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Applications of General Relativity

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Solutions 8

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Exercise 1 [Non-monochromatic gravitational waves] (4 points)

We start from Eq. (5.54) in the script.

$$
\begin{aligned}
R^{(2)}_{\mu\kappa} =& -\frac{h^{\lambda\nu}}{2}\left[\frac{\partial^2 h_{\lambda\nu}}{\partial x^\mu \partial x^\kappa} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu}\right] + \\
&+ \frac{1}{4}\left[\frac{\partial h^\nu{}_\sigma}{\partial x^\nu} + \frac{\partial h^\nu{}_\sigma}{\partial x^\nu} - \frac{\partial h^\nu{}_\nu}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x^\kappa} + \frac{\partial h^\sigma{}_\kappa}{\partial x^\mu} - \frac{\partial h_{\mu\kappa}}{\partial x_\sigma}\right] - \\
&- \frac{1}{4}\left[\frac{\partial h_{\sigma\kappa}}{\partial x^\lambda} + \frac{\partial h_{\sigma\lambda}}{\partial x^\kappa} - \frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x^\mu} - \frac{\partial h^\lambda{}_\mu}{\partial x_\sigma}\right].
\end{aligned} \tag{1}
$$

We want to impose the two conditions $\partial_\mu \gamma^{\mu\nu} = 0$ (Lorentz gauge) and $\gamma = 0$ (traceless tensor). The second condition translates in $h = 0$, since $\gamma = -h$ if $\gamma_{\mu\nu} = h_{\mu\nu} - h\eta_{\mu\nu}/2$. Therefore, the first reads $\partial_\mu h^{\mu\nu} = 0$. Moreover, we have to remember that $\Box h_{\mu\nu} = 0$, because we are considering gravitational wave solutions. From the conditions above, we automatically have that the first term of the second row in Eq. (1) is zero and hence the second row itself. Now, we can exploit the fact that we are actually interested in the temporal average $\langle R^{(2)}_{\mu\kappa}\rangle$, where $\langle\cdot\rangle = (1/T)\int_0^T \cdot \, \mathrm{d}t$ (T is the wave period). In particular, every solution of a wave-like equation will depend on the spatial and temporal coordinates through an expression like $kx - \omega t$. This means that spatial and temporal derivatives will differ only by a constant, e.g. $\partial_1 = -k_1/k_0 \partial_0$. Therefore, inside the time integral of the temporal average, we can integrate any space-time derivate ∂_μ in the same way.

We can use this to manipulate the first and third row of Eq. (1). For the first row, we have:

$$
\begin{aligned}
&h^{\lambda\nu}\left[\partial_\mu\partial_\kappa h_{\lambda\nu} + \partial_\lambda\partial_\nu h_{\mu\kappa} - \partial_\lambda\partial_\kappa h_{\mu\nu} - \partial_\mu\partial_\nu h_{\lambda\kappa}\right] = \\
=&\partial_\mu(h^{\lambda\nu}\partial_\kappa h_{\lambda\nu}) - \partial_\mu h^{\lambda\nu}\,\partial_\kappa h_{\lambda\nu} + \partial_\lambda(h^{\lambda\nu}\partial_\nu h_{\mu\kappa}) - \partial_\lambda h^{\lambda\nu}\,\partial_\nu h_{\mu\kappa} \\
&- \partial_\lambda(h^{\lambda\nu}\partial_\kappa h_{\mu\nu}) + \partial_\lambda h^{\lambda\nu}\,\partial_\kappa h_{\mu\nu} - \partial_\nu(h^{\lambda\nu}\partial_\mu h_{\lambda\kappa}) + \partial_\nu h^{\lambda\nu}\,\partial_\mu h_{\lambda\kappa}.
\end{aligned} \tag{2}
$$

The only non-zero term is $-\partial_\mu h^{\lambda\nu}\,\partial_\kappa h_{\lambda\nu}$. All the terms of the form $\partial_\alpha(...)$ can be integrated directly in the temporal average and they give 0 because of the periodicity of the wave, while the others are

0 because of the above conditions. The third row of Eq. (1) can be written as:

$$\begin{aligned}
&\left[\frac{\partial h_{\sigma\kappa}}{\partial x^\lambda}+\frac{\partial h_{\sigma\lambda}}{\partial x^\kappa}-\frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x_\lambda}+\frac{\partial h^{\sigma\lambda}}{\partial x^\mu}-\frac{\partial h^\lambda{}_\mu}{\partial x_\sigma}\right]=\\
=&\partial_\lambda h_{\sigma\kappa}\,\partial^\lambda h^\sigma_\mu+\partial_\lambda h_{\sigma\kappa}\,\partial_\mu h^{\sigma\lambda}-\partial_\lambda h_{\sigma\kappa}\,\partial^\sigma h^\lambda_\mu\\
&+\partial_\kappa h_{\sigma\lambda}\,\partial^\lambda h^\sigma_\mu+\partial_\kappa h_{\sigma\lambda}\,\partial_\mu h^{\sigma\lambda}-\partial_\kappa h_{\sigma\lambda}\,\partial^\sigma h^\lambda_\mu\\
&-\partial_\sigma h_{\kappa\lambda}\,\partial^\lambda h^\sigma_\mu-\partial_\sigma h_{\kappa\lambda}\,\partial_\mu h^{\sigma\lambda}+\partial_\sigma h_{\kappa\lambda}\,\partial^\sigma h^\lambda_\mu=\\
=&\partial_\lambda(h_{\sigma\kappa}\partial^\lambda h^\sigma_\mu)-h_{\sigma\kappa}\Box h^\sigma_\mu+\partial_\lambda(h_{\sigma\kappa}\partial_\mu h^{\sigma\lambda})-h_{\sigma\kappa}\partial_\mu\partial_\lambda h^{\sigma\lambda}\\
&+\partial_\kappa h_{\sigma\lambda}\,\partial_\mu h^{\sigma\lambda}\\
&-\partial_\sigma(h_{\kappa\lambda}\partial_\mu h^{\sigma\lambda})+h_{\kappa\lambda}\partial_\mu\partial_\sigma h^{\sigma\lambda}\\
&+\partial_\sigma(h_{\kappa\lambda}\partial^\sigma h^\lambda_\mu)-h_{\kappa\lambda}\Box h^\lambda_\mu.
\end{aligned}\tag{3}$$

With the same considerations we did above, the only term that remains is $\partial_\kappa h_{\sigma\lambda}\,\partial_\mu h^{\sigma\lambda}$. We finally have that the temporal average of $R^{(2)}_{\mu\kappa}$ is:

$$\langle R^{(2)}_{\mu\kappa}\rangle=\frac{1}{2}\langle\partial_\mu h^{\lambda\nu}\,\partial_\kappa h_{\lambda\nu}\rangle-\frac{1}{4}\langle\partial_\kappa h_{\sigma\lambda}\,\partial_\mu h^{\sigma\lambda}\rangle=\frac{1}{4}\langle\partial_\mu h^{\lambda\nu}\,\partial_\kappa h_{\lambda\nu}\rangle \tag{4}$$

Note also that:

$$\left\langle R^{(2)}\right\rangle=\frac{1}{4}\left\langle\partial^\kappa h^{\lambda\nu}\partial_\kappa h_{\lambda\nu}\right\rangle=-\frac{1}{4}\left\langle h^{\lambda\nu}\partial^\kappa\partial_\kappa h_{\lambda\nu}\right\rangle=0 \tag{5}$$

due to the wave equation and the same consideration on the integration of ∂_μ terms. From the lecture we have:

$$t^{\mu\nu}_{\text{grav}}=\frac{c^4}{16\pi G}\left[2R^{(2)}_{\mu\nu}-\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}\right]=\frac{c^4}{32\pi G}\left\langle\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\right\rangle, \tag{6}$$

and since $\gamma=-h=0$ with tracelessness assumption, we finally have:

$$t^{\mu\nu}_{\text{grav}}=\frac{c^4}{32\pi G}\left\langle\partial_\mu\gamma^{\alpha\beta}\partial_\nu\gamma_{\alpha\beta}\right\rangle. \tag{7}$$

Exercise 2 [Plugging in the numbers] (7 points)

(i) We have:

$$t_{\text{spir}}=\frac{5c^5}{256G^3}\frac{r_0^4}{M_1M_2(M_1+M_2)}, \tag{8}$$

where $M_1=M_2=m$. We can estimate r_0 as:

$$r_0=\left(\frac{2Gm}{\omega_0^2}\right)^{1/3}, \tag{9}$$

where $\omega_0=2\pi/t_0$. Putting everything together we get:

$$t_{\text{spir}}=\frac{5}{2^{31/3}\pi^{8/3}}\left(\frac{c\,t_0^{1/3}}{(Gm)^{1/3}}\right)^5 t_0\simeq 10^7\text{ yr}. \tag{10}$$

(ii) As the two black holes are merging, we are in the limiting case where the Newtonian approximation breaks down. This is typically the case when the two black holes are separated by a distance comparable to their Schwarzschild radius, i.e. $R_\mathrm{s} = 2GM_\bullet/c^2$.

(a) The frequency of the gravitational waves is approximately:

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{GM_\bullet}{R_\mathrm{s}^3}} = \frac{1}{2^{5/2}\pi}\frac{c^3}{GM_\bullet} \simeq 10^{-5}\ \mathrm{Hz}. \tag{11}$$

(b) The dimensionless strain, defined as the relative amplitude of the oscillation of a ruler when a gravitational wave passes through, is:

$$h = |h_{ij}| \approx \frac{2GM_\bullet}{c^2 D} \sim 5\times 10^{-14} \tag{12}$$

(c) We take here the formula from part a) with $r_0 = R_\mathrm{s}$ and we find:

$$t_\mathrm{spir} = \frac{5\,G\,M_\bullet}{32c^3} \simeq 800\ \mathrm{s} \simeq 13\ \mathrm{min}. \tag{13}$$

(iii) We consider binaries that decay because of emission of gravitational waves. Therefore, their typical size is $R_\mathrm{s} \simeq GM/c^2$. The dimensionless strain is approximately:

$$h \sim R_\mathrm{s}/D, \tag{14}$$

where D is the distance of the source. The angular velocity of the source is approximately given by:

$$\omega \sim \sqrt{\frac{GM}{R_\mathrm{s}^3}} \sim \frac{c^3}{GM} \tag{15}$$

Then we have that the source mass is:

$$M \sim \frac{c^3}{G\omega} \sim 160\ \mathrm{M}_\odot, \tag{16}$$

with Schwartzchild radius:

$$R_\mathrm{s} \sim \frac{c}{\omega} \sim 240\ \mathrm{km}, \tag{17}$$

at a distance:

$$D \sim \frac{c}{\omega h} \sim 7\ \mathrm{Gpc}. \tag{18}$$

(a) The energy flux of the waves is:

$$F = \frac{P}{4\pi D^2} \sim \frac{1}{4\pi D^2}\frac{c^5}{G} \sim \frac{h^2\omega^2 c^3}{4\pi G} \sim 50\ \mathrm{erg\,s^{-1}\,cm^{-2}}. \tag{19}$$

(b) With the flux calculated above, the luminosity of a source at $x = 20$ Mpc would be:

$$L = 4\pi x^2 F \sim 10^{54}\ \mathrm{erg\,s^{-1}}. \tag{20}$$

(c) The Sun is a spherical black body of radius $R_\odot = 7 \times 10^{10}$ cm and temperature $T_\odot = 5780$ K. Its luminosity is thus: $L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4 = 3.9 \times 10^{33}$ erg s^{-1}. As its electromagnetic radiation emission is spherically symmetric, the relation between the flux F and distance d is:

$$F = \frac{L}{4\pi d^2} \Rightarrow d = \sqrt{\frac{L}{4\pi F}} \sim 166 \text{ AU}. \tag{21}$$

Thus, for the Sun to generate the same flux as these waves, it should be located at 166 times the distance Sun-Earth, and about 5 times the distance Sun-Pluto.