PHY 127 FS2026

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Lecture 5
March 20th, 2026

Remember what you learned
about standing waves

LECTURE 4 reminder

(1) wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

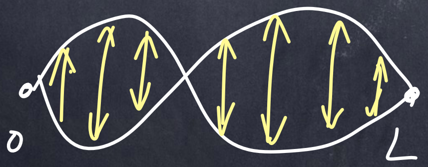
solutions:

$$y = A\sin(kx - \omega t)$$

For a standing wave on a string:

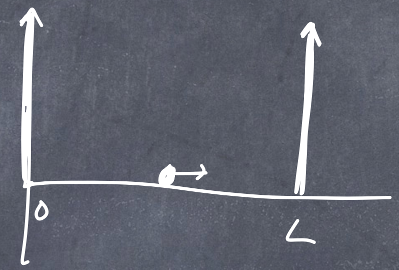
$$y = A_n \sin\frac{n\pi x}{L}$$

$$n = 1, 2, 3, \ldots$$



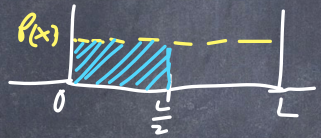
$n=2$

(2) "classical" 1-D particle in a box:



Probability is equal for it to be anywhere

$$P(x) = \frac{1}{L}$$



$$P_{0\to\frac{L}{2}} = \frac{1}{2}$$

(3) A photon has a momentum associated to its wavelength

$$p = \frac{h}{\lambda}$$

and behaves both like a particle and a wave

Today a related concept. All particles (like electrons) also behave as waves.

The wavelength of a particle is $\boxed{\lambda = \frac{h}{p}}$

p: momentum

This is known as the deBroglie wavelength.

Consider the energy associated with an electron:

$$\underset{\substack{\text{total}\\\text{energy}}}{E} = \underset{\substack{\text{kinetic}\\\text{energy}}}{K} + \underset{\substack{\text{potential}\\\text{energy}}}{U} \qquad \text{(classical formula)}$$

$$K = \frac{1}{2}mv^2$$

since $p = mv \Rightarrow K = \frac{p^2}{2m}$

and since $p = \frac{h}{\lambda} \Rightarrow K = \frac{h^2}{\lambda^2 2m}$

so $E = \frac{h^2}{\lambda^2 2m} + U$

Now we explore what the wave nature of electrons means when we measure them.

First, a reminder of some properties of waves.

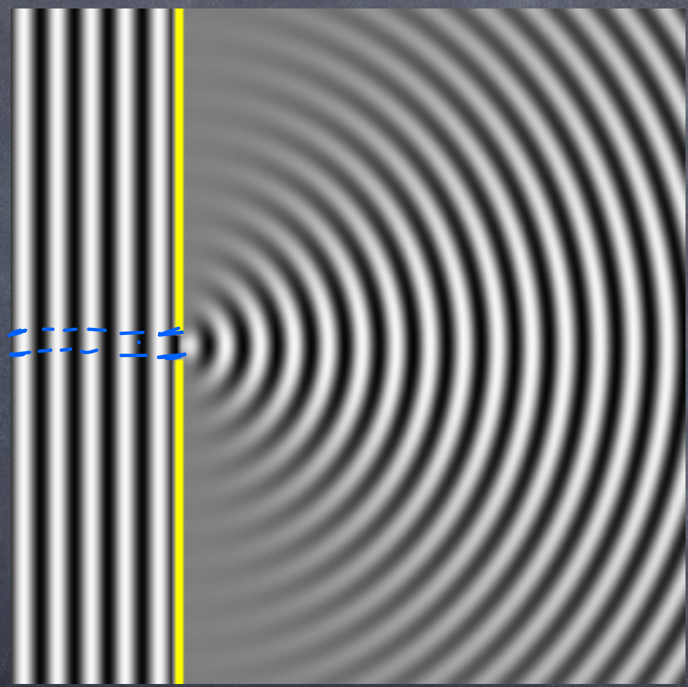
Phenomena of diffraction

plane wave moving to the right.

encounters a small hole of size

$a \sim \lambda$

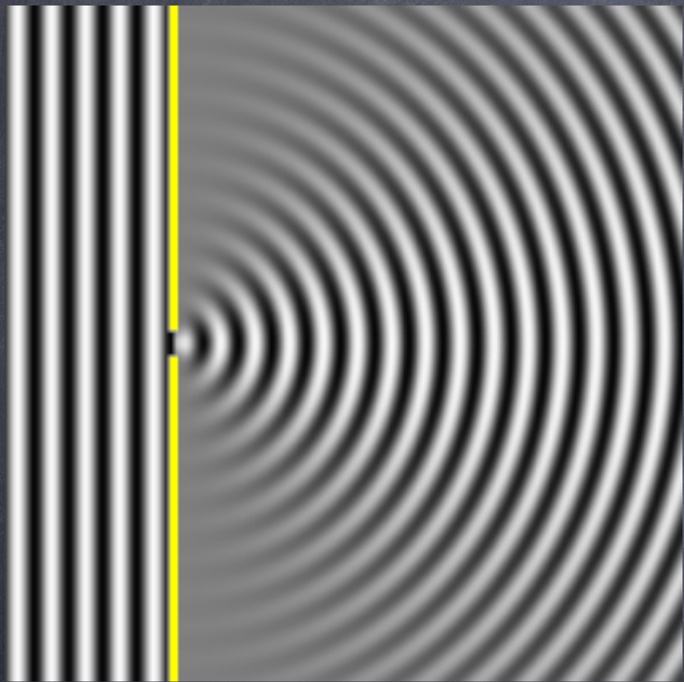
a



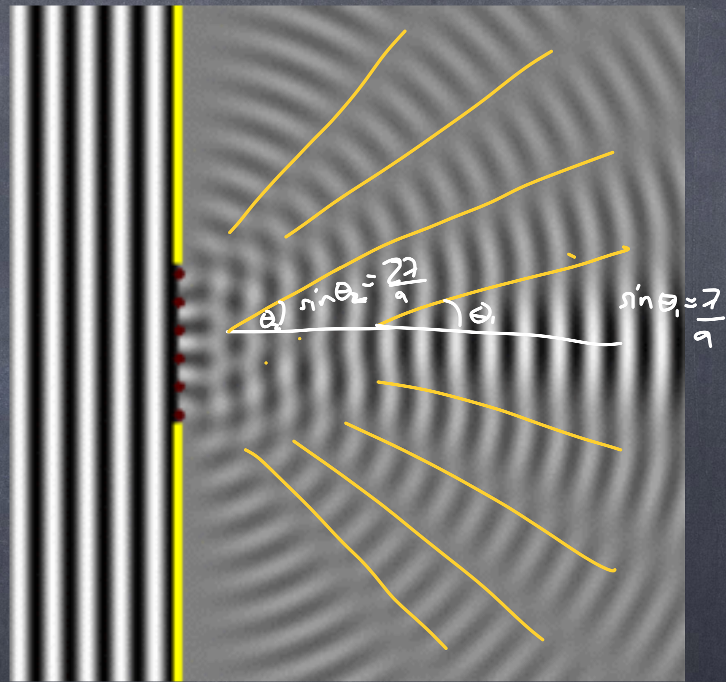
wave

Huygen's principle: every point on a wave front serves as a source of secondary spherical waves (wavelets)

$a \sim \lambda$



If $a > \lambda$, more wavelets are produced, which interfere with each other:



we observe a diffraction pattern from constructive + destructive interference

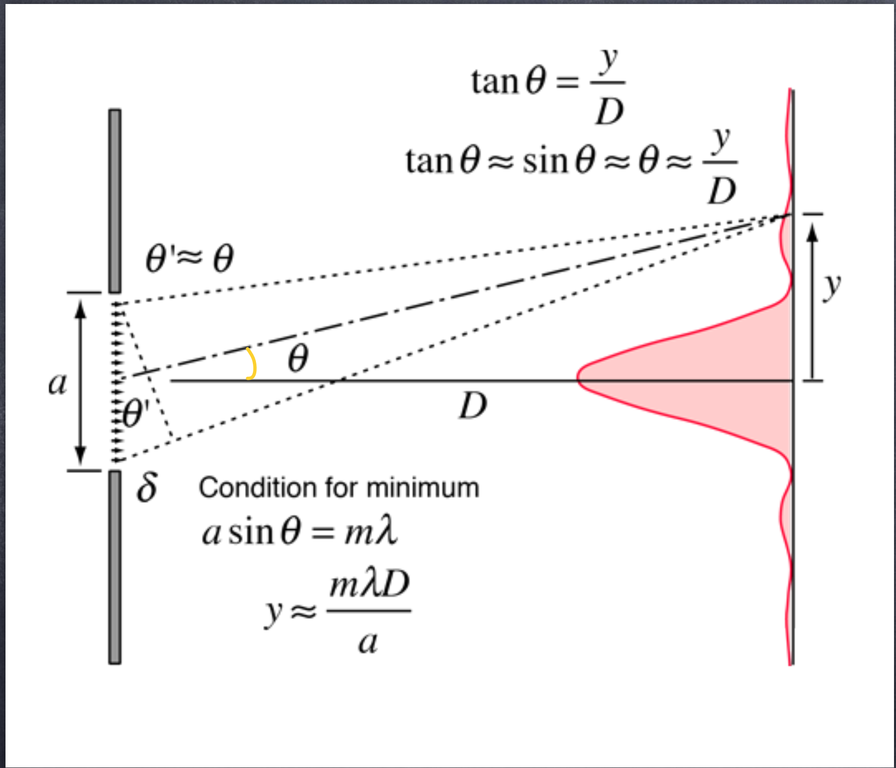
$$\sin\theta = \frac{m\lambda}{a}$$

$m = 1, 2, 3, \ldots$

(videos on lecture notes page)

If we were observing particles, we would instead see:

For a wave, we can calculate where the diffractive minima (dark spots) are



General rule:

$$\boxed{a\sin\theta = m\lambda \quad m = 1, 2, 3, \ldots} \quad (1)$$

for destructive interference → dark spots

From geometry, $\tan\theta = \frac{y}{D}$

For small angles, $\tan\theta \sim \sin\theta$

$$\boxed{\sin\theta = \frac{y}{D}} \quad (2)$$

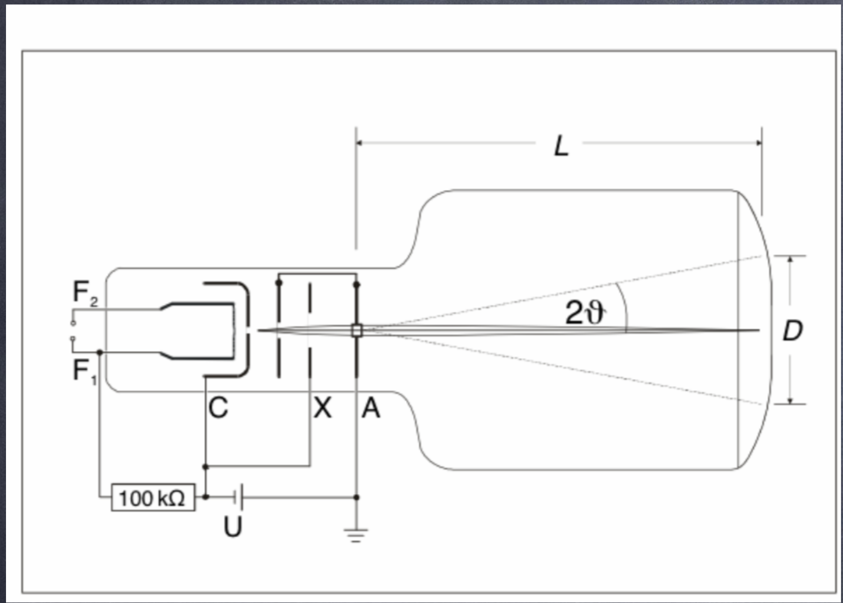
(1) + (2): $\sin\theta = \frac{y}{D} = \frac{m\lambda}{a}$

$$\boxed{y = \frac{m\lambda D}{a}}$$

$m = 1, 2, 3, \ldots$ location of dark spots.

Experiment in class: can we observe that electrons behave as waves by observing interference pattern?

Need to give electron momentum → wavelength



circular diffration with electrons.

we accelerate electrons in an electric potential, U.

$$\underbrace{eU}_{\text{potential energy}} = \underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic energy}}$$

$$v = \sqrt{\frac{2eU}{m}} \qquad p = mv = \sqrt{2emU}$$

The wavelength of the electrons is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meU}}$$

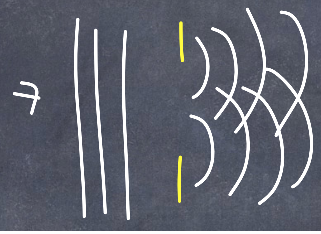
Note:

$$\lambda \propto \frac{1}{\sqrt{U}}$$

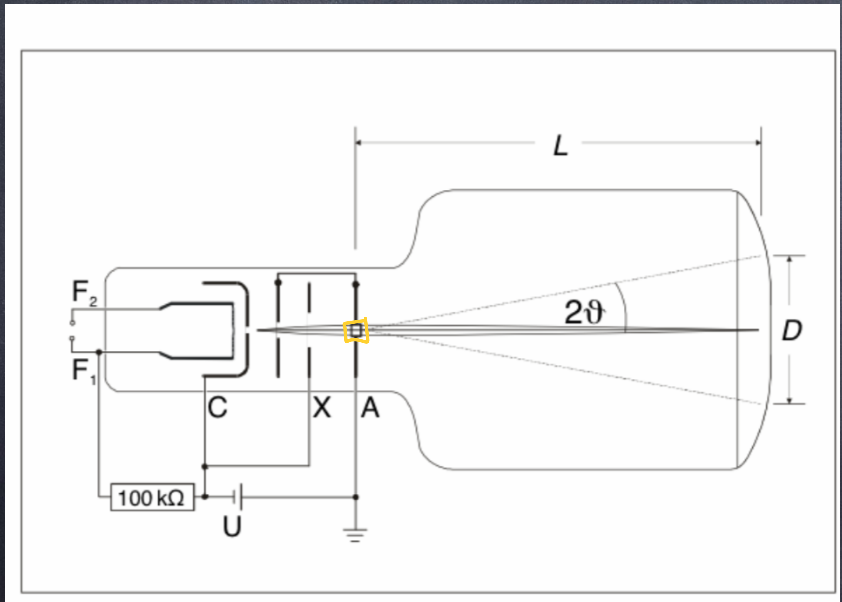
Target is 2-D hexagonal crystal (graphite)

what does 2-D diffraction look like?

1D:



2D:



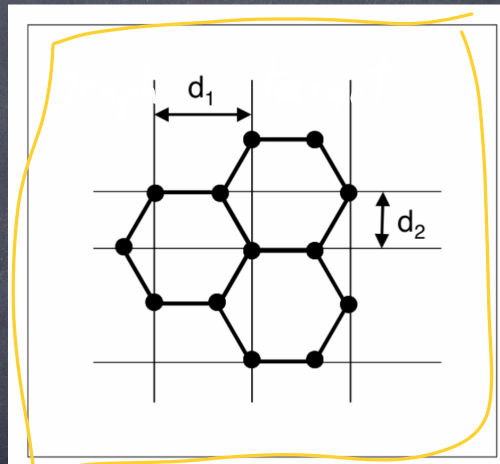
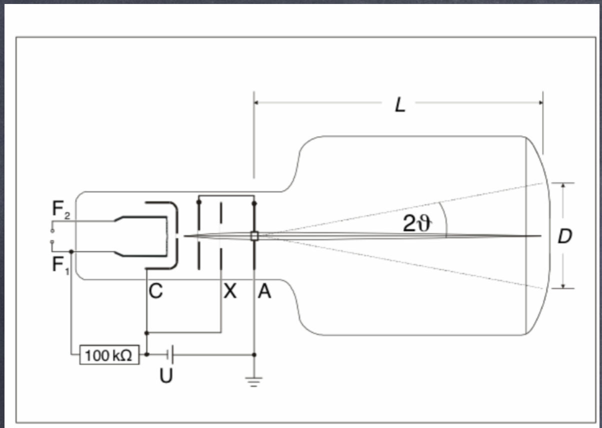


Fig. 3 Lattice plane spacings in graphite:
$d_1 = 2.13 \cdot 10^{-10}$ m
$d_2 = 1.23 \cdot 10^{-10}$ m

lattice spacings d_1, d_2



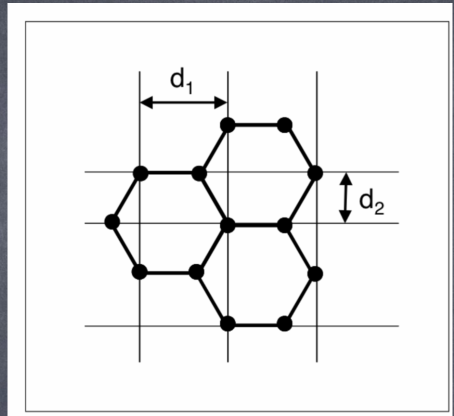


Fig. 3 Lattice plane spacings in graphite:
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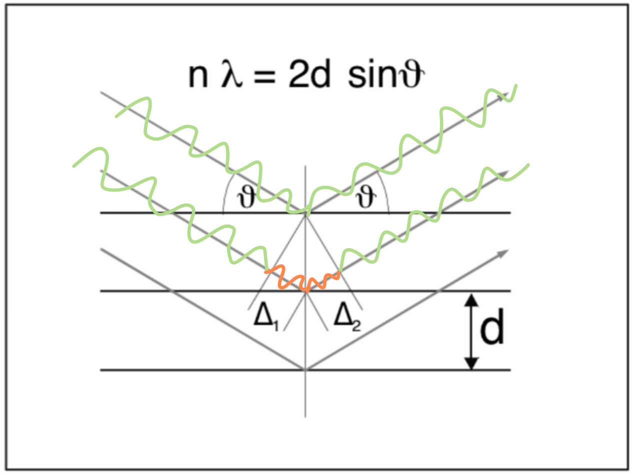
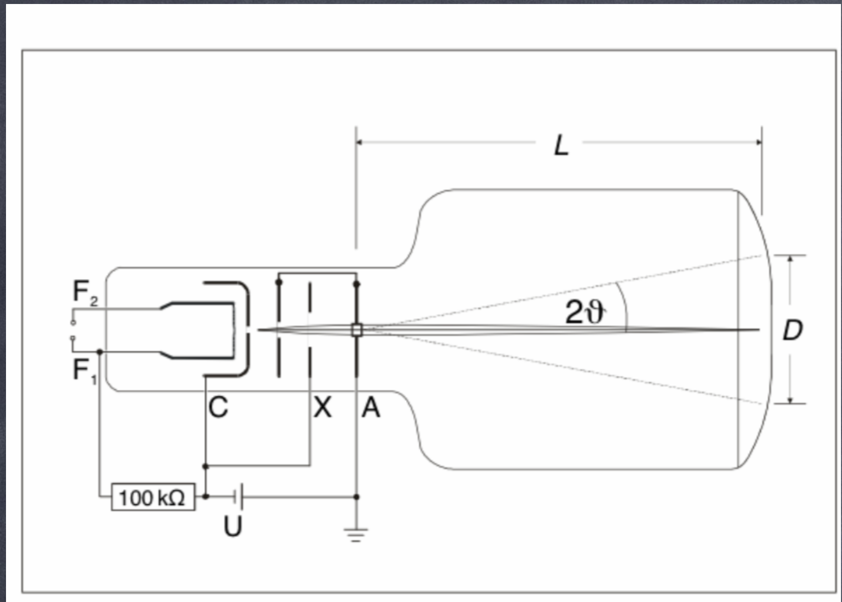


Fig. 2: Schematic representation of the Bragg condition.

Bright spot when path difference is an integer number of wavelengths:
Bragg condition $n\lambda = 2d\sin\theta$

(Here, there are two d values)



D/2

by geometry:

$$\tan 2\theta = \frac{\frac{D}{2}}{L} = \frac{D}{2L}$$

small-angle approximations:

$$\tan(2\theta) \sim \sin(2\theta) \sim 2\sin\theta$$

$$\boxed{2\sin\theta \simeq \frac{D}{2L}} \quad (a)$$

Bragg Diffraction condition for bright spots:

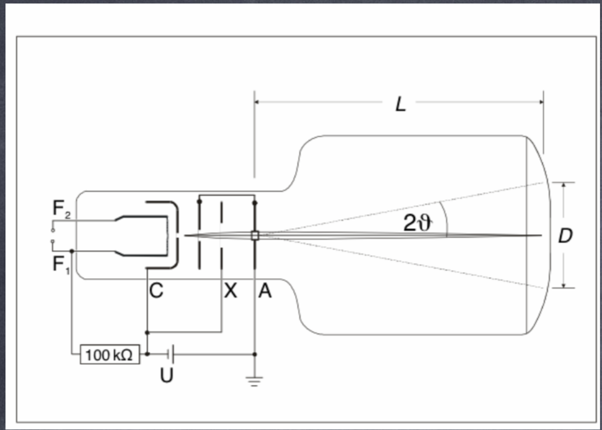
$$n\lambda = 2d\sin\theta$$

$$\boxed{2\sin\theta = \frac{n\lambda}{d}} \quad (b)$$

set (a) = (b) $\Rightarrow \dfrac{D}{2L} = \dfrac{n\lambda}{d}$

$$\boxed{D = \frac{n\lambda 2L}{d} \qquad n = 1, 2, 3, \dots}$$

This tells us where we will see bright rings.



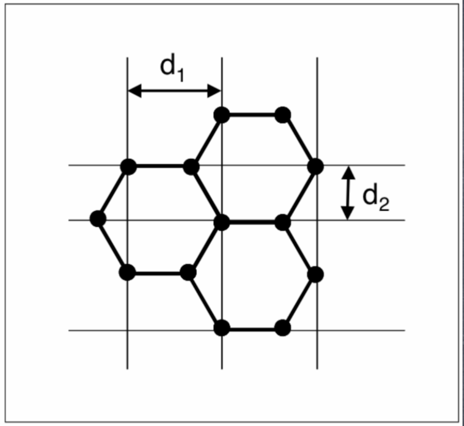


Fig. 3 Lattice plane spacings in graphite:
$d_1 = 2.13 \cdot 10^{-10}$ m
$d_2 = 1.23 \cdot 10^{-10}$ m

$$D = \frac{n \lambda 2L}{d}$$

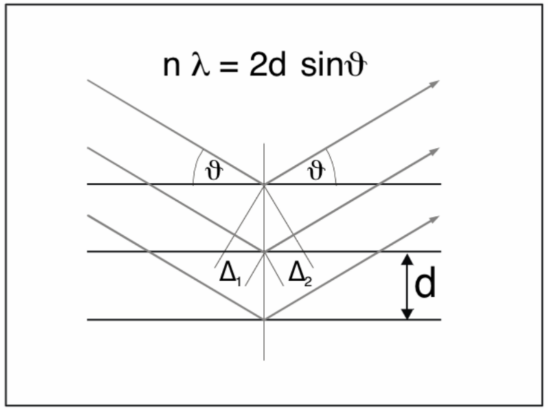


Fig. 2: Schematic representation of the Bragg condition.

for $n = 1$

$$D_1 = \frac{2L\lambda}{d_1}$$

$$D_2 = \frac{2L\lambda}{d_2}$$

If $d_1 = d_2$, then $D_1 = D_2$

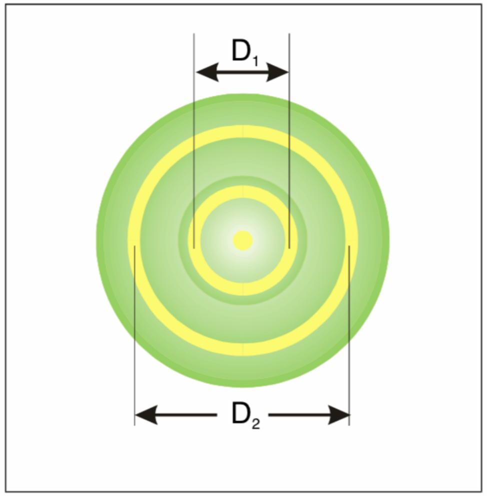


Fig. 1: Schematic representation of the observed ring pattern due to the diffraction of electrons on graphite. Two rings with diameters D_1 and D_2 are observed corresponding to the lattice plane spacings d_1 and d_2 (Fig. 3).

Note: diffraction allows us to measure atomic + molecular structure

(similar to X-ray crystallography)

If an electron is a wave, what does it look like?
where is it?

Consider a wave with a singular angular frequency ($\omega = 2\pi\nu$)
and a wave number ($k = \frac{2\pi}{\lambda}$) looks like this:
(not kinetic energy)

(perfect sine wave with exact ω & k)

It has no beginning or end in space.
we can't define where it is.

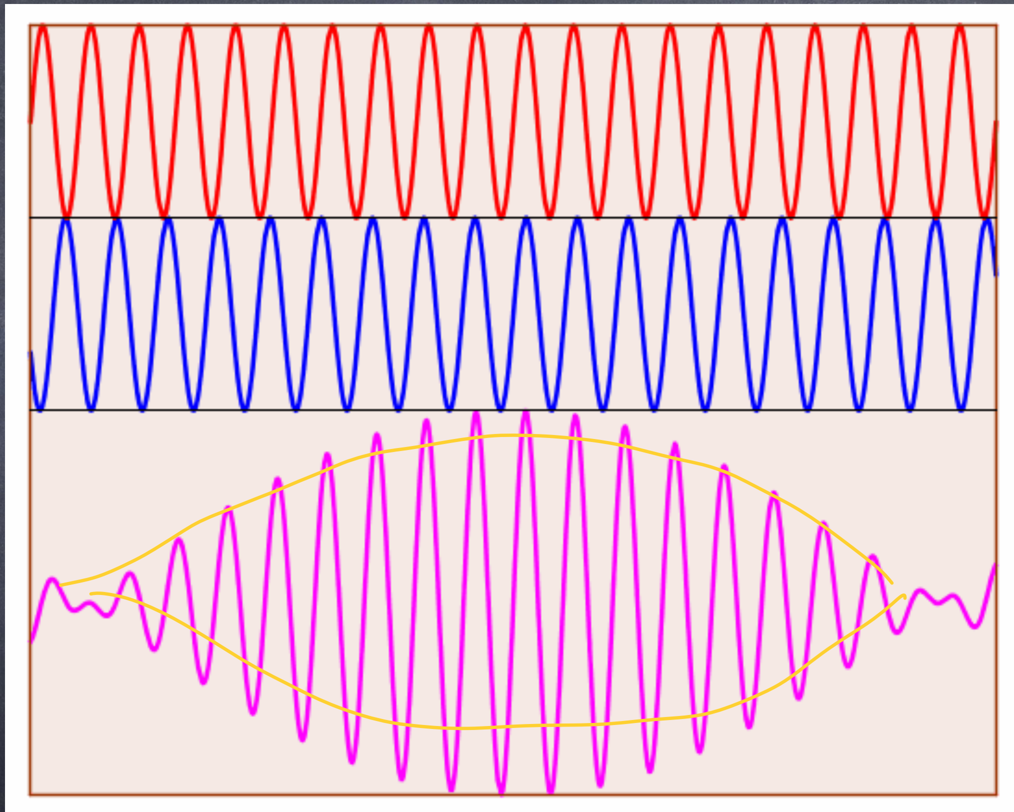
To represent a localized position of a wave,
we need a group of waves,
called a "wave packet".

Interference of 2 waves of slightly different ω + k

ω_1, k_1

ω_2, k_2

Sum

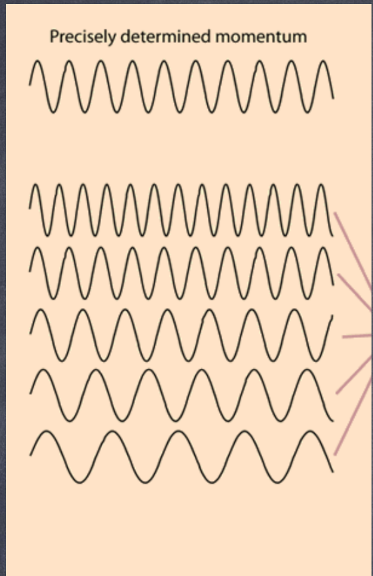


wave packet is the envelope of the added waves.

wave packet provides a location of a particle.

what if we add more waves of slightly different $\omega + k$

range of λ: λ_1, λ_2, λ_3, λ_4, λ_5, $\vdots$, λ_n



Each individual wave has its own $\lambda_i + k_i$

$\rightarrow$ own $\omega_i, k_i \Rightarrow V_i = \dfrac{\omega_i}{k_i}$

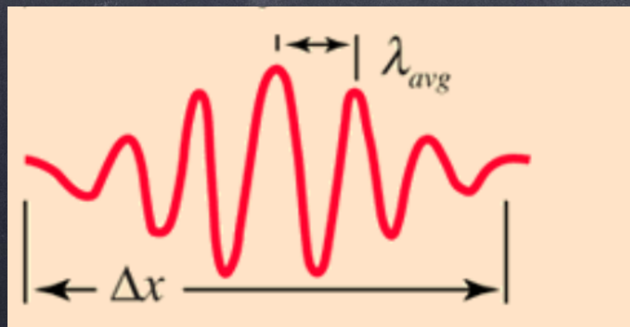
we can define

$$\Delta\lambda = \lambda_n - \lambda_1 \quad \text{and} \quad \lambda_{avg} = \sum_i^n \frac{\lambda_i}{n}$$

Also, $\Delta\omega = \omega_n - \omega_1 \qquad \omega_{avg} = \sum_i^n \dfrac{\omega_i}{n}$

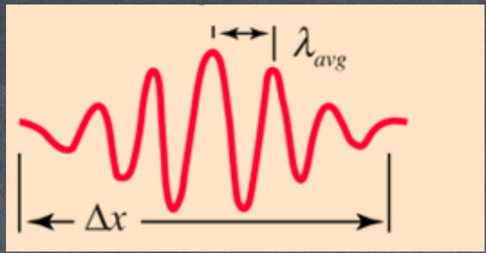
And $\Delta k = \dfrac{2\pi}{\Delta\lambda} \qquad k_{avg} = \dfrac{2\pi}{\lambda_{avg}}$

Resulting wave packet $\rightarrow$



Δx : location range of particle

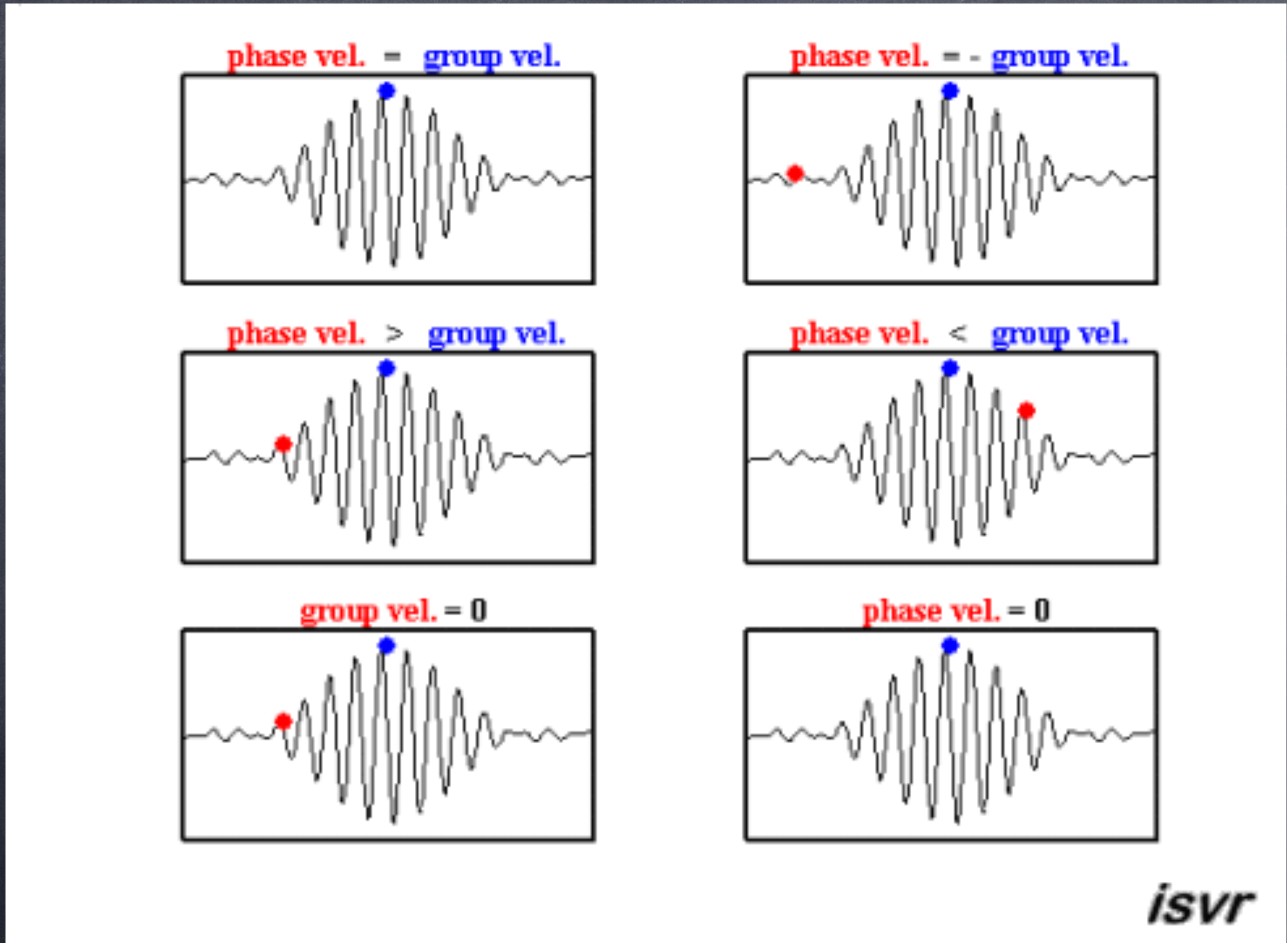
wave packet:



wave packet characterized by two velocities:
one related to $\Delta\lambda$ + one related to λ_{avg}

1) velocity of the wave packet is $\boxed{V_g = \frac{\Delta \omega}{\Delta K}}$ group velocity

2) The average of the individual waves has a velocity $\boxed{V_p = \frac{\omega_{avg}}{K_{avg}}}$ phase velocity


phase vel. = group vel.
phase vel. = - group vel.
phase vel. > group vel.
phase vel. < group vel.
group vel. = 0
phase vel. = 0
isvr

A general property of waves is that

$\Delta k \Delta x \sim 1$

$\Delta \omega \Delta t \sim 1$

Δk: range of wave numbers
Δx: location, or size of wave packet
$\Delta \omega$: range in angular frequencies
Δt: time available for measurement

If we know wavelength/wave number perfectly ($\Delta k = 0$) then $\Delta x \sim \frac{1}{0} = \infty$
so we have no idea where it is!

we can multiply these equations by a constant, $\frac{h}{2\pi} = \hbar$

$p = \frac{h}{\lambda} = \hbar k \qquad E = h\nu = \hbar\omega$

$\Delta p \Delta x \sim \hbar$

$\Delta E \Delta t \sim \hbar$

For example, if an excited atom has a lifetime τ, its energy can only be known to $\Delta E \sim \frac{\hbar}{\tau}$

These relations provide a fundamental concept.
Since particles are waves, they must obey the rules of waves. Therefore, we can't know both the p + x simultaneously. Likewise, we can't know both E + t simultaneously.

This is expressed by:

$$\boxed{\begin{aligned} \Delta p \, \Delta x &\geq \frac{\hbar}{2} \\ \Delta E \, \Delta t &\geq \frac{\hbar}{2} \end{aligned}}$$

Heisenberg's uncertainty principle(s).

Δt: time available for measurement

ΔE: uncertainty in measured energy.

Δp: uncertainty in measured momentum.

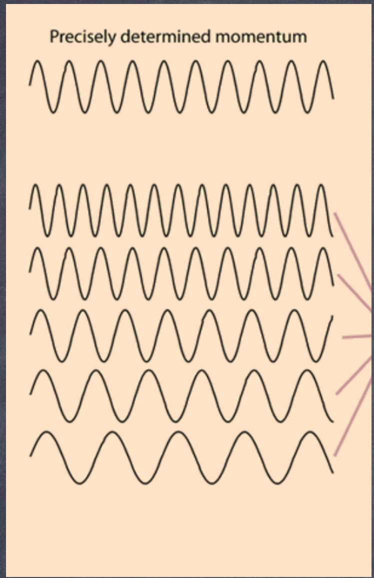
Δx: uncertainty in measured position.

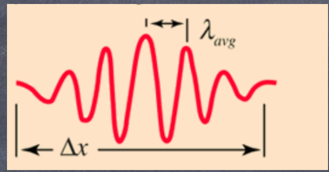
If you know λ exactly, then you know p exactly, so $\Delta p = 0$. Then $\Delta x \sim \infty$

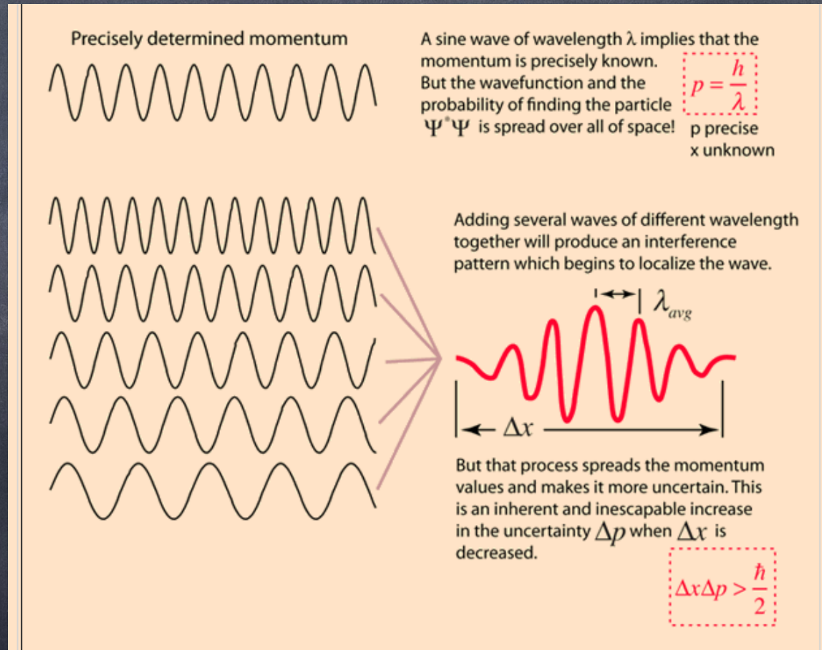
Like this example: λ is one precise value, so wave looks like

Its position x is completely unknown.

Recap:

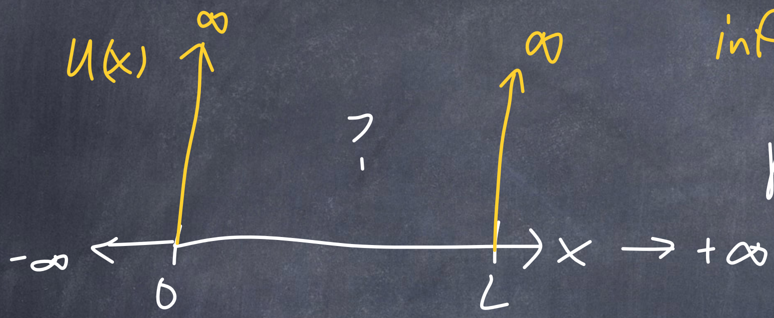

Precisely determined momentum


λ_{avg}
Δx


Precisely determined momentum
A sine wave of wavelength λ implies that the momentum is precisely known. But the wavefunction and the probability of finding the particle $\Psi^*\Psi$ is spread over all of space!
$p = \frac{h}{\lambda}$
p precise
x unknown
Adding several waves of different wavelength together will produce an interference pattern which begins to localize the wave.
λ_{avg}
Δx
But that process spreads the momentum values and makes it more uncertain. This is an inherent and inescapable increase in the uncertainty Δp when Δx is decreased.
$\Delta x \Delta p > \frac{\hbar}{2}$

Let's go back to our electron in a 1-D box.
What is its wave function? What are the allowed energies? Where is it?

infinitely tall box



particle is not outside the box

We write the wave function as $\Psi = \Psi(x)$

$P(x) = \Psi^2(x)$ is the probability of finding the particle at some x.

Boundary conditions: $\int_{-\infty}^{\infty} P(x) = 1$ particle is somewhere

But we know it is in box. $P(x=0) = 0$ $P(x=L) = 0$

The energy of the particle is $E = K + U$

The equation for quantum mechanics is:

Schroedinger wave equation (time-independent)

$$\boxed{\underbrace{\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}}_{\text{kinetic energy operator}} \Psi(x) + \underbrace{U(x)}_{\text{potential energy operator}} \Psi(x) = \underbrace{E}_{\text{total energy operator}} \Psi(x)}$$

operators operate on wave function (take derivative of it, or multiply it)

Inside the box, $U(x) = 0$, the particle has no potential energy

so we can simplify wave equation to:

$$\boxed{-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x)}{\partial x} = E\,\Psi(x)} \quad (a)$$

$$\frac{\partial^2 \Psi(x)}{\partial x^2} = -\frac{2mE}{\hbar^2} \Psi(x)$$

Now let's call $k^2 = \frac{2mE}{\hbar^2}$

k: constant, not kinetic energy (we will see that this k is wave number)

and write (a) as!

$$\boxed{E = \frac{k^2 \hbar^2}{2m}} \quad (1)$$

$$\boxed{\frac{\partial^2 \Psi(x)}{\partial x^2} = -k^2\, \Psi(x)} \quad (b)$$

To solve (b), we need a function whose 2nd derivative is $-k^2$ times the function.

Guess: If $\psi(x) = A\sin(kx)$

$$\frac{\partial \psi(x)}{\partial x} = Ak\cos(kx)$$

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -Ak^2 \sin kx$$

A: constant

k: wave number!

we can see that $\frac{\partial^2 \psi(x)}{\partial x^2} = -k^2 \psi(x)$

Also, $\psi(x) = B\cos kx$ also works.

A + B constants

So the general solution is $\psi(x) = A\sin kx + B\cos kx$

we also know that $\psi(x=0) = 0$, $\psi(x=L) = 0$

so:

$$\psi(x=0) = 0 = A\underbrace{\sin k(0)}_{0} + B\underbrace{\cos k(0)}_{1} = B$$

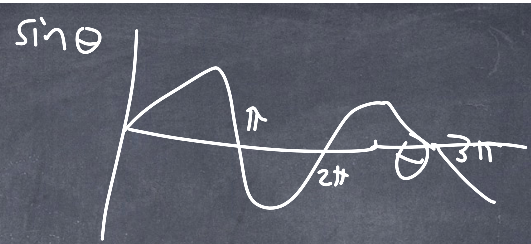
$B = 0$

B must be zero

$$\Psi(x=L) = 0 = A\sin(kL)$$

when does this happen?

It happens when $kL = n\pi$ $(n = 0, 1, 2, \ldots)$



we get solutions when $k = \dfrac{n\pi}{L}$ $(n = 0, 1, 2, \ldots)$

from ①, $E = \dfrac{k^2\hbar^2}{2m} = \dfrac{\left(\frac{n\pi}{L}\right)^2\hbar^2}{2m}$ $(n = 1, 2, \ldots)$

$$\boxed{E = \frac{n^2\hbar^2\pi^2}{2mL^2}}$$

or, since $\hbar = \dfrac{h}{2\pi}$,

$$\boxed{E_n = \frac{n^2h^2}{8mL^2} \quad \text{for } n = 1, 2, 3, \ldots}$$

The allowed energy levels of an electron in a 1-D box.

For $n=1$, $E_1 = \frac{h^2}{8mL^2}$ and $\boxed{E_n = n^2 E_1 \text{ for } n=1,2,\cdots}$

The energy is quantized!

Next, where is the electron?

$$\psi(x) = A \sin kx = A_n \sin\left(\frac{n\pi}{L}x\right) \quad \text{for } n=1,2,\cdots$$

A_n : constant for each n

what is A_n ?

$$\int_0^L \psi^2(x)\,dx = 1 \quad \text{must be true}$$

$$\int_0^L \left[A_n \sin\left(\frac{n\pi x}{L}\right)\right]^2 dx = 1 \longrightarrow A_n^2 \cdot \frac{L}{n\pi}\int_0^{n\pi} \sin^2\theta\, d\theta = 1$$

To solve this, we substitute

$$\frac{n\pi x}{L} = \theta \Rightarrow \frac{n\pi\, dx}{L} = d\theta$$

when $x=L$, $\theta = n\pi$

Solution for this integral : $$\int_a^b \sin^2 x \, dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big]_a^b$$

For our case: $$A_n^2 \frac{L}{n\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{n\pi} = A_n^2 \frac{L}{\cancel{n\pi}} \cdot \frac{\cancel{n\pi}}{2} = 1$$

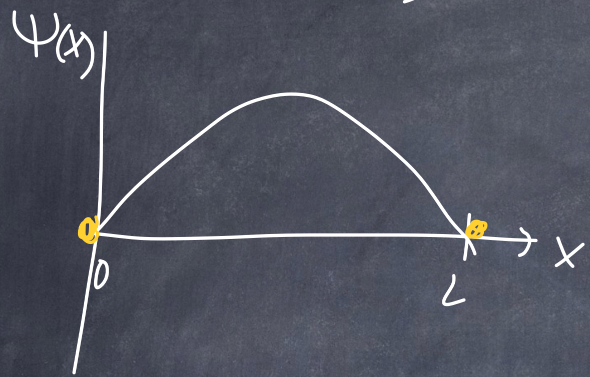
we find that $A_n = \sqrt{\frac{2}{L}}$ (independent of n)

so the solution is $$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

for $n = 1, 2, \ldots$

n: quantum numbers ($\frac{n\pi}{L}$ are wave numbers k_n)

If $n=1$, then $\psi_1(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{\pi x}{L}\right)$



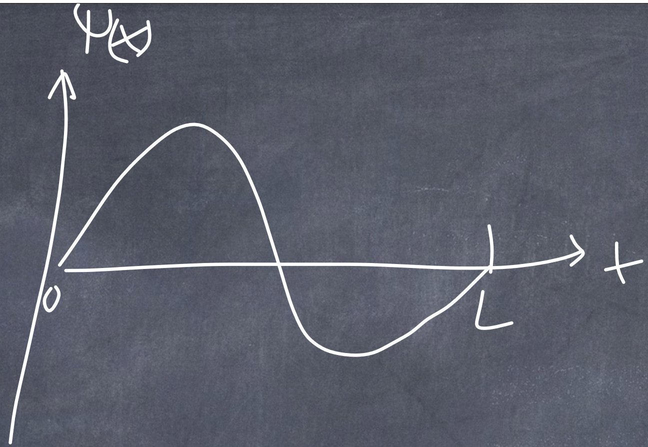
probability
$P(x) = \psi^2(x)$

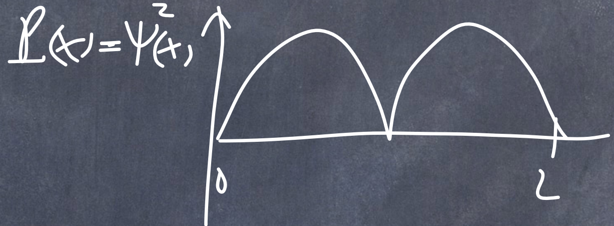
This is where the electron is!!

$$\boxed{E_n = \frac{n^2 h^2}{8 m L^2} \quad \text{for } n = 1, 2, 3, \ldots}$$

The allowed energy levels of an electron in a 1-D box.

For $n=2$





$n=2$

The classical limit $n \to$ large number