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**Exercise 1** [Gravitational field of a moving particle]

We are considering a particle of mass  $M$  moving with constant velocity  $\mathbf{V}$ . In the particles restframe  $\Sigma'$  we have

$$h'_{\mu\mu}(\mathbf{r}) = -\frac{2G}{c^2} \int d^3r' \frac{\rho(r')}{|\mathbf{r} - \mathbf{r}'|} = -\frac{2GM}{c^2} \frac{1}{|\mathbf{r} - \mathbf{r}_M|} \quad (1)$$

and  $h'_{0i} = 0$ . Here  $\mathbf{r}_M$  is the position of the mass  $M$ . If we now transfer to a general frame  $\Sigma$ , the metric changes according to  $g_{\mu\nu} = \alpha_{\mu}^{\kappa} \alpha_{\nu}^{\rho} g'_{\kappa\rho}$ . Since we are working at linear order in  $h$  and the mass  $M$  is moving on a straight line, the transformation is nothing but a Lorentz transformation  $\alpha_{\nu}^{\mu} = \Lambda_{\nu}^{\mu} + \mathcal{O}(h)$ . For the transformation matrix we have up

$$\Lambda_{\nu}^{\mu} = \begin{pmatrix} 1 & V^i/c \\ V^i/c & \mathbb{1} \end{pmatrix} \quad (2)$$

where we neglected terms  $\mathcal{O}(V^2/c^2)$ , i.e. set  $\gamma = 1$ . Note that we only need to transform the perturbation since  $\Lambda_{\mu}^{\kappa} \Lambda_{\nu}^{\rho} \eta_{\kappa\rho} = \eta_{\mu\nu}$  and are thus left with

$$h_{\mu\nu} = \Lambda_{\mu}^{\kappa} \Lambda_{\nu}^{\rho} h'_{\kappa\rho}, \quad (3)$$

Evaluating the above expression we obtain for the metric perturbation in  $\Sigma$

$$h_{\mu\mu} = h'_{\mu\mu} + \mathcal{O}(V^2/c^2) \quad h_{0i} = 2 \frac{V_i}{c} h'_{00} + \mathcal{O}(V^2/c^2) \quad (4)$$

The mass  $M$  is moving on a straight line in  $\Sigma$ , thus its position as a function of time can be written as

$$\mathbf{r}_M(t) = \mathbf{r}_{M,0} + \mathbf{V}t. \quad (5)$$

The gravitomagnetic potential is

$$\mathbf{h}(\mathbf{r}) = h_{0i}(\mathbf{r}) = -\frac{4GM}{c^3} \frac{\mathbf{V}}{|\mathbf{r} - \mathbf{r}_M(t)|}, \quad (6)$$

for the gravitomagnetic field we thus have

$$\boldsymbol{\Omega}(\mathbf{r}) = -\frac{c}{2} \nabla_{\mathbf{r}} \times \mathbf{h}(\mathbf{r}) = \frac{2GM}{c^2} \frac{\mathbf{V} \times (\mathbf{r} - \mathbf{r}_M(t))}{|\mathbf{r} - \mathbf{r}_M(t)|^3}. \quad (7)$$

The equation of motion is calculated in analogy to the derivation presented in the lecture. The velocity of the test mass  $m$  will be symbolised by a lowercase  $\mathbf{v}$ . Evaluating the geodesic equation we obtain

$$\frac{dv^i}{dt} = -\gamma_{\alpha\beta}^i u^{\alpha} u^{\beta} = -\Gamma_{00}^i u^0 u^0 - 2\Gamma_{0j}^i u^0 u^j \quad (8)$$

$$= -c^2 \Gamma_{00}^i - 2\Gamma_{0j}^i c v_m^j \quad (9)$$

where we have neglected terms  $\mathcal{O}(v^2/c^2)$ . For the Christoffel symbols we have

$$\Gamma_{00}^i = \frac{1}{2}\eta^{ij} (\partial_0 h_{j0} + \partial_0 h_{0j} - \partial_j h_{00}) = \partial_0 h_0^i - \frac{1}{2}\partial^i h_{00} \quad (10)$$

$$\Gamma_{0j}^i = \frac{1}{2}\eta^{ik} (\partial_0 h_{jk} + \partial_j h_{0k} - \partial_k h_{0j}) = \frac{1}{2}\partial_0 h_j^i + \frac{1}{2}\eta^{ik} (\partial_j h_{0k} - \partial_k h_{0j}) \quad (11)$$

In contrast to the lecture, we now have a time dependent metric perturbation. The time dependence arises via  $\mathbf{r}_M(t)$ , such that the time derivative can be rewritten as

$$\frac{\partial h_{\mu\nu}}{\partial t} = -\mathbf{V} \cdot \nabla_{\mathbf{r}} h_{\mu\nu}. \quad (12)$$

Hence we have for the equation of motion of the test mass  $m$

$$\frac{dv_i}{dt} = -c^2 \partial_0 h_{i0} + \frac{c^2}{2} \partial_i h_{00} - \frac{c}{2} v^j \partial_0 h_{ij} - \frac{c}{2} v^j (\partial_j h_{0i} - \partial_i h_{0j}) \quad (13)$$

$$= -c^2 \partial_0 h_{i0} + \frac{c^2}{2} \partial_i h_{00} - \frac{c}{2} v^j \partial_0 h_{ij} + \epsilon_{ikj} \Omega^k v^j \quad (14)$$

Neglecting  $\partial_0 h_{i0} = \mathcal{O}(V^2/c^2)$  we finally have

$$\frac{d\mathbf{v}}{dt} = -GM \frac{(\mathbf{r} - \mathbf{r}_M) \cdot \mathbf{V}}{|\mathbf{r} - \mathbf{r}_M|^3} \mathbf{v} - GM \frac{\mathbf{r} - \mathbf{r}_M}{|\mathbf{r} - \mathbf{r}_M|^3} + \boldsymbol{\Omega} \times \mathbf{v} \quad (15)$$

We see that the gravitomagnetic forces due to moving masses are of order  $\mathcal{O}(vV/c^2)$ .

### Exercise 2 [Particles in the field of a gravitational wave]

We start from the parametric form of an ellipse centered on the origin

$$r(\varphi) = \frac{b}{\sqrt{1 - e^2 \cos^2(\varphi)}}. \quad (16)$$

Thus we have for  $e^2 \ll 1$

$$r^2(\varphi) = b^2 \left( 1 + \frac{e^2}{2} (1 + \cos(2\varphi)) \right). \quad (17)$$

Our goal is to bring this in the form of the distortion of a circle due to an incoming gravitational wave

$$r^2(\varphi) = R^2 (1 - 2h \cos(\omega t) \cos(2\varphi)) \quad (18)$$

which can be achieved by setting

$$R^2 = b^2 \left( 1 + \frac{e^2}{2} \right) = \text{const.}, \quad e^2 = -4h \cos(\omega t). \quad (19)$$

Using  $b^2 = R^2(1 - e^2/2)$  and  $a^2 - b^2 = a^2 e^2$  we obtain

$$a = R(1 - h \cos(\omega t)) \quad b = R(1 + h \cos(\omega t)) \quad (20)$$

Equations 19 and 20 describe the time dependence of the ellipticity and the semi-minor and -major axis. It might seem troublesome that for  $-\pi/2 \leq \omega t \leq \pi/2$  we have  $b > a$  and  $e^2 < 0$ . But this can be interpreted as a phase shift by  $\pi/2$  which means that the ellipse is rotated by  $90^\circ$  with respect to its standard orientation.

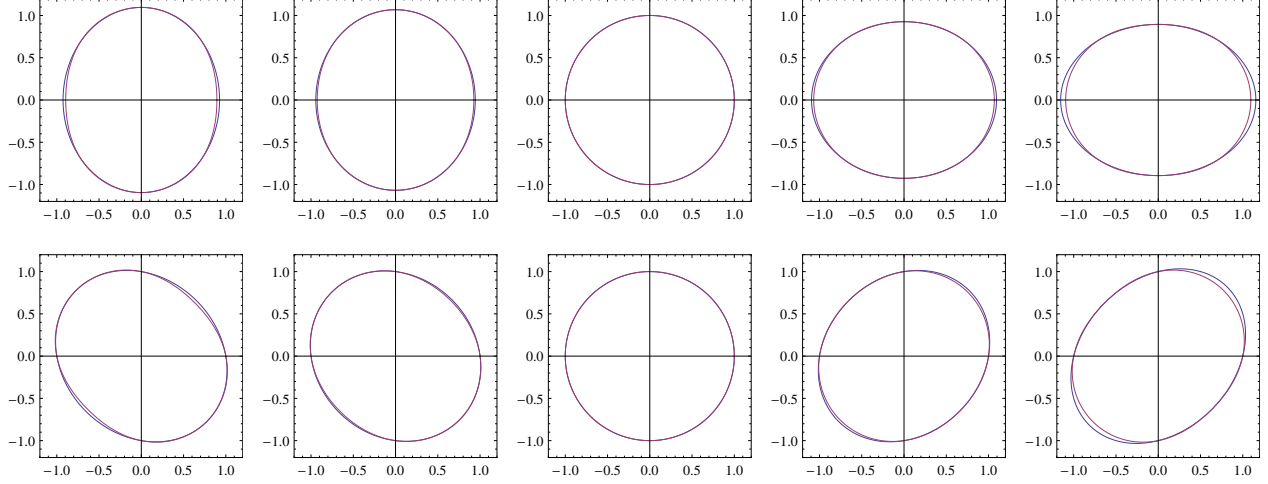


Figure 1: Time dependence of the elliptical distortion for  $\omega t = 0, \pi/4, \pi/2, 3\pi/4, \pi$ . *Top row:  $h_+$  polarization* *Bottom row:  $h_x$  polarization*

For the second polarization we have to add a phase factor of  $-\pi/4$  to the ellipse Eq. (16).

### Exercise 3 [Gravitational Bremsstrahlung]

The parabolic orbit of the mass  $m$  scattering on  $M \gg m$  is described by the parametric form

$$\mathbf{r}(\varphi) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2b}{1 + \cos(\varphi)} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \quad (21)$$

where the time dependence is described by

$$\dot{\varphi} = \sqrt{\frac{GM}{8b^3}} [1 + \cos(\varphi)]^2 \quad (22)$$

Thus we have for the quadrupole tensor

$$I_{ij}(t) = \int d^3x' \rho(\mathbf{x}') x'_i x'_j = \frac{4b^2 m}{(1 + \cos(\varphi))^2} \begin{pmatrix} \cos^2(\varphi) & \sin(\varphi) \cos(\varphi) & 0 \\ \sin(\varphi) \cos(\varphi) & \sin^2(\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (23)$$

where we have used  $\rho(\mathbf{x}') = M\delta(\mathbf{x}') + m\delta(\mathbf{x}' - \mathbf{r}(\varphi))$ . Note that the angular position  $\varphi$  is not integrated over since it is given by the Newtonian equation of motion for  $m$ . Now we can use that for any function  $f(\varphi)$  and  $\dot{\varphi} = \dot{\varphi}(\varphi)$  we have

$$\frac{d^2 f(\varphi)}{dt^2} = \frac{d^2 \varphi}{dt^2} \frac{df}{d\varphi} + \frac{d^2 f}{d\varphi^2} \left( \frac{d\varphi}{dt} \right)^2 \quad (24)$$

$$= \dot{\varphi} \frac{d\dot{\varphi}}{d\varphi} \frac{df}{d\varphi} + \dot{\varphi}^2 \frac{d^2 f}{d\varphi^2} \quad (25)$$

$$= \frac{1}{2} \frac{d\dot{\varphi}^2}{d\varphi} \frac{df}{d\varphi} + \dot{\varphi}^2 \frac{d^2 f}{d\varphi^2} \quad (26)$$

With the above relation and Eq. (22) the second derivative of the quadrupole tensor simplifies to

$$\frac{d^2 I_{ij}}{dt^2} = \frac{GM}{8b^3} \left( -2 \sin(\varphi) (1 + \cos(\varphi))^3 \frac{dI_{ij}}{d\varphi} + (1 + \cos(\varphi))^4 \frac{d^2 I_{ij}}{d\varphi^2} \right). \quad (27)$$

For the first and second term in the above equation we obtain

$$\sin(\varphi) (1 + \cos(\varphi))^3 \frac{dI_{ij}}{dt} = \quad (28)$$

$$4b^2 \begin{pmatrix} -2 \sin^2(\varphi) \cos(\varphi) & \sin(\varphi) (\cos(2\varphi) + \cos(\varphi)) & 0 \\ \sin(\varphi) (\cos(2\varphi) + \cos(\varphi)) & 2 \sin^2(\varphi) (\cos(\varphi) + 1) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (29)$$

and

$$(1 + \cos(\varphi))^4 \frac{d^2 I_{ij}}{dt^2} = \quad (30)$$

$$8b^2 \begin{pmatrix} \cos^2\left(\frac{\varphi}{2}\right) (-6 \cos(\varphi) + \cos(2\varphi) + 3) & 2 \left( \sin\left(\frac{3\varphi}{2}\right) - 5 \sin\left(\frac{\varphi}{2}\right) \right) \cos^3\left(\frac{\varphi}{2}\right) & 0 \\ 2 \left( \sin\left(\frac{3\varphi}{2}\right) - 5 \sin\left(\frac{\varphi}{2}\right) \right) \cos^3\left(\frac{\varphi}{2}\right) & -4 \cos^4\left(\frac{\varphi}{2}\right) (\cos(\varphi) - 2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (31)$$

Finally we have

$$\gamma_{ij} = \frac{2G^2 m M}{r b c^4} \begin{pmatrix} (\sin^2(\varphi) - \cos^2(\varphi) (\cos(\varphi) + 1)) & -8 \sin\left(\frac{\varphi}{2}\right) \cos^5\left(\frac{\varphi}{2}\right) & 0 \\ -8 \sin\left(\frac{\varphi}{2}\right) \cos^5\left(\frac{\varphi}{2}\right) & 4 \cos^4\left(\frac{\varphi}{2}\right) \cos(\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (32)$$

We immediately see that the metric perturbation in the trace reversed Lorentz gauge is not traceless. The Lorentz gauge condition relates the time components to the spatial components

$$k^\mu \gamma_{\mu\nu} = k(\gamma_{0\nu} - \gamma_{3\nu}) = 0 \quad (33)$$

Thus we have  $\gamma_{0i} = \gamma_{3i} = 0$  and  $\gamma_{00} = \gamma_{03} = 0$ .

In the vacuum we can project to transverse traceless gauge

$$h_{ij}^{\text{TT}} = \Lambda_{ij}^{kl} \gamma_{kl}, \quad (34)$$

where the projection tensor is given by

$$\Lambda_{ij}^{kl} = P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl}. \quad (35)$$

Using  $P_i^j = \delta_i^j - n_i n^j$  we have for an observer on the  $z$ -axis with  $\mathbf{n} = (0, 0, 1)$

$$h_{11}^{\text{TT}} = \frac{1}{2} (\gamma_{11} - \gamma_{22}) \quad h_{22}^{\text{TT}} = \frac{1}{2} (\gamma_{22} - \gamma_{11}) = -h_{11}^{\text{TT}} \quad h_{12}^{\text{TT}} = h_{21}^{\text{TT}} = \gamma_{12} \quad (36)$$

and all the other components vanish. Figure 2 shows the time dependence of the two independent polarizations  $h_+ = h_{11}^{\text{TT}}$  and  $h_\times = h_{12}^{\text{TT}}$ .

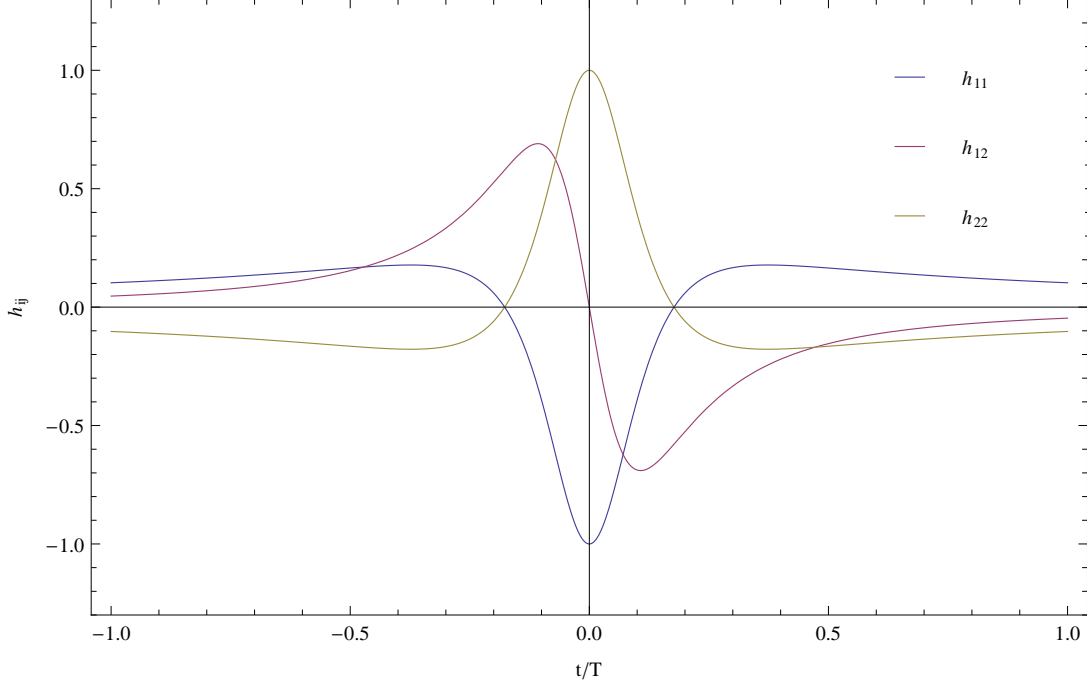


Figure 2: Time dependence of the  $h_{11}^{\text{TT}}$  and  $h_{22}^{\text{TT}}$   $h_{12}^{\text{TT}}$  components normalised to  $h_{11}(\varphi = 0)$ . The time is expressed in terms of  $T_o$

The timescale of the transit is given by the orbital time  $T_o = 2\pi\sqrt{\frac{b^3}{GM}}$ , whereas the amplitude is affected by  $T_{\text{gw}}^2/T_o^2$ , where we defined

$$T_{\text{gw}} = \sqrt{\frac{Gmb^2}{rc^4}} \quad (37)$$

and the time dependence of the orbit was obtained solving the implicit equation for  $\varphi(t)$

$$t(\varphi) = \sqrt{\frac{2b^3}{GM}} \left( \tan\left(\frac{\varphi}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\varphi}{2}\right) \right). \quad (38)$$