QFT methods for BSM theories
Solution 2.

HS 2019
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Exercise 1. *The scalar sector of the Standard Model*

The scalar and the Yukawa sectors of the Standard Model Lagrangian are given by,

$$\mathcal{L} = \bar{Q}_L i\not{D} Q_L + \bar{d}_R i\not{D} d_R + \bar{u}_R i\not{D} u_R + |D_\mu \Phi|^2 + \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger\Phi)^2 - y_u \bar{Q}_L \tilde{\Phi} u_R - y_d \bar{Q}_L \Phi d_R + h.c. \quad (1)$$

where $\Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(h - i\phi^0) \end{pmatrix}$ *is the Higgs scalar* $SU(2)_L$ *doublet,* $\phi^\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm i\phi_2)$ *and* $\tilde{\Phi} = i\tau_2 \Phi^*$. *The* D_μ *is the* $SU(2)_L \otimes U(1)_Y$ *gauge-covariant derivative is,*

$$D_\mu = \partial_\mu + ig\frac{\vec{\tau}}{2}\vec{W}_\mu - ig'\frac{\tau_3}{2}B_\mu. \quad (2)$$

where τ^i *are the Pauli-matrices.*

1. *In the limit of vanishing Yukawa and hypercharge interactions, which is the symmetry of the fermion sector?*
2. *Let us define the scalar field,*
$$\Sigma = h\mathbb{1} + i\vec{\tau} \cdot \vec{\phi}. \quad (3)$$
where $\vec{\phi} = (\phi^2 \ \phi^1 \ \phi^0)^T$.
Show that the scalar sector in terms of Σ,
$$\mathcal{L} = \frac{1}{4} Tr(D_\mu \Sigma^\dagger D^\mu \Sigma) + \frac{\mu^2}{4} Tr(\Sigma^\dagger\Sigma) - \frac{\lambda}{16}(Tr(\Sigma^\dagger\Sigma))^2, \quad (4)$$
is equivalent to the scalar sector in Eq. (1).
3. *Under which condition is the Yukawa sector of Eq. (1) equivalent to the term* $-\frac{y}{\sqrt{2}}\bar{Q}_L \Sigma\,(u_R \ d_R)^{\mathrm{T}}$? *How should* Σ *transform if we wish to keep this term invariant under the symmetry? How do the fields* h *and* $\vec{\phi}$ *transform in this case? Show that the scalar sector is then also invariant under the symmetry.*
4. *It is convenient to work with fields with nonzero vacuum expectation value. To this end we perform the shift,*
$$h \to h + v \quad and \quad \vec{\phi} \to \vec{\phi}, \quad (5)$$
where $v = \sqrt{\frac{\mu^2}{\lambda}}$. *Which is the symmetry of the vacuum? This symmetry is called custodial symmetry.*
5. *Expand the Lagrangian after the shift (5) and identify the mass spectrum. For simplicity, neglect the electroweak interactions.*
6. *Compute the amplitude for the scattering* $\phi^+\phi^- \to \phi^+\phi^-$ *at tree-level. Confirm that a basic property of Goldstone interactions is manifested. Then examine the behaviour of each diagram in the low-energy limit.*

We define a more convenient realisation of the model by performing a field redefinition of the form,

$$\Sigma(x) = (v + H(x))U(G(x)), \quad U(G(x)) = \exp\left(i\vec{\tau}\cdot\vec{G}(x)/v\right) \tag{6}$$

where $\vec{G} = (G^2\ G^1\ G^0)^T$.

7. *How do* H *and* U *transform under the symmetry? How do the fields* $\vec{G}$ *transform under the custodial symmetry?*
8. *Show that using Eq. (6) the Lagrangian (neglecting the Yukawa interaction) takes the form,*

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu H)^2 - 2\lambda v^2 H^2\right] + \frac{(v+H)^2}{4} Tr(D_\mu U^\dagger D^\mu U) - \lambda v H^3 - \frac{\lambda}{4}H^4. \tag{7}$$

9. *Verify explicitly that the amplitude for the scattering* $G^+G^- \to G^+G^-$ *at tree-level is the same with the full result of part 6. In the low-energy limit, what difference do you observe with what you found in part 6?*
10. *Consider now a theory described solely by the Lagrangian,*

$$\mathcal{L} = \frac{v^2}{4} Tr(D_\mu U^\dagger D^\mu U), \tag{8}$$

Calculate the cross-section of the process $G^+G^- \to G^+G^-$ *at tree-level, take the high energy limit and determine at which point the theory loses unitarity. Can you comment on the necessity of the field* H *in Eq. (7) with respect to the unitarization of the theory.*

Solution.

1. If $y_u = y_d = g' = 0$ then the fermion sector is invariant under the transformation,

$$\begin{aligned}\psi_L \to \psi' = L\psi_L, \quad L = \exp(i\vec{a}_L\cdot\vec{\tau})\\ \psi_R \to \psi' = R\psi_R, \quad R = \exp(i\vec{a}_R\cdot\vec{\tau})\end{aligned} \tag{S.1}$$

The corresponding symmetry $\mathcal{G} = SU(2)_L \times SU(2)_R$ is called chiral symmetry.

The Lagrangian is also invariant under a phase shift $\psi_{L/R} \to \psi'_{L/R} = e^{iBa_B}\psi_{L/R}$ and the corresponding symmetry $U(1)_B$ is recognized as the baryon number.

2. Notice that the field Σ can be written as the following vector,

$$\Sigma = \sqrt{2}(\tilde{\Phi}\ \Phi)^T \tag{S.2}$$

We then find,

$$\begin{aligned}Tr(\Sigma^\dagger\Sigma) = 2Tr(\tilde{\Phi}^\dagger\tilde{\Phi} + \Phi^\dagger\Phi) = 4\Phi^\dagger\Phi \quad \text{and}\\ Tr(D_\mu\Sigma^\dagger D^\mu\Sigma) = 2Tr(D_\mu\tilde{\Phi}^\dagger D^\mu\tilde{\Phi} + D_\mu\Phi^\dagger D^\mu\Phi) = 4D_\mu\Phi^\dagger D^\mu\Phi\end{aligned} \tag{S.3}$$

from which the equivalence of the two Lagrangian follows.

3. From Eq. (S.2) we immediately see that the condition is $y_u = y_d = y$. Furthermore, this term is invariant under the chiral symmetry if the scalar field Σ acquires the following transformation property,

$$\Sigma \to \Sigma' = L\Sigma R^\dagger \tag{S.4}$$

and the invariance of the scalar sector follows trivially[1].

Now, using the properties of Pauli matrices one can show that,

$$h = \frac{1}{2}Tr(\Sigma) \quad \text{and} \quad \phi^i = \frac{1}{2}Tr(\tau^i \Sigma) \tag{S.5}$$

and then determine the transformation properties,

$$h \to h' = h + \frac{1}{2}(a_L^i - a_R^i)\phi^i \quad \text{and} \quad \phi'^i = \phi^i - \frac{1}{2}(a_L^i - a_R^i)h - \frac{1}{2}\epsilon^{ijk}(a_L^j + a_R^j)\phi^k. \tag{S.6}$$

Now, we see that the fields h and $\vec{\phi}$ mix under chiral transformations. This is an important point since it means that we cannot integrate out any of the constituent fields of Σ without spoiling explicitly the chiral symmetry.

4. The vacuum is invariant,

$$\langle \Sigma \rangle = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \to \langle \Sigma' \rangle = L \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} R^\dagger \tag{S.7}$$

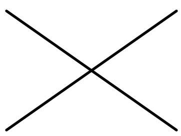
only if $L = R$. The custodial symmetry is then defined by the so-called isospin group $SU(2)_V$ (where V stands for vectorial). Looking back at Eq. (S.6), we now see that h is invariant under the custodial symmetry, while $\vec{\phi}$ transforms in the adjoint representation of $SU(2)_V$. h is then what we call an isospin singlet and $\vec{\phi}$ an isospin triplet.

5. After dropping irrelevant constant terms, we obtain:

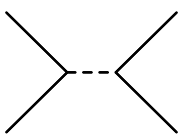
$$\begin{aligned} \mathcal{L} = \bar{\psi} i \not{\partial} \psi - \frac{yv}{\sqrt{2}}\bar{\psi}\psi + \frac{1}{2}\left[(\partial_\mu h)^2 - 2\lambda v^2 h^2\right] + \frac{1}{2}(\partial_\mu \vec{\phi})^2 \\ - \frac{y}{\sqrt{2}}\bar{\psi}(h\mathbb{1} + i\vec{\tau}\cdot\vec{\phi})\psi - \frac{\lambda}{4}(h^2 + \vec{\phi}^2)^2 - \lambda v h(h^2 + \vec{\phi}^2) \end{aligned} \tag{S.8}$$

where $\psi = (u_L + u_R, d_L + d_R)^T$. You may recognize this Lagrangian as the famous linear-sigma model. As expected, the fields $\vec{\phi}$ remain massless and they correspond to the Goldstone bosons of the SM, while the scalar h acquires the mass $m_h^2 = 2\lambda v^2$ and can be identified with the massive radial component of the Higgs boson.

6. According to the interactions described by Lagrangian, the two diagrams contributing to the tree-level scattering are according,



$$= -4i\lambda$$



$$= (-2i\lambda v)^2 \sum_{s,t} \frac{i}{(p+p')^2 - m_h^2} = -4i\lambda^2 v^2 \left(\frac{1}{s - m_h^2} + \frac{1}{t - m_h^2}\right), \tag{S.9}$$

[1] Notice that the covariant derivative $D_\mu \Sigma$ transforms the same way as Σ by construction.

The full amplitude of the scattering is,

$$\mathcal{M}(\phi^+\phi^- \to \phi^+\phi^-) = -2\lambda\left(2 + \frac{2\lambda v^2}{s - m_h^2} + \frac{2\lambda v^2}{t - m_h^2}\right) = -2\lambda\left(\frac{s}{s - 2\lambda v^2} + \frac{t}{t - 2\lambda v^2}\right). \tag{S.10}$$

In the low-energy limit $m_h^2 \gg s$ we get,

$$\mathcal{M}(\phi^+\phi^- \to \phi^+\phi^-) \approx \frac{s+t}{v^2} + \frac{s^2 + t^2}{2m_h^2 v^2} + \mathcal{O}\left(\frac{1}{m_h^4}\right) \tag{S.11}$$

which makes it clear that the interaction is a derivative interaction characteristic of Goldstone bosons. However, the individual diagrams involving only the Goldstone bosons do not exhibit this feature and one always has to take into account the diagrams with the heavy field h. This is a consequence of the conclusion we have drawn at the end of part 3.

7. Note that $H^2 = \mathrm{Tr}(\Sigma^\dagger\Sigma)/2$ and therefore H is a singlet under the chiral symmetry[2]. As a result, U must transform the same way Σ does,

$$U \to LUR^\dagger. \tag{S.12}$$

We may use the Baker-Campbell-Hausdorff formula and write,

$$LUR^\dagger = \exp\left(i\tau^i \frac{G^j}{v}\left[\delta_{ij} + \epsilon_{ijk}(a_L^k + a_R^k)\right] - i\left[\frac{G^i G^j}{3v^2}(\delta_{ij}\tau^k - \delta_{ik}\tau^j) - \tau^k\right](a_L^k - a_R^k)\right) \tag{S.13}$$

As it is evident, the fields $\vec{G}$ transform non-linearly under the chiral symmetry. However, a $SU(2)_V$ transformation (i.e. $a_L = a_R$) is linear. Therefore we have reproduced one of the main results of the lecture, namely that the Goldstone degrees of freedom, transform linearly under the unbroken subgroup.

We also see then that in this parametrization the custodial-symmetry singlet H and the the triplet U no longer transform jointly under the chiral symmetry. In other words, the fields $\vec{G}$ in the non-linear representation transform non-linearly under the chiral transformation, to compensate for the non-transformation properties of H. Therefore we expect that H can be integrated out while preserving the chiral symmetry and hence obtain a pure theory of Goldstone fields.

8. Direct computation yields the required Lagrangian form. We confirm that the fields $\vec{G}$ remain massless and the field H acquires the mass $m_H^2 = 2\lambda v^2$.

9. We expand the exponential,

$$U = \mathbb{1} + i\frac{\vec{\tau}\cdot\vec{G}}{v} - \frac{|\vec{G}|^2}{2v^2} - i\frac{|\vec{G}|^2}{6v^3}(\vec{\tau}\cdot\vec{G}) + \dots\,, \tag{S.14}$$

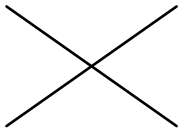
and insert it into Eq. (7) one gets for the second term,

$$\begin{aligned}\Delta\mathcal{L} &= \frac{1}{6v^2}\left[(\vec{G}\cdot\partial_\mu\vec{G})^2 - G^2(\partial_\mu\vec{G}\cdot\partial^\mu\vec{G})\right] + \frac{H}{v}(\partial_\mu\vec{G})^2 + \dots \\ &= \frac{1}{6v^2}\left(G^-\partial_\mu G^+ - G^+\partial_\mu G^-\right)^2 + \frac{2H}{v}(\partial_\mu G^+)(\partial^\mu G^-) + \dots\,.\end{aligned} \tag{S.15}$$

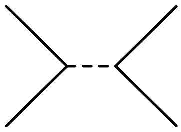
where we have shown only the terms relevant to our process. In contrast to the linear-sigma model, the derivative interactions of the Goldstone fields are now manifested at the Lagrangian level.

[2] This relation holds of course, before expanding around the chiral-symmetry breaking vacuum.

One may now read off the Feynman rules and compute the amplitudes:



$$= i\frac{2}{6v^2}\sum_{s,t}(p-p')^2 = i\frac{s+t}{v^2}$$



$$= \frac{i^2}{v^2}\sum_{s,t}(2pp')^2\frac{i}{(p+p')^2-m_H^2} = -i\left(\frac{s^2}{s-m_H^2}+\frac{t^2}{t-m_H^2}\right), \tag{S.16}$$

The full amplitude of the scattering is,

$$\mathcal{M}(G^+G^- \to G^+G^-) = \frac{1}{v^2}\left(\frac{sm_H^2}{s-m_H^2}+\frac{tm_H^2}{t-m_H^2}\right) \tag{S.17}$$

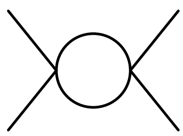
Once again we ascertain that this parametrization is more suitable to describe a theory of Goldstone fields at low-energies, since the diagrams are independently momentum-dependent without the need to invoke any "strange" cancellation between them as in the linear case.

Moreover we notice that the full amplitude is the same with the one computed in the linear-sigma model (S.11). This a direct consequence of the so-called *Representation invariance Theorem* (or *Haag's Theorem*), which states that if two fields are related non-linearly and the Jacobian of the transformation is 1 then the same experimental observables result if one calculates with the original or the redefined field. The proof consists basically of demonstrating that (i) two S-matrices are equivalent if they have the same single particle singularities, and (ii) that the two fields have the same free field behaviour and single particle singularities.

10. You may recognize the Lagrangian (8) as the leading-order term of the χPT Lagrangian. The cross-section for the Goldstone scattering in this theory is given in terms of the amplitude generated by the diagram (S.5 a). We may write,

$$\sigma(G^+G^- \to G^+G^-) = \frac{|\mathcal{M}|^2}{16\pi s} = \frac{(s+t)^2}{16\pi v^4 s} \tag{S.18}$$

and thus the cross-section grows scales with the energy, which implies a violation of unitarity at high-energies. The exact point when this happens can be found quantitatively by performing a decomposition of the amplitude in partical waves and consequently employing the optical theorem (the derivation of which, relies on the definition of unitarity). A rough estimate can be given alternatively by the condition that the higher order-contributions must be smaller than the tree-level ones. For example consider the one-loop approximation,



$$\approx i\frac{N_G}{16\pi^2}\frac{s^2}{v^4}\log(s/\mu^2) \tag{S.19}$$

where N_G is the number of Goldstone degrees of freedom. Perturbativity is lost when this amplitude approaches s/v^2 and so the resulting limit is,

$$\sqrt{s} \approx \frac{4\pi v}{\sqrt{N_G}}$$

As you have seen in the lecture, in χPT Lagrangian strongly-coupled resonances (e.g. the ρ meson) appear at energies below $\Lambda_{\rm QCD} \approx 4\pi f_\pi$ and restore unitarity in the pion-scattering. In our case, the case of the SM, the naive estimate is $\Lambda_{\rm SM} \approx 4\pi v \approx 1.8$ TeV. A more detailed calculation, involving the partial wave analysis and several coupled channels, gives a bound $\Lambda_{\rm SM} \lesssim 700$ GeV.

Now, in order to appreciate the above bound, one may recall that according to the *Equivalence Theorem*, amplitudes involving longitudinally polarized $W_L^\pm$ gauge bosons are equal to amplitudes with the corresponding Goldstone bosons $G^\pm$ as external particles up to corrections of $\mathcal{O}(m_W^2/s)$. This means that in order to unitarize W-boson scattering something has to happen at energies $\lesssim 700$ GeV or equivalently, the electroweak chiral Lagrangian necessarily breaks down and has to be replaced by a high-energy theory. This is of course the theory described by the Lagrangian (7) involving the SM Higgs boson H. The amplitude in this case, as calculated in Eq. (S.17), no longer grows at high energies. The Higgs thus unitarizes the SM.