CHAPTER 1

Introduction to Effective Field Theories

"We demand rigidly defined areas of doubt and uncertainty!"

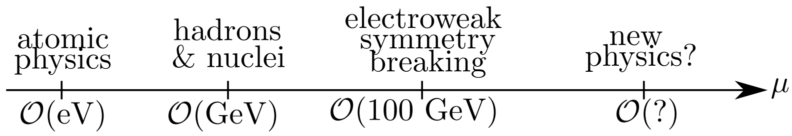
— *Douglas Adams*, The Hitchhiker's Guide to the Galaxy

If one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles.

— *Steven Weinberg*, Phenomenological Lagrangians [1]

1.1 Why Effective Field Theories?

The Standard Model (SM) of particle physics [2] is a powerful theory, capable of describing phenomena across a wide range of scales:



This makes it a powerful and versatile tool. But are all of these scales always important? The answer is clearly no: When we are describing low-energy physics, like for example the spectrum of hydrogen, we typically ignore effects of particles of weak-scale masses completely and expand around the limit of the proton being infinitely heavy. Similarly, when describing processes at very high energy, masses of light particles can typically be ignored. Of course, these statements depend on the level of precision with which we want to describe physics. At high precision, we should include also the small corrections from subleading effects.

The core concept goes beyond just quantum field theories: When asked to compute the velocity of an apple hitting the ground after falling from a tree, we would simply compute

$$mgh = \frac{mv^2}{2}\,, \qquad\qquad \Rightarrow v = \sqrt{2gh}\,, \tag{1.1}$$

even though we know that the gravitational potential is not linear in h. However, since the height of the tree is small compared to the scale over which the force of gravity changes (the radius of the earth R), the above result is accurate. Corrections to it would arise with a parametrical suppression of the relative order of $\mathcal{O}(h/R)$, which is roughly $\sim 10^{-6}$ for a typical apple tree on earth. The linear gravitational potential can be thought of an effective theory for the more complete Newtonian theory of gravity (which in turn, can be thought of as an effective theory for General Relativity).

Effective Field Theories (EFTs) are quantum field theories (QFTs) that are less general by construction. They focus on an isolated region compared to a more complete QFT (for example the SM), for which they are designed and treat effects from other regions as perturbations in a well-defined and systematic way. As an example, consider a theory of two real scalars, ϕ and φ with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{g}{3!}\varphi^3\Phi\,. \tag{1.2}$$

Let Φ be much heavier than φ, meaning $M \gg m$ and let us consider a process at very low energy $E \ll M$. Processes with intermediate Φ particles will then be suppressed by the propagator

$$\langle 0|T\{\Phi(0)\Phi(x)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2 - M^2}\,, \tag{1.3}$$

where $k^2 \sim \mathcal{O}(E^2) \ll M^2$. We can see immediately, that neglecting k^2 makes the expressions we are dealing with structurally simpler while still being a good approximation up to corrections of order $\mathcal{O}(k^2/M^2)$.

The next important point is that at low energies the heavy scalar Φ cannot be produced as a real particle. We should therefore be able to describe physics with a Lagrangian that contains only φ:

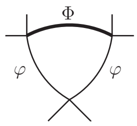
$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \Delta\mathcal{L}\,. \tag{1.4}$$

Here $\Delta\mathcal{L}$ is a new ingredient with interactions of φ that were previously not part of the Lagrangian (1.2). While in the full theory we had processes of the form $\varphi^3 \to \varphi^3$ through virtual Φ particles, the interaction terms generating these amplitudes are missing from the effective Lagrangian since it does not contain Φ. Therefore, we must include an interaction of the form

$$\Delta\mathcal{L} \supset \frac{\mathcal{C}_6}{M^2}\varphi^6\,, \tag{1.5}$$

to describe this process. Note how this operator needs to have a prefactor with two inverse powers of mass. We have chosen the heavy mass as a prefactor $1/M^2$ with no further explanation other than the propagator of Φ being of this form in the low-energy limit, but we will justify this later on in more detail.

You might now ask, why we need an effective Lagrangian when we can simply compute amplitudes in the full theory and expand them in the relevant limits we are interested in. And in fact, most of the times we need to do just that anyway to determine the coupling coefficients in what we called $\Delta\mathcal{L}$ above. The answer seems technical at first, but it is an important one. The issue hides at the loop-level, when we are computing radiative corrections. As an example, take the Lagrangian (1.2) again. At one loop, the interactions in this theory generate a contribution to the $\varphi^3 \to \varphi^3$ process of the form:



$$\begin{aligned} &= \frac{ig^2}{M^2}\left\{(i\lambda)M^2 \int \frac{d^d l}{(2\pi)^d}\frac{i}{l^2 - M^2}\left(\frac{i}{l^2 - m^2}\right)^2\right\} \\ &= \frac{ig^2}{M^2}\left\{-\frac{\lambda}{16\pi^2}\left[1 + \log\frac{m^2}{M^2} + \mathcal{O}\left(\frac{m^2}{M^2}\right)\right] + \mathcal{O}(\lambda^2)\right\}\,, \end{aligned} \tag{1.6}$$

where we have factored out the tree-level expression ig^2/M^2. While this contribution is formally of higher order in the expansion in the coupling λ, it exhibits a logarithm of the two masses in the process. For extreme enough values of the mass ratio m^2/M^2, this logarithm can become large enough to overpower the suppression of the coupling constant and spoil the convergence of our perturbation expansion. In this case, we should count these *large logarithms* as inverse powers in the coupling and assign them to the leading-order contribution.

We can already anticipate that these offending terms are absent in the effective theory: There will not be propagators of the form $1/(l^2 - M^2)$ so the matrix elements there cannot give rise to logarithms of this mass. We will later see how EFTs provide a way of reorganizing the perturbation expansion and correctly including these large logarithmic contributions to any order in $[\lambda \log(m^2/M^2)]^n$ through the means of the renormalization group. In the jargon of the field, this goes by the term of *resummation of large logarithms*.

1.2 Core concepts

Let us now cast our collection of vague ideas into a more systematic approach. What goes into developing an EFT? The three key ingredients are:

1. **Identify the limit in which we want to work**
   Once we decide on the kind of approximation we want to work in (for example a particle being very heavy, a particle moving a very low velocities, etc), we need to find a way to turn this idea into a parametric limit in the theory. This means that we need to indentify the parameters that we wish to treat as small or large so that we can systematically expand in it. In our example of a heavy particle at low energies, this expansion parameter would be E/M.

   In this sense we define the *power-counting* of the effective theory and our Lagrangian, along with the fields are defined as expansions in this parameter.

The power-counting is a profound ingredient and being consistent in it is just as important as the counting in the coupling constants of the Lagrangian.

2. **Determine the relevant degrees of freedom**
In this step, one defines the building blocks out of which the effective Lagrangian is pieced together. This step can be as simple as dropping heavy particles and keeping only the light ones. However, there are EFT constructions in which this step is considerably more involved. We will encounter an example later when we discuss Chiral Perturbation Theory (χPT), the low-energy effective theory of QCD. Other examples of EFTs with non-trivial field content are the Heavy Quark Effective Theory (HQET) [3] and the Soft-Collinear Effective Theory (SCET) [4].

3. **Determine the symmetries of the theory**
As with any theory in physics, knowing its symmetries is crucial in order to constrain possible interactions, especially in cases in which we do not know the full theory and we simply write down all possible operators in agreement with the symmetries and the power-counting of the theory. In other cases, we might gain new symmetries from the expansion one employs in the first step. An example would be HQET, a theory build for the physics of B-mesons. These mesons consist of a heavy quark (in this case the b-quark) and a light flavor and one expands in the heavy b-quark mass. At leading order, this theory predicts equal masses for the B-meson and its excited spin-1 counterpart, the B^*-meson, because spin-effects are suppressed by the b-mass. Indeed, the mass difference between the B and the B^* is tiny compared to their mass. This *heavy quark spin symmetry* is a new symmetry of the effective theory at leading power in the expansion parameter.

Before formalizing our ideas more, we should also clarify some more terminology. First off, we will very often refer to the region of high energy as *ultraviolet (UV)* and analogously to the low-energy region as *infrared (IR)*. Alternative jargon used for UV and IR are *hard* and *soft*. Another set of phrases are *top-down* and *bottom-up*:

- Doing a **top-down** construction means that we know the full theory is known and we find it useful (or necessary) to neglect UV effects. Most of the ideas and examples discussed to this point followed this philosophy: Using the full theory makes computations unnecessarily complicated and can even spoil perturbation theory and we therefore go to an effective description. Typical examples are HQET, SCET, and the so-called *Fermi theory*.

- On the other hand, we could also construct an EFT by deciding on the degrees of freedom, power-counting and symmetries and then simply writing down all possible operators in accordance with the symmetries up to a desired order in power-counting, with undetermined couplings. This type of construction is referred to as **bottom-up**. A typical example of this is the Standard Model Effective Field Theory (SMEFT) and χPT, and we will discuss both later.

1.3 Construction

We now want to formalize our core concepts and create a step-by-step program.

1. First, we choose a cutoff, which we will call Λ, and divide the field content of the theory. Let ϕ be a generic placeholder for fields of the theory. Then:

$$\phi = \phi_H + \phi_S \,, \tag{1.7}$$

where the subscripts H and S correspond to *hard* and *soft*, respectively. In this way, ϕ_S contains all Fourier modes of fields with frequencies below the cutoff $\omega < \Lambda$, whereas ϕ_H denotes the modes living above the cutoff, $\omega > \Lambda$.

2. Physics at low energies are by construction now described exclusively by the modes denoted by ϕ_S. Matrix elements derived at low energies are then given by n-point vacuum correlators of the soft modes, which are of the form:

$$\langle 0|T\{\phi_S(x_1)\dots\phi_S(x_n)\}|0\rangle = \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J_S(x_1)}\right)\dots\left(-i\frac{\delta}{\delta J_S(x_n)}\right) Z[J_S]\bigg|_{J_S=0} \,, \tag{1.8}$$

where the generating function of the theory is

$$Z[J_S] = \int \mathcal{D}\phi_S\,\mathcal{D}\phi_H \exp\left\{iS(\phi_S,\phi_H) + i\int d^dx J_S(x)\phi_S(x)\right\} \,. \tag{1.9}$$

From this formula, we can define the Wilsonian effective action:

$$\int \mathcal{D}\phi_H\, \exp\{iS(\phi_S,\phi_H)\} \equiv \exp\{iS_\Lambda(\phi_S)\} \,. \tag{1.10}$$

In computing this quantity, the path integral over the hard modes is performed and they are removed from the theory - they are "integrated out". The Wilsonian action encodes all the interactions at the hard scales. Once these are integrated out, the generating functional becomes:

$$Z[J_S] = \int \mathcal{D}\phi_S \exp\left\{iS_\Lambda(\phi_S) + i\int d^dx J_S(x)\phi_S(x)\right\} \,. \tag{1.11}$$

3. The object S_Λ encodes interactions at the hard scale Λ. In position space this means that it is non-local at scales $\Delta x \sim 1/\Lambda$ by definition. Since it no longer depends on the hard modes ϕ_H, it can be written as a series of interactions between fields ϕ_S with positions x_1 to x_n and $\Delta x^\mu_{ij} = (x_i - x_j)^\mu \sim \mathcal{O}(1/\Lambda)$. Since the wavelengths of the fields are large compared to Δx, we can expand around $\Delta x = 0$. In this limit, the product of operators is local, meaning they all depend on a single position x. This means, our next step is to write S_Λ as a local operator product expansion (OPE):

$$S_\Lambda(\phi_s) = \int d^dx\,\mathcal{L}_{\text{eff}}(x)\,, \qquad \text{with} \quad \mathcal{L}_{\text{eff}} = \sum_i g_i(\mu)\mathcal{O}_i\,, \tag{1.12}$$

where we introduced the effective Lagrangian $\mathcal{L}_{\text{eff}}$. This object is in principle an infinite series of operator products allowed by the symmetries of the theory. Once we have identified or decided on the power-counting parameter of the EFT, we can assign a power-counting to each operator $\mathcal{O}_i$ and truncate the sum at a given order.

Some comments are in order. An important point is the separation in step 1. As stated previously, this largely determines the construction of the effective theory. The common way to decide whether a mode is hard or soft is through its contribution to the matrix elements we are computing. In terms of Feynman diagrams, the contribution of a field to an amplitude is through propagators, so it is intuitive to determine the power-counting from the two-point correlators. At the example of a scalar field this would be:

$$\langle 0|T\{\varphi(0)\varphi(x)\}|0\rangle = \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot x} \frac{i}{k^2 - m_\varphi^2}\,. \tag{1.13}$$

Counting the powers of some expansion parameter λ in this expression tells you the order of two insertions of φ. Let us clarify this by a few examples: Consider again the theory with the Lagrangian (1.2). We want to build an effective theory for scattering processes of the light fields ϕ at very low energies E, much lower than the mass of the heavy fields Φ. This means, the momenta in the above formula are all $k^2 \sim \mathcal{O}(E^2)$ and we are power-counting in $\lambda = E/M$. Then we find the power-counting for the light fields by (setting $d = 4$ and counting $m^2 \sim E^2$):

$$\begin{aligned} \langle 0|T\{\phi(0)\phi(x)\}|0\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - m^2} \\ &\sim \quad \mathcal{O}(E^4) \quad \cdot \quad \mathcal{O}(E^{-2}) = \mathcal{O}(E^2) = \mathcal{O}(\lambda^2 M^2)\,. \end{aligned} \tag{1.14}$$

We thus see that two insertions of ϕ count as $\lambda^2 M^2$ and consequently a single field operator counts as λM. On the other hand for the heavy scalar we find:

$$\begin{aligned} \langle 0|T\{\Phi(0)\Phi(x)\}|0\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - M^2} \approx \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{-M^2} \\ &\sim \quad \mathcal{O}(E^4) \quad \cdot \quad \mathcal{O}(M^{-2}) = \mathcal{O}(\lambda^4 M^2)\,, \\ &\Rightarrow \ \Phi^2 \sim \lambda^4 M^2 \quad \Rightarrow \quad \Phi \sim \lambda^2 M\,. \end{aligned} \tag{1.15}$$

Therefore, if we take the hard scale M as reference, the heavy fields count as λ^2 and the light fields count as λ. The two take-away messages from this back-of-the-envelope calculation are:

1. We have seen directly that the heavy fields yield subleading contributions. This gives us a parametrical justification as to why we can remove them from the theory.

2. The power-counting of the modes is directly connected to their frequencies. The light fields have low invariant mass and thus count as soft. The heavy fields count as hard, since even for low momenta, their correlators are governed by the hard scale M.

Going back to our example, we now also know how to count the operators in the infinite sum in eq. 1.12: Each insertion of a soft field counts as one power of λ. Similarly, a derivative acting on the fields will contribute a power of the field momentum to the amplitude. Since we are counting $k^\mu \sim \lambda$, derivatives consequently also count as one power of λ. The result (and this hold for fermions and gauge fields as well), is that the order of an operator in our example is simply λ^{n-4} for n being the mass dimension of the operator. Therefore we can tweak our notation for $\mathcal{L}_{\text{eff}}$ in eq. 1.12 by turning the couplings g_i into dimensionless numbers and organize the OPE in our power-counting:

$$\mathcal{L}_{\text{eff}} = \sum_i g_i(\mu)\mathcal{O}_i = \sum_k^{\infty} \frac{1}{\Lambda^{k-4}} \sum_i^{n_k} \mathcal{C}_{(i,k)}(\mu)\mathcal{O}_{(i,k)}\,. \tag{1.16}$$

Here n_k denotes the number of operators at order k. The dimensionless coupling coefficients of the operators $\mathcal{C}_j$ are called *Wilson coefficients*. They encode the short-distance physics that we have integrated out, hence they are sometimes called *short-distance coefficients*. At a given order k, the number of operators n_k is finite. Once we truncate the sum over k at a finite value N, $\mathcal{L}_{\text{eff}}$ will contain a finite number of operators.

While the example of heavy and light particles in low-energy processes is by far the most common type of EFT construction, let us briefly digress into a different example to point out some points that are generic to the idea of EFTs and some points that are specific to the example at hand.

The light fields in our previous example counted as soft because of their low invariant mass. We immediately knew that their invariant mass would be small since we assumed every component of their momenta to be small, $k \sim E = \lambda M$. However, when looking at eq. (1.14), it is clear that it suffices to have k^2 to be small. This can be the case for massless, highly energetic fields with $k^\mu \sim (M, 0, 0, M)$. In this example, the field would still count as soft, since $k^2 \sim 0$ even though individual components of its momentum k^μ are large. Note that in contrast to our previous example, derivatives in the operators do not necessarily yield a power-suppression, since the derivative along the direction of k is actually of the high scale. Similarly, a completely massless field can count as hard, if it is highly off-shell. Its propagator would then be of the form $1/k^2$ with $k^2 \sim M^2$. This example leads to the Soft-Collinear Effective Theory, an effective theory developed for hard processes in QCD.

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