitational Bremsstrahlung]
The parabolic orbit of the mass m scattering on $M \gg m$ is described by the parametric form

$$\boldsymbol{r}(\varphi) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2b}{1+\cos(\varphi)} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \tag{21}$$

where the time dependence is described by

$$\dot{\varphi} = \sqrt{\frac{GM}{8b^3}} \left[1 + \cos(\varphi)\right]^2 \tag{22}$$

Thus we have for the quadrupole tensor

$$I_{ij}(t) = \int \mathrm{d}^3x' \rho(\boldsymbol{x}') x'_i x'_j = \frac{4b^2 m}{\left(1+\cos(\varphi)\right)^2} \begin{pmatrix} \cos^2(\varphi) & \sin(\varphi)\cos(\varphi) & 0 \\ \sin(\varphi)\cos(\varphi) & \sin^2(\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

where we have used $\rho(\boldsymbol{x}') = M\delta(\boldsymbol{x}') + m\delta\big(\boldsymbol{x}' - \boldsymbol{r}(\varphi)\big)$. Note that the angular position φ is not integrated over since it is given by the Newtonian equation of motion for m. Now we can use that for any function $f(\varphi)$ and $\dot{\varphi} = \dot{\varphi}(\varphi)$ we have

$$\frac{\mathrm{d}^2 f(\varphi)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\frac{\mathrm{d}f}{\mathrm{d}\varphi} + \frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 \tag{24}$$

$$= \dot{\varphi}\frac{\mathrm{d}\dot{\varphi}}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}\varphi} + \dot{\varphi}^2\frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2} \tag{25}$$

$$= \frac{1}{2}\frac{\mathrm{d}\dot{\varphi}^2}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}\varphi} + \dot{\varphi}^2\frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2} \tag{26}$$

With the above relation and Eq. (22) the second derivative of the quadrupole tensor simplifies to

$$\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = \frac{GM}{8b^3}\left(-2\sin(\varphi)\big(1+\cos(\varphi)\big)^3\frac{\mathrm{d}I_{ij}}{\mathrm{d}\varphi} + (1+\cos(\varphi))^4\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}\varphi^2}\right). \tag{27}$$

For the first and second term in the above equation we obtain

$$\sin(\varphi)\big(1+\cos(\varphi)\big)^3\frac{\mathrm{d}I_{ij}}{\mathrm{d}t} = \tag{28}$$

$$4b^2\begin{pmatrix} -2\sin^2(\varphi)\cos(\varphi) & \sin(\varphi)\left(\cos(2\varphi)+\cos(\varphi)\right) & 0 \\ \sin(\varphi)\left(\cos(2\varphi)+\cos(\varphi)\right) & 2\sin^2(\varphi)(\cos(\varphi)+1) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

and

$$\big(1+\cos(\varphi)\big)^4\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = \tag{30}$$

$$8b^2\begin{pmatrix} \cos^2\left(\frac{\varphi}{2}\right)\left(-6\cos(\varphi)+\cos(2\varphi)+3\right) & 2\left(\sin\left(\frac{3\varphi}{2}\right)-5\sin\left(\frac{\varphi}{2}\right)\right)\cos^3\left(\frac{\varphi}{2}\right) & 0 \\ 2\left(\sin\left(\frac{3\varphi}{2}\right)-5\sin\left(\frac{\varphi}{2}\right)\right)\cos^3\left(\frac{\varphi}{2}\right) & -4\cos^4\left(\frac{\varphi}{2}\right)\left(\cos(\varphi)-2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{31}$$

Finally we have

$$\gamma_{ij} = \frac{2G^2 mM}{rbc^4}\begin{pmatrix} \left(\sin^2(\varphi)-\cos^2(\varphi)(\cos(\varphi)+1)\right) & -8\sin\left(\frac{\varphi}{2}\right)\cos^5\left(\frac{\varphi}{2}\right) & 0 \\ -8\sin\left(\frac{\varphi}{2}\right)\cos^5\left(\frac{\varphi}{2}\right) & 4\cos^4\left(\frac{\varphi}{2}\right)\cos(\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

We immediately see that the metric perturbation in the trace reversed Lorentz gauge is not traceless. The Lorentz gauge condition relates the time components to the spatial components

$$k^\mu \gamma_{\mu\nu} = k(\gamma_{0\nu} - \gamma_{3\nu}) = 0 \tag{33}$$

Thus we have $\gamma_{0i} = \gamma_{3i} = 0$ and $\gamma_{00} = \gamma_{03} = 0$.

In the vacuum we can project to transverse traceless gauge

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij}^{kl}\gamma_{kl}, \tag{34}$$

where the projection tensor is given by

$$\Lambda_{ij}^{kl} = P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}. \tag{35}$$

Using $P_i^j = \delta_i^j - n_i n^j$ we have for an observer on the z-axis with $\boldsymbol{n} = (0,0,1)$

$$h_{11}^{\mathrm{TT}} = \frac{1}{2}\left(\gamma_{11} - \gamma_{22}\right) \qquad h_{22}^{\mathrm{TT}} = \frac{1}{2}\left(\gamma_{22} - \gamma_{11}\right) = -h_{11}^{\mathrm{TT}} \qquad h_{12}^{\mathrm{TT}} = h_{21}^{\mathrm{TT}} = \gamma_{12} \tag{36}$$

and all the other components vanish. Figure 2 shows the time dependence of the two independent polarizations $h_+ = h_{11}^{\mathrm{TT}}$ and $h_\times = h_{12}^{\mathrm{TT}}$.

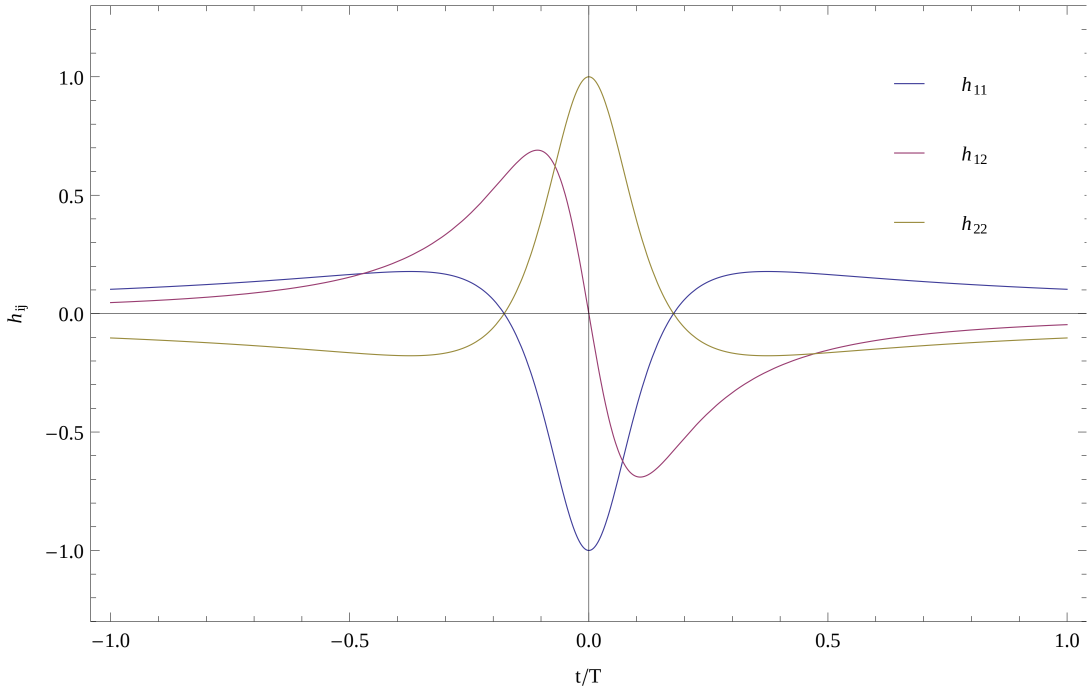

Figure 2: Time dependence of the h_{11}^{TT} and h_{22}^{TT} h_{12}^{TT} components normalised to $h_{11}(\varphi = 0)$. The time is expressed in terms of T_{o}

The timescale of the transit is given by the orbital time $T_{\mathrm{o}} = 2\pi\sqrt{\frac{b^3}{GM}}$, whereas the amplitude is affected by $T_{\mathrm{gw}}^2/T_{\mathrm{o}}^2$, where we defined

$$T_{\mathrm{gw}} = \sqrt{\frac{Gmb^2}{rc^4}} \tag{37}$$

and the time dependence of the orbit was obtained solving the implicit equation for $\varphi(t)$

$$t(\varphi) = \sqrt{\frac{2b^3}{GM}}\left(\tan\left(\frac{\varphi}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\varphi}{2}\right)\right). \tag{38}$$