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# Applications of General Relativity

# Solutions 8

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Website: `http://www.physik.uzh.ch/en/teaching/PHY519/`

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**Exercise 1** [Hamiltonian geodesic formulation in PN framework]

a) The Lagrangian for the full theory is

$$L = \frac{1}{2}\left[-\left(1-\frac{r_s}{r}\right)\dot{t}^2 + \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right]. \tag{1}$$

The Hamiltonian for the full theory is

$$\mathcal{H} = \frac{1}{2}\left[-p_t^2\left(1-\frac{r_s}{r}\right)^{-1} + p_r^2\left(1-\frac{r_s}{r}\right) + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right]. \tag{2}$$

b) This gives us the equation of motion:

$$\dot{t} = -p_t\left(1-\frac{r_s}{r}\right)^{-1}, \tag{3}$$

$$\dot{r} = p_r\left(1-\frac{r_s}{r}\right), \tag{4}$$

$$\dot{\theta} = \frac{p_\theta}{r^2}, \tag{5}$$

$$\dot{\phi} = \frac{p_\phi}{r^2\sin^2\theta}, \tag{6}$$

$$\dot{p}_t = 0, \tag{7}$$

$$\dot{p}_r = \frac{r_s}{r^2}\left(-\frac{p_t^2}{2}\left(1-\frac{r_s}{r}\right)^{-2} - \frac{p_r^2}{2} + \frac{p_\theta^2}{r r_s} + \frac{p_\phi^2}{r r_s\sin^2\theta}\right), \tag{8}$$

$$\dot{p}_\theta = \frac{p_\phi^2\cot\theta}{r^2\sin^2\theta}, \tag{9}$$

$$\dot{p}_\phi = 0. \tag{10}$$

(c) Let's introduce a parameter $\epsilon$ to help us keep track of our orders. Let's choose $r_s = \mathcal{O}(\epsilon^2)$. For a massive particle we have $v \ll c$, so we find $p_t = \mathcal{O}(1)$, $p_r, p_\theta, p_\phi = \mathcal{O}(\epsilon)$, while for a massless particle we have $p_\mu = \mathcal{O}(1)$. To expand the EOM we e.g. make the replacements $r_s \to \epsilon^2 r_s, p_r \to \epsilon\, p_r, p_\theta \to \epsilon\, p_\theta, p_\phi \to \epsilon\, p_\phi$ and then expand to second order around $\epsilon = 0$. Applying this procedure we find the following equations of motion:

| massless | massive | |
|---|---|---|
| $\dot{t} = -p_t\left(1+\frac{r_s}{r}\right)$ | $\dot{t} = -p_t\left(1+\frac{r_s}{r}\right)$ | (11) |
| $\dot{r} = p_r\left(1-\frac{r_s}{r}\right)$ | $\dot{r} = p_r$ | (12) |
| $\dot{\theta} = \frac{p_\theta}{r^2}$ | $\dot{\theta} = \frac{p_\theta}{r^2}$ | (13) |
| $\dot{\phi} = \frac{p_\phi}{r^2\sin^2\theta}$ | $\dot{\phi} = \frac{p_\phi}{r^2\sin^2\theta}$ | (14) |
| $\dot{p}_t = 0$ | $\dot{p}_t = 0$ | (15) |
| $\dot{p}_r = -\frac{p_t^2 r_s}{2r^2} - \frac{p_r^2 r_s}{2r^2} + \frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3\sin^2\theta}$ | $\dot{p}_r = -\frac{p_t^2 r_s}{2r^2} + \frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3\sin^2\theta}$ | (16) |
| $\dot{p}_\theta = \frac{p_\phi^2\cot\theta}{r^2\sin^2\theta}$ | $\dot{p}_\theta = \frac{p_\phi^2\cot\theta}{r^2\sin^2\theta}$ | (17) |
| $\dot{p}_\phi = 0$ | $\dot{p}_\phi = 0$ | (18) |

(d) To plot trajectories we first fix the orbital plane by setting $\theta = \pi/2$ and therefore $p_\theta = 0$. Then we can fix two more constants, $p_t$ corresponding to energy and $p_\phi$ corresponding to angular momentum. Afterwards we can integrate the equations for $\dot{r}$, $\dot{\phi}$ and $\dot{p}_r$ numerically with suitable initial conditions. Knowing $r$ and $\phi$ we can plot some trajectories in the orbital plane. We can see that the closer to the horizon we come the more the approximate solution deviates from the full solution. Note also that the full solution breaks down at the horizon due to the coordinate singularity of Schwarzschild coordinates, whereas the approximate solution is just physically invalid near the horizon.

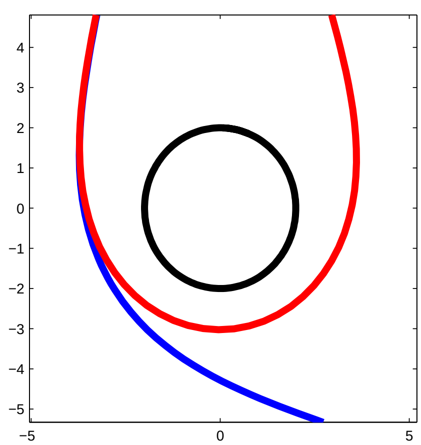


Figure 1: Two photons, each with the same impact parameter around a Schwarzschild black hole. Black shows the event horizon surface. Both photons have the same initial conditions, yet the red photon has been calculated using the full metric, and the blue the PN metric. Note that the red photon grazes the *photon sphere* at $r = 3r_s$.