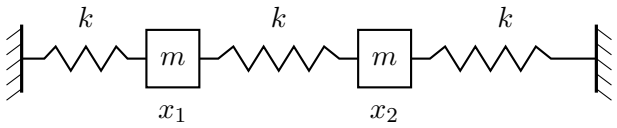
Mechanics
Exercise Sheet 8

HS 2026
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Exercise 1: Normal modes of two coupled masses

Two identical masses m move without friction along a horizontal line. They are connected to fixed walls and to each other by three identical springs with spring constant k, as shown in the figure. We denote by x_1 and x_2 the displacements of the masses from equilibrium.



a) Write the Lagrangian and derive the equations of motion for x_1 and x_2.

b) Determine the normal frequencies and the corresponding eigenvectors. Explain how the two masses move in each normal mode.

c) Find the normal coordinates which diagonalize the Lagrangian, choosing their normalization such that the kinetic term takes the form $\frac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2)$.

d) Initially, the first mass is displaced by a while the second is at equilibrium, and both masses are released from rest. Find their motion as a function of time.

Solution:

a) The two outer springs are stretched by x_1 and x_2, while the central spring is stretched by $x_2 - x_1$. The Lagrangian is therefore

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2}\left[x_1^2 + x_2^2 + (x_2 - x_1)^2\right]. \tag{1.1}$$

We obtain

$$m\ddot{x}_1 + k(2x_1 - x_2) = 0\,, \tag{1.2}$$

$$m\ddot{x}_2 + k(2x_2 - x_1) = 0\,, \tag{1.3}$$

or, in matrix form

$$m\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} + k\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 0. \tag{1.4}$$

b) Normal frequencies and eigenvectors can be obtained from the eigenvalue problem

$$\begin{pmatrix}2k/m & -k/m\\ -k/m & 2k/m\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \omega^2\begin{pmatrix}a_1\\ a_2\end{pmatrix}. \tag{1.5}$$

The existence of non-trivial solutions requires the determinant of the system to be non-zero

$$\det \begin{pmatrix} 2k/m - \omega^2 & -k/m \\ -k/m & 2k/m - \omega^2 \end{pmatrix} = 0. \tag{1.6}$$

We find the following frequencies and eigenvectors by solving the determinant condition and substituting back into (1.5)

$$\omega_1^2 = \frac{k}{m}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \omega_2^2 = \frac{3k}{m}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.7}$$

The first mode corresponds to both masses moving together, leaving the central spring unstretched. In the second mode they move in opposite directions, which also deforms the central spring and gives a higher frequency.

c) The normal coordinates which diagonalize the Lagrangian are those given by the eigenvectors. Choosing their normalization such that the kinetic term takes the form $\frac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2)$

$$Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2), \qquad Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2). \tag{1.8}$$

The Lagrangian becomes a sum of two independent oscillators,

$$L = \frac{1}{2}\left(\dot{Q}_1^2 - \omega_1^2 Q_1^2\right) + \frac{1}{2}\left(\dot{Q}_2^2 - \omega_2^2 Q_2^2\right). \tag{1.9}$$

Alternatively, define the matrix that converts from the original coordinates $\mathbf{x} = (x_1, x_2)$ to the normal coordinates $\mathbf{q} = (x_1 + x_2, x_1 - x_2)/\sqrt{2}$,

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{U}\mathbf{x} = \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}. \tag{1.10}$$

This matrix is orthogonal $\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbb{I}$, therefore $\mathbf{x} = \mathbf{U}^\mathbf{T}\mathbf{q}$. We can then write the Lagrangian in matrix form and use (1.10) to change basis

$$L = \frac{m}{2}\dot{\mathbf{x}}^\mathbf{T}\dot{\mathbf{x}} - \frac{k}{2}\mathbf{x}^\mathbf{T}\mathbf{K}\mathbf{x} = \frac{m}{2}\dot{\mathbf{q}}^\mathbf{T}\dot{\mathbf{q}} - \frac{k}{2}\mathbf{q}^\mathbf{T}\mathbf{U}\mathbf{K}\mathbf{U}^\mathbf{T}\mathbf{q} = \frac{m}{2}\dot{\mathbf{q}}^\mathbf{T}\dot{\mathbf{q}} - \frac{k}{2}\mathbf{q}^\mathbf{T} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{q}. \tag{1.11}$$

Finally, we achieve canonical normalization with the redefinition $q_i = Q_i/\sqrt{m}$ to recover (1.9).

d) Each normal coordinate describes an independent harmonic oscillator with its own normal frequency $Q_i(t) = A_i \cos(\omega_i t + \delta_i)$. The initial conditions $x_1(0) = a$, $x_2(0) = 0$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$ give $Q_i(0) = a\sqrt{m/2}$ and $\dot{Q}_i = 0$, therefore

$$\dot{Q}_i(0) = -\omega_i A_i \sin(\delta_i) = 0 \Rightarrow \delta_i = 0\,. \tag{1.12}$$

$$Q_i(0) = A_i = a\sqrt{m/2}\,. \tag{1.13}$$

Therefore

$$Q_1(t) = \sqrt{\frac{m}{2}}\, a \cos(\omega_1 t)\,, \qquad Q_2(t) = \sqrt{\frac{m}{2}}\, a \cos(\omega_2 t) \tag{1.14}$$

Inverting (1.8) we finally obtain

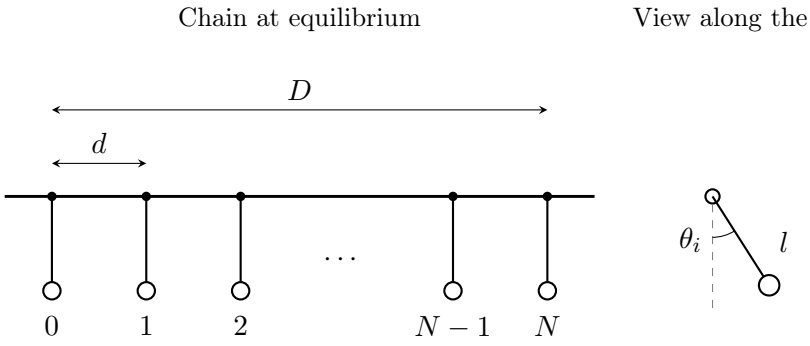
$$x_1(t) = \frac{1}{\sqrt{2m}} \left[Q_1(t) + Q_2(t) \right] = \frac{a}{2} \left[\cos(\omega_1 t) + \cos(\omega_2 t) \right], \tag{1.15}$$

$$x_2(t) = \frac{1}{\sqrt{2m}} \left[Q_1(t) - Q_2(t) \right] = \frac{a}{2} \left[\cos(\omega_1 t) - \cos(\omega_2 t) \right]. \tag{1.16}$$

Exercise 2: A chain of coupled pendulums

Consider $N+1$ identical pendulums attached at equal intervals d to a horizontal bar that can twist along its length. Each pendulum consists of a point mass m at the end of a massless rod of length l. The rods rotate about the axis of the bar, so the masses swing in planes perpendicular to it. The figure shows the chain at equilibrium and an end view of a displaced pendulum.

We denote by $\theta_i(t)$ the angle from the downward vertical, with $i = 0, \ldots, N$. A relative rotation of neighbouring pendulums twists the bar, giving a potential energy $\frac{\kappa}{2}(\theta_{i+1} - \theta_i)^2$. Neglect friction and the inertia of the bar. The distance between the first and last pendulum is $D = Nd$.



a) Write the Lagrangian of the chain and derive the equations of motion for the interior pendulums, $i = 1, \ldots, N-1$.

b) At distances much larger than d, the system can be described by a continuous field $\phi(x,t)$, with $\theta_i(t) = \phi(id, t)$. Take the limit $d \to 0$ at fixed D, keeping $\rho = m/d$ and $\eta = \kappa d$ constant. Find the equation of motion for $\phi(x,t)$.

c) Find the Lagrangian density in the continuum limit and verify that it gives the same equation of motion.

d) For small oscillations, derive the dispersion relation. The pendulums at the two ends are now held fixed, so that $\phi(0,t) = \phi(D,t) = 0$. Find the allowed frequencies and the general solution satisfying $\phi(x,0) = 0$, leaving the mode amplitudes undetermined. Where are the nodes of the first two normal modes?

Solution:

a) The kinetic energy of each pendulum is $\frac{1}{2}ml^2\dot{\theta}_i^2$. Including gravity and the torsion of the bar, the Lagrangian, up to a constant, is

$$L = \sum_{i=0}^{N} \left[\frac{ml^2}{2}\dot{\theta}_i^2 + mgl\cos\theta_i \right] - \frac{\kappa}{2}\sum_{i=0}^{N-1}(\theta_{i+1} - \theta_i)^2. \tag{2.1}$$

For an interior pendulum, the equation of motion is

$$ml^2\ddot{\theta}_i + mgl\sin\theta_i - \kappa(\theta_{i+1} - 2\theta_i + \theta_{i-1}) = 0. \tag{2.2}$$

b) System described by continous field with $\theta_i(t) = \phi(x = id, t)$. The angles at neighbouring sites are Taylor-expanded for small d,

$$\theta_{i+1} = \phi(x+d,t) = \phi(x,t) + d\,\partial_x\phi(x,t) + \frac{d^2}{2}\partial_x^2\phi(x,t) + \frac{d^3}{3}\partial_x^2\phi(x,t) + \mathcal{O}(d^4) \tag{2.3}$$

$$\theta_{i-1} = \phi(x-d,t) = \phi(x,t) - d\,\partial_x\phi(x,t) + \frac{d^2}{2}\partial_x^2\phi(x,t) - \frac{d^3}{3}\partial_x^2\phi(x,t) + \mathcal{O}(d^4) \tag{2.4}$$

Therefore, the torsion term becomes

$$\theta_{i+1} - 2\theta_i + \theta_{i-1} = d^2\partial_x^2\phi(x,t) + \mathcal{O}(d^4). \tag{2.5}$$

Neglecting the d^4 terms, with $m = \rho d$ and $\kappa = \eta/d$, the continuum field equation becomes

$$\rho l^2\partial_t^2\phi - \eta\partial_x^2\phi + \rho g l\sin\phi = 0. \tag{2.6}$$

The spatial derivative describes the torsional coupling, while gravity restores each pendulum towards the downward position.

c) The chain of length D is divided in N segments of length d, such that $D/N = d$. We can imagine the continuum limit as taking $N \to \infty$. As N grows, the number of pendulums increases within the fixed chain of length D, so d has to become small and eventually infinitesimal, $d \to dx$, and the sum becomes an integral,

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} f(x = id, t)\frac{D}{N} = \int_0^D dx f(x)\,, \qquad \frac{D}{N} = d\,. \tag{2.7}$$

For a sum $\sum_{i=0}^{N} f(x)d = \sum_{i=0}^{N-1} f(x)d + f(D)d$, the second term vanishes as $d \to 0$ assuming $f(D)$ is finite. In practice, the replacement rule is $\sum_{i=0}^{N} d \to \int_0^d dx f(x)$. Taking into account $\theta_{i+1} - \theta_i \simeq d\partial_x\phi$ at leading order, we obtain

$$T = \frac{1}{2}ml^2\sum_{i=0}^{N}\dot\theta_i^2 \to \int_0^D dx\frac{\rho l^2}{2}(\partial_t\phi)^2 \tag{2.8}$$

$$V_e = \frac{\kappa}{2}\sum_{i=0}^{N-1}(\theta_{i+1} - \theta_i)^2 \to \int_0^D dx\frac{\eta}{2}(\partial_x\phi)^2 \tag{2.9}$$

$$V_g = -mgl\sum_{i=0}^{N}\cos\theta_i \to -\int_0^D dx\rho g l\cos\phi \tag{2.10}$$

Therefore, the Lagrangian density is

$$L = \int_0^D \mathrm{d}x\,\mathcal{L}, \qquad \mathcal{L} = \frac{\rho l^2}{2}(\partial_t\phi)^2 - \frac{\eta}{2}(\partial_x\phi)^2 + \rho g l\cos\phi. \tag{2.11}$$

The field Euler-Lagrange equation,

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right) + \partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0\,, \tag{2.12}$$

reproduces Eq. (2.6).

d) At small angles $\sin\phi \simeq \phi$, the Euler-Lagrange equation of $\phi(x,t)$ is that of a linear wave equation with additional restoring force,

$$\partial_t^2\phi - \frac{\eta}{\rho l^2}\partial_x^2\phi + \rho g l\phi = 0 \tag{2.13}$$

Substituting a wave ansatz $\phi = Ae^{i(qx-\omega t)}$ gives

$$\omega^2 = \frac{g}{l} + \frac{\eta}{\rho l^2}q^2. \tag{2.14}$$

Since the dispersion relation depends on q^2, a wave travelling in the opposite direction is also a solution. We then try the ansatz

$$\phi(x,t) = A\cos(qx-\omega t) + B\cos(qx+\omega t)\,, \tag{2.15}$$

and apply the boundary conditions

$$\phi(0,t) = A\cos(-\omega t) + B\cos(\omega t) = 0 \Rightarrow A = -B \Rightarrow \phi(x,t) = 2A\sin(qx)\sin(\omega t) \tag{2.16}$$

$$\phi(D,t) = 2A\sin(qD)\sin(\omega t) = 0 \Rightarrow \sin(qD) = 0 \Rightarrow q = \frac{n\pi}{\pi}\,, \tag{2.17}$$

with $n = 0, 1, 2, ...$ an integer number. The dispersion relation then becomes

$$\omega_n^2 = \frac{g}{l} + \frac{\eta}{\rho l^2}\frac{n^2\pi^2}{D^2}. \tag{2.18}$$

The initial displacement already vanishes $\phi(x,0) = 0$, so the general motion is

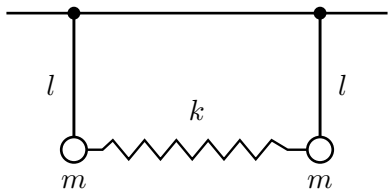
$$\phi(x,t) = \sum_{n=1}^{\infty} B_n \sin\Big(\frac{n\pi x}{D}\Big)\sin(\omega_n t). \tag{2.19}$$

where the amplitudes B_n remain undetermined as the exercise suggests. The first mode has spatial profile $\sin(\pi x/D)$ and hence no nodes inside the chain. The second has profile $\sin(2\pi x/D)$, hence one node at $x = D/2$.

Exercise 3: Class: Two pendulums coupled by a spring

Two identical pendulums consist of point masses m attached to fixed pivots by massless rods of length l. The pivots are at the same height. The masses are connected by a light spring with spring constant k, which is horizontal and unstretched when both pendulums hang vertically.

The pendulums swing in the plane of the figure. Denote their angular displacements from the downward vertical by $\theta_1(t)$ and $\theta_2(t)$, positive towards the right. Neglect friction and consider small oscillations.



a) Show that, to first order in the angles, the extension of the spring is $l(\theta_2 - \theta_1)$. Write the Lagrangian to quadratic order in the angles and their velocities.

b) Derive the equations of motion and find the two normal frequencies and their associated motions. Explain why one mode is unaffected by the spring.

c) Initially, $\theta_1(0) = \alpha$ and $\theta_2(0) = 0$, and both pendulums are released from rest. Find $\theta_1(t)$ and $\theta_2(t)$.

d) What happens to the frequencies when $k \to 0$? Explain the result.

Solution:

a) The horizontal displacement of each mass from equilibrium is

$$x_i = l \sin\theta_i \simeq l\theta_i. \tag{3.1}$$

The extension of the spring to first order in the angles is therefore

$$\Delta L \simeq x_2 - x_1 = l(\theta_2 - \theta_1). \tag{3.2}$$

The kinetic energy is

$$T = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2). \tag{3.3}$$

The gravitational potential energy, measured relative to the lowest position of each pendulum, is

$$V_g = mgl(1 - \cos\theta_1) + mgl(1 - \cos\theta_2) \simeq \frac{1}{2}mgl(\theta_1^2 + \theta_2^2). \tag{3.4}$$

The elastic potential energy is

$$V_s = \frac{1}{2}k(\Delta L)^2 \simeq \frac{1}{2}kl^2(\theta_2 - \theta_1)^2. \tag{3.5}$$

The Lagrangian to quadratic order is consequently

$$L = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}mgl(\theta_1^2 + \theta_2^2) - \frac{1}{2}kl^2(\theta_2 - \theta_1)^2. \tag{3.6}$$

b) Applying the Euler–Lagrange equations gives

$$ml^2\ddot{\theta}_1 + mgl\theta_1 + kl^2(\theta_1 - \theta_2) = 0, \tag{3.7}$$
$$ml^2\ddot{\theta}_2 + mgl\theta_2 + kl^2(\theta_2 - \theta_1) = 0. \tag{3.8}$$

In matrix form,

$$ml^2 \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} mgl + kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0. \tag{3.9}$$

The normal frequencies then satisfy

$$\det \begin{pmatrix} g/l + k/m - \omega^2 & -k/m \\ -k/m & g/l + k/m - \omega^2 \end{pmatrix} = 0. \tag{3.10}$$

This gives

$$\omega_+^2 = \frac{g}{l}, \qquad \omega_-^2 = \frac{g}{l} + \frac{2k}{m}. \tag{3.11}$$

The corresponding normalized eigenvectors are

$$v_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.12}$$

The first mode corresponds to the two pendulums oscillating in phase, $\theta_1 = \theta_2$. The spring is not stretched, so its restoring force vanishes and the frequency is that of a single pendulum.

In the second mode the pendulums oscillate in opposite directions, $\theta_1 = -\theta_2$. The spring is stretched or compressed and produces an additional restoring force, giving a higher frequency.

c) We introduce the normal coordinates corresponding to the two eigenvectors,

$$q_+ = \frac{\theta_1 + \theta_2}{\sqrt{2}}, \qquad q_- = \frac{\theta_1 - \theta_2}{\sqrt{2}}. \tag{3.13}$$

The inverse transformation is

$$\theta_1 = \frac{q_+ + q_-}{\sqrt{2}}, \qquad \theta_2 = \frac{q_+ - q_-}{\sqrt{2}}. \tag{3.14}$$

The equations of motion in these coordinates are decoupled,

$$\ddot{q}_+ + \omega_+^2 q_+ = 0, \qquad \ddot{q}_- + \omega_-^2 q_- = 0. \tag{3.15}$$

Each normal coordinate therefore describes an independent harmonic oscillator, with general solution

$$q_\pm(t) = A_\pm \cos(\omega_\pm t) + B_\pm \sin(\omega_\pm t). \tag{3.16}$$

The initial conditions in the original coordinates are

$$\theta_1(0) = \alpha, \qquad \theta_2(0) = 0, \qquad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0. \tag{3.17}$$

Transforming them into normal coordinates gives

$$q_+(0) = q_-(0) = \frac{\alpha}{\sqrt{2}}, \qquad \dot{q}_+(0) = \dot{q}_-(0) = 0. \tag{3.18}$$

Therefore

$$q_+(t) = \frac{\alpha}{\sqrt{2}}\cos(\omega_+ t), \qquad q_-(t) = \frac{\alpha}{\sqrt{2}}\cos(\omega_- t). \tag{3.19}$$

Returning to the original coordinates, we obtain

$$\begin{aligned} \theta_1(t) &= \frac{\alpha}{2}\left[\cos(\omega_+ t) + \cos(\omega_- t)\right], \\ \theta_2(t) &= \frac{\alpha}{2}\left[\cos(\omega_+ t) - \cos(\omega_- t)\right]. \end{aligned} \tag{3.20}$$

These expressions satisfy the initial displacements and the vanishing initial angular velocities.

d) In the limit $k \to 0$, the two frequencies become equal,

$$\omega_+ = \omega_- = \sqrt{\frac{g}{l}}. \tag{3.21}$$

The spring no longer couples the two pendulums. Each therefore oscillates independently with the frequency of a single pendulum. The frequencies are degenerate, so the symmetric and antisymmetric combinations are no longer uniquely distinguished as normal modes.