Applications of General Relativity

Sheet 10

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Exercise 1 [Post-Newtonian Lagrangian for a binary system] (10 points)

The gravitational action for a particle a propagating in the metric $g_{\mu\nu}$ is given by

$$S_a = -m_a c \int \mathrm{d}t \left(-g_{\mu\nu} \frac{\mathrm{d}x_a^\mu}{\mathrm{d}t} \frac{\mathrm{d}x_a^\nu}{\mathrm{d}t} \right)^{1/2}, \tag{1}$$

where $t = x^0/c$ (not the proper time of the particle) and $\mathrm{d}x^\mu/\mathrm{d}t = (-c, \mathbf{v})$. In this problem we are considering two masses m_1 and m_2 with separation r. Note that the total action is given by $S = S_1 + S_2$ (paying attention to not double count the terms of the potential energy) and that particle 1 propagates in the metric generated by particle 2 and vice versa.

(i) Expand the action to order 1PN. Using the definitions:

$$^{(2)}g_{00} = -2\phi, \qquad ^{(2)}g_{ij} = -2\phi\delta_{ij}, \tag{2}$$

$$^{(3)}g_{0i} = \zeta_i + \partial_i\partial_0\chi, \qquad ^{(4)}g_{00} = -2\left(\phi^2 + \psi\right), \tag{3}$$

you should finally get, for e.g. particle m_1:

$$\mathcal{L} = m_1 c^2 \left\{ \frac{1}{2}\left(\frac{v_1}{c}\right)^2 - \phi + \frac{1}{8}\left(\frac{v_1}{c}\right)^4 - \frac{\phi^2}{2} - \psi - \frac{3\phi}{2}\left(\frac{v_1}{c}\right)^2 + \zeta_i \frac{v_1^i}{c} + \frac{v_1^i}{c}\partial_i\partial_0\chi \right\}. \tag{4}$$

(Hint: remember that 1PN means to expand up to ϵ^4, where $\epsilon = v/c$).

(ii) Expand the energy momentum tensor:

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_a \frac{\mathrm{d}t}{\mathrm{d}\tau_a} m_a \frac{\mathrm{d}x_a^\mu}{\mathrm{d}t} \frac{\mathrm{d}x_a^\nu}{\mathrm{d}t} \delta^{(3)}\left(\boldsymbol{x} - \boldsymbol{x}_a(t)\right) \tag{5}$$

to order 1PN. Here τ_a is the proper time of particle a.
(Hint: remember that the metric in which m_1 propagates is given by m_2 only, and so $T^{\mu\nu}$. Moreover, only terms up to ϵ^2 are necessary in the expansion of $T^{\mu\nu}$. Note also that $\mathrm{d}\tau/\mathrm{d}t = \mathcal{L}$.)

(iii) From the results at point (ii), compute the actual values of the terms in $\mathcal{L}$, using the definitions:

$$\phi(t, \boldsymbol{x}) = -\frac{G}{c^4} \int \frac{\mathrm{d}^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \, ^{(0)}T^{00}(t, \boldsymbol{x}'), \tag{6}$$

$$\zeta_i(t, \boldsymbol{x}) = -\frac{4G}{c^4} \int \frac{\mathrm{d}^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \, ^{(1)}T^{0i}(t, \boldsymbol{x}'), \tag{7}$$

$$\psi(t, \boldsymbol{x}) = -\frac{G}{c^4} \int \frac{\mathrm{d}^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \left[^{(2)}T^{00}(t, \boldsymbol{x}') + ^{(2)}T^{ii}(t, \boldsymbol{x}') \right], \tag{8}$$

$$\chi(t, \boldsymbol{x}) = -\frac{G}{2c^4} \int \mathrm{d}^3x' \, |\boldsymbol{x} - \boldsymbol{x}'| \, ^{(0)}T^{00}(t, \boldsymbol{x}'). \tag{9}$$

Hint: while computing ψ, there is a term in ${}^{(2)}T^{00}$ that depends on ϕ. For a system of 2 particles only, it contributes as 0 in computing ψ and you can neglect it.)

(iv) Plug the metric perturbations into the Lagrangian. For e.g. particle m_1, you will get:

$$\mathcal{L} = \frac{1}{2}m_1 v_1^2 + \frac{G m_1 m_2}{r_{12}} + \frac{1}{8}m_1\frac{v_1^4}{c^2} - \frac{G^2 m_2^2 m_1}{r_{12}^2 c^2} \tag{10}$$

$$+ \frac{G m_1 m_2}{r_{12}}\left\{3\left[\left(\frac{v_1}{c}\right)^2 + \left(\frac{v_2}{c}\right)^2\right] - 7\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2} - \frac{(\mathbf{v}_1\cdot\hat{\mathbf{r}}_{12})(\mathbf{v}_2\cdot\hat{\mathbf{r}}_{12})}{c^2}\right\}. \tag{11}$$

Sum it up with the equivalent lagrangian for particle m_2 (do not forget to symmetrize all the terms in which both index 1 and 2 appear simmetrically, which means counting them only once!). You will get finally:

$$\mathcal{L} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{G m_1 m_2}{r_{12}} + \frac{1}{8}m_1\frac{v_1^4}{c^2} + \frac{1}{8}m_2\frac{v_2^4}{c^2} \tag{12}$$

$$- \frac{G^2 m_2 m_1 (m_1 + m_2)}{2 r_{12}^2 c^2} \tag{13}$$

$$+ \frac{G m_1 m_2}{r_{12}}\left\{3\left[\left(\frac{v_1}{c}\right)^2 + \left(\frac{v_2}{c}\right)^2\right] - 7\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2} - \frac{(\mathbf{v}_1\cdot\hat{\mathbf{r}}_{12})(\mathbf{v}_2\cdot\hat{\mathbf{r}}_{12})}{c^2}\right\}. \tag{14}$$