MMP I
Solution Sheet 12

HS 21
Prof. Ph. Jetzer

L. Buonocore, M. Loechner, X. Liu, M. Ebersold
https://www.physik.uzh.ch/en/teaching/PHY312

Issued: 09.12.2021
Due: 16.12.2021

Exercise 1 [Multipole decomposition (6 points)]

a)

$$|\vec{x}-\vec{x}'|^2 = (\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}') = \vec{x}^2+\vec{x}'^2-2\vec{x}\cdot\vec{x}' = r^2+r'^2-2rr'\cos(\gamma)$$

$|\vec{x}| \equiv r$; $|\vec{x}'| \equiv r'$ and γ the angle between $\vec{x}$ and $\vec{x}'$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos(\gamma)}} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma)$$

$$|\vec{x}'|<|\vec{x}| \quad \Rightarrow \quad r_> = |\vec{x}| \equiv r$$
$$r_< = |\vec{x}'| \equiv r'$$

$$\Rightarrow \phi(x) = \int_{\mathbb{R}^3}\frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|}d^3x' = \int_{\mathbb{R}^3}\rho(\vec{x}')\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma)d^3x'$$
$$= \sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_{\mathbb{R}^3}\rho(\vec{x}')r'^l P_l(\cos\gamma)d^3x'$$

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{l,m}(\theta,\varphi)Y^*_{l,m}(\theta',\varphi')$$

$$\Rightarrow \phi(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\underbrace{\int_{\mathbb{R}^3}\sqrt{\frac{4\pi}{2l+1}}\rho(\vec{x}')r'^l Y^*_{l,m}(\theta',\varphi')d^3x'}_{q_{l,m}}\frac{Y_{l,m}(\theta,\varphi)}{r^{l+1}}$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}q_{l,m}\frac{Y_{l,m}(\theta,\varphi)}{r^{l+1}}$$

with

$$q_{l,m} = \int_{\mathbb{R}^3}\sqrt{\frac{4\pi}{2l+1}}\rho(\vec{x}')r'^l Y^*_{l,m}(\theta',\varphi')d^3x'$$

$$= \iiint \rho(r', \theta', \varphi') r'^{l+2} \sqrt{\frac{4\pi}{2l+1}} Y^*_{l,m}(\theta', \varphi') \sin\theta' dr' d\theta' d\varphi'$$

b)

$$q_{0,0} = \int_{\mathbb{R}^3} \rho(\vec{x}') \sqrt{4\pi} Y^*_{0,0}(\theta', \varphi') d^3x'$$

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}$$

$$\Rightarrow q_{0,0} = \int_{\mathbb{R}^3} \rho(\vec{x}') d^3x' \mathrel{\widehat{=}} \begin{cases} \text{total charge (electrostatics)} \\ \text{total mass (gravitation)} \end{cases}$$

Exercise 2 [Distributions (3 points)]]

a) A distrubution T is a *linear* and *continuous* functional which is defined for all test functions satisfying

- $F(x) \in \mathbb{R}$, $x \in \mathbb{R}$
- $F(x) \in C_\infty(\mathbb{R})$
- $\forall n > 0$: $F^{(n)}(x) = \mathcal{O}\left(\frac{1}{|x|^n}\right)$ for $x \to \pm\infty$ (function and its derivatives vanish at infinity)

Functional: $\langle T, F(x) \rangle = \alpha \in \mathbb{R}$

Linearity: $\langle T, \alpha F_1 + \beta F_2 \rangle = \alpha \langle T, F_1 \rangle + \beta \langle T, F_2 \rangle$

The Dirac delta distribution is defined through $\langle \delta(x), F(x) \rangle = F(0)$ or, explicitly: $\int\limits_{-\infty}^{\infty} \delta(x) F(x) dx = F(0)$

b) spherical coordinates: $\vec{r} = \begin{pmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\varphi \end{pmatrix}$

$\to r \in [0, \infty)$, $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi) \Rightarrow \sin\theta = |\sin\theta|$

Integration measure: $d^3x = |J| dr d\varphi d\theta = r^2 \sin\theta dr d\varphi d\theta$

$\delta(y(x)) = \sum\limits_i \frac{\delta(x - x_i)}{|y'(x_i)|}$ for all i where $y(x_i) = 0$

$\Rightarrow \delta(\underbrace{\cos\theta - \cos\theta'}_{y(\theta)}) = \frac{\delta(\theta - \theta')}{|\sin\theta'|}$

Where $y'(\theta) = \sin\theta$ and for $\theta \in [0, \pi]$:

$y(\theta) = 0 \Leftrightarrow \cos\theta = \cos\theta' \Leftrightarrow \theta = \theta'$

$$\Rightarrow \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_0^{\infty} dr \delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\varphi - \varphi_0)\frac{1}{r^2} r^2 \sin\theta$$

$$\overset{\sin\theta = |\sin\theta|}{=} \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_0^{\infty} dr \delta(r - r_0)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0)$$

$$= \underbrace{\int_0^{\pi} d\theta \delta(\theta - \theta_0)}_{1} \underbrace{\int_0^{2\pi} d\varphi \delta(\varphi - \varphi_0)}_{1} \underbrace{\int_0^{\infty} dr \delta(r - r_0)}_{1} = 1$$

by the definition of the delta distribution.