Applications of General Relativity
Solutions 4

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Exercise 1 [Out of plane precession of S2 orbit]

The time derivative of the orbital momentum $\boldsymbol{l} = \boldsymbol{r} \wedge \boldsymbol{v}$ is

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} &= \boldsymbol{r} \wedge \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} && (1)\\ &= -\boldsymbol{r} \wedge \nabla\phi + 2\boldsymbol{r} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{v}) && (2)\\ &= 2\boldsymbol{r} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{v}) && (3)\end{aligned}$$

We assume now $\boldsymbol{l} \perp \boldsymbol{S}$ and circular orbits. A good choice for the radius of the circular orbit may be the actual mean distance between focal point and S2 as obtained by integrating over the true anomaly[1]. This results in

$$\bar{r} = \frac{r_0(1-e^2)}{2\pi} \int \frac{\phi}{1 + e\cos\phi} = r_0\sqrt{1-e^2} = 2.2 \times 10^{-3} \text{ pc}, \tag{4}$$

where r_0 is the semi-major axis and e is the ellipticity.
The orbital time for S2 is given by

$$T_{\mathrm{o}} \approx 2\pi\sqrt{\frac{r_0^3}{GM}} = 16.86 \text{ yr}. \tag{5}$$

It is reasonable to assume that the frequency of the orbital plane precession is much smaller than the frequency of the orbital motion of S2 (this can be checked a posteriori). Under this assumption we can calculate the precession integrating over the unperturbed orbit. With $\boldsymbol{S} = S\boldsymbol{e}_y$, $\boldsymbol{r} = \bar{r}\,[\cos(\varphi)\,\boldsymbol{e}_x + \sin(\varphi)\,\boldsymbol{e}_y]$ and $\boldsymbol{v} = \bar{r}\omega\,[-\sin(\varphi)\boldsymbol{e}_x + \cos(\varphi)\boldsymbol{e}_y]$ we obtain

$$\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} = \frac{4GS\omega}{c^2\bar{r}}\left(\sin^2(\varphi)\boldsymbol{e}_x - \sin(\varphi)\cos(\varphi)\boldsymbol{e}_y\right) \tag{6}$$

After one orbit the change in angular momentum is

$$\Delta\boldsymbol{l} = \int \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t}dt = \frac{T_{\mathrm{o}}}{2\pi}\int \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t}d\varphi = 4\pi\frac{G\,S}{c^2\bar{r}}\boldsymbol{e}_x \tag{7}$$

The total change in the orbital angular momentum is perpendicular to the angular momentum, so the gravitomagnetic field does not change the magnitude of the angular momentum vector, but just its direction. The maximal angular frequency of the precession can now be estimated as

$$\Omega_{\max} = \frac{\Delta l}{lT_{\mathrm{o}}} = 2\frac{G^2M^2}{c^3\bar{r}^3}a, \tag{8}$$

[1] The true anomaly is the angle between the direction of periapsis and the current position of the body, as seen from the main focus of the ellipse.

where $a \in [0, 1]$ is the spin parameter of the central black hole. A comparison with the orbital frequency $\Omega_o = \frac{2\pi}{T_o}$ gives, setting $a = 1$,

$$\frac{\Omega_{\max}}{\Omega_o} \sim 10^{-6} \ll 1, \tag{9}$$

which is in agreement with our a priori assumption. Estimating the "real" out-of-plane precession as $\Omega_{\text{real}} = \frac{75}{360} \frac{2\pi}{t_{S2}}$, we obtain the constraint on the black hole spin parameter:

$$a \geq \left(4\pi \frac{G^2 M^2}{c^3 \bar{r}^3} \right)^{-1} \Omega_{\text{real}} \approx 0.40. \tag{10}$$

Since the lower bound lies within the interval allowed by General Relativity, the estimation we have performed is not able to exclude "gravitomagnetic" precession as a source of S2 inclination. Remember, however, that the approximations we have done are rather drastic, and we are not able, with these simple calculations, to say how realistic is the hypotesis of gravitomagnetic precession.